Boundary of recursion relation

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based on work with Qingjun Jin, Chenkai Qiao, Junjie Rao, Kang Zhou.

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- Background and motivation
- Proposal for boundary contribution
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Part I: Background

- In last ten years, we have seen an extraordinary development of on-shell technique for scattering amplitude
- It is the renascence of old S-matrix program, where analyticity has played a crucial role.
- However, old S-matrix program treats amplitude as functions of all complex external momenta, so the analyticity of multiple complex variables is much more difficult to be understood
- A natural approach of avoiding above complexities is to consider only constrained variations. This is the key of new on-shell technique.

Let us consider on-shell tree-level amplitudes:

- They are rational functions of all external momenta.
- However, we keep all other momenta invariant and choose only a pair of momenta p_i, p_j, doing following deformation

$$p_i(z) = p_i + zq, \qquad p_j(z) = p_j - zq$$

  which satisfies momentum conservation automatically.
- To keep on-shell conditions for all z values, we impose conditions

$$q^2 = q \cdot p_i = q \cdot p_j = 0$$

What we have achieved under above deformation?

- First, the tree amplitude $\mathcal{A}(z)$ becomes rational functions of single complex variables z, for which we have many powerful results provided by mathematicians.
- Physically, by Feynman diagrams, only possible singularities comes from putting propagators on-shell.
- More explicitly, by $(P+zq)^2 = P^2 + z(2P\cdot q)$, i.e., poles can only be single poles.

- Let us consider the contour integration $I = \oint dz A(z)/z$. One can evaluate by two ways:
  - Doing it along the point $z = \infty$, we get the **"boundary contribution"** which will denote as $\mathcal{B}$.
  - Doing it for big cycle around $z = 0$, we have $I = A(0) + \sum_\alpha \mathrm{Res}(A(z)/z)|_{z_\alpha}$.
- Combining above we have

$$A(z=0) = B - \sum_{poles\ z_\alpha} \mathrm{Res}\left(\frac{A(z)}{z}\right)_{z=z_\alpha}$$

[Britto, Cachazo, Feng , 2004] [Britto, Cachazo, Feng , Witten, 2004]

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Residue of finite pole z_α:

- **Location:** It can be found by solving $P^2 + z_\alpha(2P\cdot q) = 0$.
- **Residue:** there is an important Factorization property: when one propagator goes to on-shell, i.e., $P^2 - m^2 \to 0$, we have

$$A^{tree}(1,..,n) \quad \to \quad \sum_\lambda A_{m+1}(1,..,m,P^\lambda)\frac{1}{P_{1m}^2 - m^2} A_{n-m+1}(-P^{-\lambda}, m+1,...,n)$$

Using it we get

$$\left(\frac{A(z)}{z}\right)_{z=z_\alpha} = \sum_\lambda A^L_{m+1}(1,..,m,P^\lambda(z_\alpha)) \frac{1}{P^2} A^R_{n-m+1}(-P^{-\lambda}(z_\alpha), m+1,...,n)$$

How about the boundary contribution?

- It is well known that when $z \to \infty$, $A(z) \to \sum_{i=0}^{k} c_i z^i + \mathcal{O}(1/z)$ with $c_0 \neq 0$, there is nonzero boundary contribution
- But how to determine if there is a boundary contribution for a given theory ? A nice method is the background field method. [Arkani-Hamed, Kaplan 2008]
- Using above method, one can see that for many interesting applications, we can choose deformation to avoid boundary.

Let us have a better look of the analytic structure of boundaries:

- It could be a rational function. The key is that under the chosen deformation of (p_i, p_j), there are propagators not depending on both p_i, p_j, so they are z-independent. We will call them undetectable under the given deformation!

- It could be pure polynomial of external momenta. Since our principle is to determine them using pole structure, it is very natural that the method can not work for these terms. Then what is the effective method?

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- Finding other deformations to avoid boundary contributions
- Using new deformations to detect these undetectable poles
- Using roots, which is very natural for polynomial structures

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Part II: Avoiding Boundary

Key idea: taking more momenta to do deformation

- The first application of the idea is to prove the MHV-expansion by Risager, where anti-spinor parts of all particles with negative helicities have been deformed as $|m] + z\alpha_m|\eta]$ so on-shell condition has been kept! [Risager, 2005]
- Then all-line deformation has been discussed by Cohen, Elvang, Kiermaier to discuss the on-shell constructibility for general theories (where anti-spinor parts of all particles have been deformed).

[Cohen, Elvang, Kiermaier, 2010]

- Above multiple line deformation still makes each propagator as linear function of z, so does not improve the convergence too much.
- Cheung, Shen, and Trnka have considered more general deformations, where all particles have been divided into two groups. In one group spinor parts have been deformed $|i\rangle + z|\eta_i\rangle$ while in another group, anti-spinor parts have been deformed $|i] + z|\widetilde{\eta}_i]$.

  [Cheung, Shen, and Trnka, 2015]
- The key of above deformation is that now in general propagator will be quadratic of z, so it has increased the z-power in denominator and improved the convergence a lot.

Remarks:

- Although power of z in denominator has increased a lot, it is not guaranteed to eliminate boundary!
- With more deformed momenta, recursion relation will contains more terms. Also expression of each term becomes more complicated with explicit square root in general. Only after proper summing, we get rational expression.

Part III: Multiple deformations

- As we have remarked, for a given BCFW-deformation, only a subset of poles can be detected
- Thus to detect boundary, which contains undetected poles, we should use new deformation

Let us demonstrate above idea with the initial deformation $\underline{0} \equiv \langle i_0|j_0]$:

- Under the deformation, some physical propagators will depend on deformed parameter z_0 (which will be called **detectable propagators** and denoted by $\mathcal{D}^{\underline{0}}$), while some physical propagators will NOT depend on deformed parameter z_0 (which will be called **undetectable propagators** and denoted by $\mathcal{U}^{\underline{0}}$)
- Observation: Boundary could contain poles only from undetectable propagators $\mathcal{U}^{\underline{0}}$ as well as **spurious poles** (denoted by $\mathcal{S}^{\underline{0}}$)

- More explicitly, the expansion

$$-A_n^{\underline{0}}(z_{\underline{0}}) = \frac{N(z_{\underline{0}})}{\prod P_t^2(z_{\underline{0}})} = \mathcal{R}^{\underline{0}}(z_{\underline{0}}) + \mathcal{B}^{\underline{0}}(z_{\underline{0}}).$$

with **recursive part** as

$$\mathcal{R}^{\underline{0}}(z_{\underline{0}}) = \sum_{P_t \in \mathcal{D}^{\underline{0}}} \frac{A_{t;L}(\widehat{z}_{\underline{0},t}) A_{t;R}(\widehat{z}_{\underline{0},t})}{P_t^2(z_{\underline{0}})},$$

and the **regular part** as

$$\mathcal{B}^{\underline{0}}(z_{\underline{0}}) = C_0^{\underline{0}} + \sum C_i^{\underline{0}} z_{\underline{0}}^i .$$

- **Key observation:** the poles $P_t \in \mathcal{D}^{\underline{0}}$ will appear once and only once with power one in $\mathcal{R}^{\underline{0}}$, i.e., they cannot be the poles of coefficients $\mathcal{B}^{\underline{0}}$
- **Pole structure of boundary:** (I) It belongs to $\mathcal{U}^{\underline{0}}$ or $\mathcal{S}^{\underline{0}}$; (II) The powers of spurious poles in $\mathcal{B}^{\underline{0}}$ may be larger than one.
- **Fact:** The part $\mathcal{R}^{\underline{0}}$ is **known** by recursion relation, while the part $\mathcal{B}^{\underline{0}}$ is not known.
- Natural idea: Consider a new deformation, which can detect poles from the set $\mathcal{U}^{\underline{0}}$ and $\mathcal{S}^{\underline{0}}$, so we will determine part of rational part of boundary!!

Now we explain the strategy: Let us consider the second deformation $\underline{1} \equiv \langle i_1 | j_1]$:

- The full amplitude can be calculated by two ways:
  - **The first way:** Using the recursion relation

$$-A_n^{\underline{1}}(z_{\underline{1}}) = \mathcal{R}^{\underline{1}}(z_{\underline{1}}) + \mathcal{B}^{\underline{1}}(z_{\underline{1}})$$

  - **The second way:** Using expression $-A_n^{\underline{0}}(z_{\underline{0}} = 0)$ to make the deformation and the expansion

$$\begin{aligned} \mathcal{R}^{\underline{0}}(z_{\underline{1}}) &= \mathcal{R}\mathcal{R}^{\underline{0},\underline{1}}(z_{\underline{1}}) + \mathcal{R}\mathcal{B}^{\underline{0},\underline{1}}(z_{\underline{1}}) \\ \mathcal{B}^{\underline{0}}(z_{\underline{1}}) &= \mathcal{B}\mathcal{R}^{\underline{0},\underline{1}}(z_{\underline{1}}) + \mathcal{B}^{\underline{01}}(z_{\underline{1}}), \end{aligned}$$

- **Key observation:** Identifying two ways,

$$\mathcal{R}^{\underline{1}}(z_{\underline{1}}) = \mathcal{R}\mathcal{R}^{\underline{0},\underline{1}}(z_{\underline{1}}) + \mathcal{B}\mathcal{R}^{\underline{0},\underline{1}}(z_{\underline{1}}).$$

- Using two deformations, we can find part of unknown boundary $\mathcal{B}^{\underline{01}}$, which depends on poles $P_t \in \mathcal{U}^{\underline{0}} \bigcap \mathcal{D}^{\underline{1}}$.
- It is easy to see our strategy: using enough deformations to detect all possible poles of unknown boundary $\mathcal{B}^{\underline{01}}$, thus we can determine it **up to polynomial part.**

Example: $A(1^+, 2, 3, 4, 5^+)$ of the color ordered Yukawa Theory

- Possible dependence of physical poles $\{\langle 1|2\rangle, \langle 4|5\rangle, \langle 5|1\rangle\}$
- With $\underline{0} = \langle 1|5]$

$$-A^{\underline{0}} = g^3 \frac{\langle 2|4\rangle}{\langle 2|1\rangle \langle 5|4\rangle} + \mathcal{B}^{\underline{0}}$$

with sets $\mathcal{D}^{\underline{0}} = \{\langle 1|2\rangle\}, \mathcal{U}^{\underline{0}} = \{\langle 4|5\rangle, \langle 5|1\rangle\}, \mathcal{S}^{\underline{0}} = \emptyset$

- With $\underline{1} = \langle 5|4]$,

$$\mathcal{R}^{\underline{1}}(z_{\underline{1}}) = g\lambda \frac{1}{\langle 1|5\rangle - z_{\underline{1}} \langle 1|4\rangle},$$

so we get

$$\mathcal{B}\mathcal{R}^{\underline{0},\underline{1}} = g\lambda \frac{1}{\langle 1|5\rangle}.$$

- After the deformation $\underline{1}$ we get

$$-A_5 = g^3 \frac{\langle 2|4\rangle}{\langle 2|1\rangle \langle 5|4\rangle} - g\lambda \frac{1}{\langle 5|1\rangle} + \mathcal{B}^{\underline{01}},$$

  with the corresponding sets

$$\mathcal{D}^{\underline{01}} = \{\langle 1|2\rangle, \langle 5|1\rangle\}, \quad \mathcal{U}^{\underline{01}} = \{\langle 4|5\rangle\}, \quad \mathcal{S}^{\underline{01}} = \emptyset.$$

- To continue, we need to perform another deformation, *e.g.*, $\underline{2} = \langle 5|1]$ to detect $\langle 4|5\rangle$. However, it can be checked that under $\underline{2}$ the pole part of $\mathcal{B}^{\underline{01}}$ is zero. Since all physical poles have been detected, we can conclude that $\mathcal{B}^{\underline{01}} = 0$, and the correct answer is

$$-A_5 = g^3 \frac{\langle 2|4\rangle}{\langle 2|1\rangle \langle 5|4\rangle} - g\lambda \frac{1}{\langle 5|1\rangle},$$

Now let us give a more abstract description of our algorithm:

- Let us define following two operators acting on rational functions:

$$
\begin{aligned}
P_i[R] &\equiv -\sum_{\text{finite}} \oint \frac{dz_i}{z_i} R(\lambda_{a_i} - z_i\lambda_{b_i}, \widetilde{\lambda}_{b_i} + z_i\widetilde{\lambda}_{a_i}), \\
C_i[R] &\equiv \oint_{\infty} \frac{dz_i}{z_i} R(\lambda_{a_i} - z_i\lambda_{b_i}, \widetilde{\lambda}_{b_i} + z_i\widetilde{\lambda}_{a_i}),
\end{aligned}
$$

  They satisfy

$$
P_i + C_i = I, \quad C_i^2 = C_i, \quad P_i^2 = P_i
$$

- P_i gives the recursive part, so we know its action
- If we know the R, we know how to find action of C_i too.

- Now we do following iteration:

$$
\begin{aligned}
I &= P_n + C_n = P_n + C_n(P_{n-1} + C_{n-1}) \\
&= P_n + C_n P_{n-1} + C_n C_{n-1}(P_{n-2} + C_{n-2}) \\
&= P_n + C_n P_{n-1} + \ldots + C_n C_{n-1} \cdots C_2 P_1 \\
&\quad + C_n C_{n-1} \cdots C_2 C_1 P_0 + C_n C_{n-1} \cdots C_2 C_1 C_0
\end{aligned}
$$

- If with proper choice of C_i sequence, we can show that $C_n C_{n-1} \cdots C_2 C_1 C_0 = 0$, then we have found the full boundary part.

Our algorithm is general, but there are two key issues we need to address:

- **Workability:** There is a sequence of deformations such that $C_n C_{n-1} \cdots C_2 C_1 C_0 = 0$?
- **Efficiency:** How to choose deformations in consequence to make the calculation most efficient?

Now we study the workability. First we need to have a better understanding of pole structure of boundary terms:

- Let us consider the deformation $\langle 1|n]$. Under large z-limit, we can expand

$$\frac{1}{(P_J+p_1-z\lambda_n\widetilde{\lambda}_1)^2}=\frac{1}{-z\langle n|P_J+p_1|1]}\sum_{i=0}\left(\frac{(P_J+p_1)^2}{z\langle n|P_J+p_1|1]}\right)^i$$

Thus for general amplitudes,

$$A(z)=\frac{f(z)}{\prod_{I\subset\mathcal{T}}P_I^2}\prod_{J\subset\mathcal{T}}\left[\frac{1}{-z\langle n|P_J+p_1|1]}\sum_{i=0}\left(\frac{(P_J+p_1)^2}{z\langle n|P_J+p_1|1]}\right)^i\right]$$

- Thus, possible poles of boundary terms can only be

$$P^2_{I\subset\mathcal{T}}, \qquad \langle n|P_{J\subset\mathcal{T}}|1]^a$$

- Using the observation, one can show that there is a sequence of deformations, such that the remaining boundary terms will be the form

$$\frac{\text{(polynomial)}}{\langle i_1|i_2\rangle^m\,[i_3|i_4]^{\overline{m}}}\times\text{(remaining factor)},$$

with arbitrary choices of i_1, i_2, i_3, i_4 and the remaining factor is helicity neutral and dimensionless.

Now consider the kinematic mass dimension of boundary:

- At one side, it is given by $D = 4 - n - \sum_i (D_c)_i$ where $(D_c)_i$ are mass dimension of coupling constants.
- At another side, using the schematic form of boundary terms

$$\frac{1}{\langle 12\rangle^m [34]^{\overline{m}}} \prod_{i=1}^{n} \langle i|^{\alpha_i} \prod_{i=1}^{n} [i|^{\beta_i},$$

we can write

$$D' = -(m+\overline{m}) + T'_{1234} + \sum_{i=5}^{n} |h_i| + \sum_{i=1}^{n} \min(\alpha_i, \beta_i) \geq 0$$

where

$$T'_{1234} \equiv \sum_{i=1,2} \left| h_i - \frac{m}{2} \right| + \sum_{i=3,4} \left| h_i + \frac{\overline{m}}{2} \right| \geq (m+\overline{m})$$

with proper choice of $1, 2, 3, 4$.

Now by comparing D, D' we can check if boundary is allowed after a sequence of deformations

- A first trivial implication is that when $D_c \geq 0$, our method can find all boundary terms for $n \geq 5$.
- For YM-theory and Einstein gravity, one can show that $A_n(\pm + + + ..+) = 0$ without using other arguments (like supersymmetry)
- Einstein Gravity has $D = 2$ and is solvable by our method.
- Even with coupling with negative mass dimensions, we can classify remaining freedom according to the value of $D \geq 0$. The classification does not depend on details of effective theories.

Efficiency: Although naively it seems one need to take a lot of deformations to find all boundary terms, in fact, it can be very simple.
Let us consider $A(1^{-1}, 2^{+1}, 3^{-1}, 4^{+1}, 5^{-1}, 6^{+1})$ in Einstein-Maxwell theory.

- First factorization limits, mass dimension and helicities together fix its schematic form to be

$$\frac{\langle 1|^2[2|^2\langle 3|^2[4|^2\langle 5|^2[6|^2 \times \prod^{32} |\bullet\rangle[\bullet|}{P_{12}^2 P_{14}^2 P_{16}^2 P_{32}^2 P_{34}^2 P_{36}^2 P_{52}^2 P_{54}^2 P_{56}^2} \times \frac{1}{P_{132}^2 P_{134}^2 P_{136}^2 P_{152}^2 P_{154}^2 P_{156}^2 P_{352}^2 P_{354}^2 P_{356}^2},$$

- After first deformation $\langle 3|1]$, similar trick of pole concentration gives its schematic form of boundary term

$$\frac{\langle 1|^{14}[2|^2[3|^{10}[4|^2\langle 5|^2[6|^2 \times \prod^{22}|\bullet\rangle[\bullet|}{P_{52}^2 P_{54}^2 P_{56}^2 P_{132}^2 P_{134}^2 P_{136}^2 \langle 12\rangle^2\langle 14\rangle^2\langle 16\rangle^2[32]^2[34]^2[36]^2}$$
$$\frac{1}{\langle 1|5+2|3]^2\langle 1|5+4|3]^2\langle 1|5+6|3]^2},$$

- Under the second deformation, schematic form of boundary term is

$$\frac{\langle 1|^{20}[2|^2[3|^{10}[4|^2[5|^4[6|^2 \times \prod^{18}|\bullet\rangle[\bullet|}{\langle 12\rangle^3\langle 14\rangle^3\langle 16\rangle^3[32]^2[34]^2[36]^2[52][54][56]\langle 1|3+2|5]} \frac{1}{\langle 1|3+4|5]\langle 1|3+6|5]\langle 1|5+2|3]^2\langle 1|5+4|3]^2\langle 1|5+6|3]^2},$$

which can not exist since $\prod^{18}|\bullet\rangle$ can never saturate $\langle 1|^{20}$ to form non-vanishing spinorial products.

Part III-A: Recursion relation for boundary contribution

Having above general discussions, now we focus on a special type of deformation $\langle i|n]$ with $i = 2, ..., n-1$

- All spurious poles will be the form $\langle n|P_{J\subset\mathcal{T}}|i]$. Furthermore, they are invariant under deformations $\langle i|n]$.
- The most important thing is that we can establish corresponding **on-shell recursion relation for boundary contribution**

Derivation:

- First, the boundary is defined as

$$B_0^1(\{\lambda_1,\widetilde{\lambda}_1\}, p_2, ..., p_{n-1}, \{\lambda_n,\widetilde{\lambda}_n\})$$
$$= \oint_{w=\infty} \frac{dw}{w} A_n(\{\lambda_1 - w\lambda_n, \widetilde{\lambda}_1\}, p_2, ..., p_{n-1}, \{\lambda_n, \widetilde{\lambda}_n + w\widetilde{\lambda}_1\})$$

- Now using the contour integration $\oint_{|z|=R\to\infty} dz \frac{B_0^1(z)}{z}$ we arrive

$$B_0^1 = B_0^{12} - \sum_{z_{\mathcal{I}}} \text{Res}\left(\frac{B_0^1}{z}\right)_{z=z_{\mathcal{I}}}$$

where the second deformation is $\langle 2|n]$ and $z_{\mathcal{I}} = \frac{(p_2+P_{\mathcal{I}})^2}{\langle n|P_{\mathcal{I}}|2]}$ and $\mathcal{I}\bigcup\overline{\mathcal{I}} = \{3, 4, ..., n-1\}$.

- Evaluation of residue part is given by

$$
\begin{aligned}
&\text{Res}\left(\frac{B_0^1}{z}\right)_{z=z_{\mathcal{I}}} \\
= &\oint_{z=z_{\mathcal{I}}} \frac{dz}{z} B_0^1(\{\lambda_1,\widetilde{\lambda}_1\},\{\lambda_2 - z\lambda_n,\widetilde{\lambda}_2\},...,p_{n-1},\{\lambda_n,\widetilde{\lambda}_n + z\widetilde{\lambda}_2\}) \\
= &\oint_{z_{\mathcal{I}}} \frac{dz}{z} \oint_{\infty} \frac{dw}{w} A_n(\{\lambda_1 - w\lambda_n,\widetilde{\lambda}_1\},\{\lambda_2 - z\lambda_n,\widetilde{\lambda}_2\}, \\
&p_3,...,p_{n-1},\{\lambda_n,\widetilde{\lambda}_n + z\widetilde{\lambda}_2 + w\widetilde{\lambda}_1\})
\end{aligned}
$$

- The key is then to use the **Fubini-Tonelli theorem** to exchange the ordering of two integrations

- Now we have

$$
\begin{aligned}
& \oint_{w=\infty}\frac{dw}{w}\oint_{z=z_{\mathcal{I}}}\frac{dz}{z}A_n(\{\lambda_1-w\lambda_n,\widetilde{\lambda}_1\},\{\lambda_2-z\lambda_n,\widetilde{\lambda}_2\},p_3,\ldots,p_{n-1},\{\lambda_n,\widetilde{\lambda}_n+z\widetilde{\lambda}_2+w\widetilde{\lambda}_1\}) \\
= & \oint_{w=\infty}\frac{dw}{w}\sum_h A_L(\widehat{p}_2(z_{\mathcal{I}}),\mathcal{I},-P^h(z_{\mathcal{I}}))\frac{-1}{(p_2+P_{\mathcal{I}})^2}A_R(\{\lambda_1-w\lambda_n,\widetilde{\lambda}_1\},\overline{\mathcal{I}},\{\lambda_n,\widetilde{\lambda}_n+z_{\mathcal{I}}\widetilde{\lambda}_2+w\widetilde{\lambda}_1 \\
= & \sum_h A_L(\widehat{p}_2(z_{\mathcal{I}}),\mathcal{I},-P^h(z_{\mathcal{I}}))\frac{-1}{(p_2+P_{\mathcal{I}})^2} \\
& \oint_{w=\infty}\frac{dw}{w}A_R(\{\lambda_1-w\lambda_n,\widetilde{\lambda}_1\},\overline{\mathcal{I}},\{\lambda_n,\widetilde{\lambda}_n+z_{\mathcal{I}}\widetilde{\lambda}_2+w\widetilde{\lambda}_1\},P^{-h}(z_{\mathcal{I}})) \\
= & \sum_h A_L(\widehat{p}_2(z_{\mathcal{I}}),\mathcal{I},-P^h(z_{\mathcal{I}}))\frac{-1}{(p_2+P_{\mathcal{I}})^2}B_0^1(p_1,\widehat{p}_n(z_{\mathcal{I}}),\overline{\mathcal{I}},P^{-h}(z_{\mathcal{I}}))
\end{aligned}
$$

Remarks:

- In the derivation, the commutativity of two integrations is crucial. In general with arbitrary pair of deformations, it is not true, but with our special choice of the type $\langle i|n]$, it is true.
- For it to be useful, one should show by other ways that after finite steps, there is no boundary left anymore. We have shown for **standard like model**, i.e., similar matter contents and similar interaction except all particles are massless, at most four steps are enough for getting complete answer.

Example II: Six scalars in scalar-Yang-Mills theory

$$L = \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D_\mu\bar{\Phi}D^\mu\Phi - \frac{g^2}{2}[\Phi,\bar{\Phi}]^2\right)$$

- Step 1: With deformation $\underline{0} = \langle 1|6]$, the recursive part is given by

$$R_6^{\underline{0}} = -\frac{\langle 16\rangle[35]^2[4|1+6|2\rangle^2}{\tau_{612}\langle 12\rangle[34][45][5|1+6|2\rangle[3|1+2|6\rangle}$$
$$+\frac{[13]^2\langle 46\rangle^2[1|2+3|5\rangle^2[2|1+3|6\rangle^2}{\tau_{123}[12][23]\langle 45\rangle\langle 56\rangle[1|2+3|4\rangle[3|1+2|6\rangle[1|2+3|6\rangle^2}$$
$$+\frac{[16]\langle 24\rangle^2[5|1+6|3\rangle^2}{\tau_{234}[56]\langle 23\rangle\langle 34\rangle[1|2+3|4\rangle[5|1+6|2\rangle}$$

- Next under the deformation $\underline{1} = \langle 2|6]$ and using

$$\mathcal{B}^{\underline{0}}(g^-(k_7), 4, 5, 6, 1) = \frac{[14]\Big(-2[15][46] + [14][56]\Big)}{[16][17][45][57]}$$

$$\mathcal{B}^{\underline{0}}(\bar{\Phi}(k_7), 5, 6, 1) = \frac{-\langle 17\rangle\langle 56\rangle - 2\langle 15\rangle\langle 67\rangle}{\langle 16\rangle\langle 57\rangle}$$

$$\mathcal{B}^{\underline{0}}(g^-(k_7), 6, 1) = \frac{\langle 17\rangle\langle 67\rangle}{\langle 16\rangle}$$

we find

$$\mathcal{BR}^{\underline{01}} = A(\hat{2}, 3, -\hat{p}_{23})\frac{1}{p_{23}^2}\mathcal{B}^{\underline{0}}(\hat{p}_{23}, 4, 5, \hat{6}, 1)$$
$$+A(\hat{2}, 3, 4, -\hat{p}_{234})\frac{1}{p_{234}^2}\mathcal{B}^{\underline{0}}(\hat{p}_{234}, 5, \hat{6}, 1)$$
$$+A(\hat{2}, 3, 4, 5, \hat{p}_{16})\frac{1}{p_{16}^2}\mathcal{B}^{\underline{0}}(-\hat{p}_{16}, \hat{6}, 1)$$

- At the third step, using the deformation $\underline{2} = \langle 3|6]$ and

$$\mathcal{B}^{\underline{01}}(g^-(k_7),5,6,1,2) = \frac{[25]}{[27][57]}, \quad \mathcal{B}^{\underline{01}}(\Phi(k_7),6,1,2) = -1$$

we find

$$\begin{aligned}\mathcal{BR}^{\underline{012}} &= A(\hat{3},4,-\hat{p}_{34})\frac{1}{p_{34}^2}\mathcal{B}^{\underline{01}}(\hat{p}_{34},5,\hat{6},1,2) \\ &\quad + A(\hat{3},4,5,-\hat{p}_{345})\frac{1}{p_{345}^2}\mathcal{B}^{\underline{01}}(\hat{p}_{345},\hat{6},1,2)\end{aligned}$$

- At the fourth step with deformation $\underline{3} = \langle 4|6]$, using

$$\mathcal{B}^{\underline{012}}(g^-(k_7), 6, 1, 2, 3) \quad = \frac{[13]^2[27]}{[23][37][17]^2}$$

we find

$$\mathcal{BR}^{\underline{0123}} \quad = \quad A(\hat{4}, 5, -\hat{p}_{45})\frac{1}{p_{45}^2}\mathcal{B}^{\underline{012}}(\hat{p}_{45}, \hat{6}, 1, 2, 3)$$

- Finally $\mathcal{R}_6^{\underline{0}} + \mathcal{BR}^{\underline{01}} + \mathcal{BR}^{\underline{012}} + \mathcal{BR}^{\underline{0123}}$ is equal to the total amplitude.

Remarks:

- Again, multiple deformation does not guarantee to get all results up to polynomial part.
- One good thing is that each term is rational by construction
- There is relation between multiple deformation and deformation with multiple particles, but the details have not been spelled out

Part IV: Using roots

- Recent years, there are many studies of soft theorem

  [Cachazo, Strominger, 2014]
- The vanishing at the soft limit gives, in fact, the information of roots of amplitudes.
- Using root, one can solve the boundary in principle as shown by Benincasa and Conde

  [Benincasa, Conde, 2011]

Roots of amplitude

To see the details:

- $$M_n(z) = \sum_{k\in\mathcal{P}^{(i,j)}} \frac{M_L(z_k)M_R(z_k)}{p_k^2(z)} + C_0 + \sum_{l=1}^{v} C_l z^l$$
  $$= c\frac{\prod_s (z-w_s)^{m_s}}{\prod_{k=1}^{N_p} p_k^2(z)}$$
- Split all roots into two groups $\mathcal{I}, \mathcal{J}$. For $n_{\mathcal{I}} < N_p$

$$c\frac{\prod_{s=1}^{n_{\mathcal{I}}}(z-w_s)}{\prod_{k=1}^{N_p} p_k^2(z)} = \sum_{k\in\mathcal{P}^{(i,j)}} \frac{c_k}{p_k^2(z)}$$

$$M_n(z) = \sum_{k\in\mathcal{P}^{(i,j)}} \frac{c_k}{p_k^2(z)} \prod_{t=1}^{n_{\mathcal{J}}}(z-w_t)$$

Roots of amplitude

- Perform a contour integration around the pole z_k and obtain

$$\frac{M_L(z_k)M_R(z_k)}{(-2p_k\cdot q)} = \frac{c_k}{(-2p_k\cdot q)}\prod_{t=1}^{n_{\mathcal{J}}}(z_k - w_t),$$

so

$$c_k = \frac{M_L(z_k)M_R(z_k)}{\prod_{t=1}^{n_{\mathcal{J}}}(z_k - w_t)}$$

and finally

$$M_n(z) = \sum_{k\in\mathcal{P}^{(i,j)}} \frac{M_L(z_k)M_R(z_k)}{p_k^2(z)} \prod_{t=1}^{v+1} \frac{(z-w_t)}{z_k - w_t}$$

by setting $n_{\mathcal{I}} = N_p - 1$.

- Although root method is very general and useful for theoretical discussions. However, it is very hard to find root recursively, especially roots are in general not rational function
- However, soft theorem tells us the roots under the limit!

Now we discuss how to use it

[Cheung, Kampf, Novotny, Shen, and Trnka, 2015]

- First we need to find deformation detecting the soft limit:

$$p_i \to p_i(1 - za_i)$$

with $\sum_i a_i p_i = 0$ for momentum conservation. This "rescaling shift" keeps on-shell conditions too.

- Secondly, we identify the soft limit of amplitudes:

$$A_n(z) \sim (1 - za_i)^{\sigma_i}, \quad z \to \frac{1}{a_i}$$

- Now we consider the contour integration

$$\oint \frac{dz}{z} \frac{A_n(z)}{F_n(z)}$$

with $F_n(z) = \prod_{i=1}^{n}(1 - za_i)^{\sigma_i}$.

- Since the soft limit, poles introduced by $F_n(z)$ do not exist eventually. Thus only physical poles exist and we can use the factorization limit to find residues.
- One key role of $F_n(z)$ is to introduce z in denominator, thus improve the convergence!

Remarks:

- Using roots information (even only under some limits) we can improve the convergence. Thus it is imaginable that there could be other interesting limits one can try.
- Again, it is not guaranteed that it provides enough power of z to avoid the boundary.
- The pole is in general quadratic function of z, which gives more complicated expressions.

Part V: The boundary Lagrangian and Form factors

What is the physical meaning of boundary contributions?

- A key observation is that the boundary comes from the **large z-limit** of deformation parameter. Thus two momenta $p_i + zq, p_j - zq$ become **infinity**, i.e., we have **two very heavy particles**
- The two heavy particles can be taken as classical background, while other fields as soft (quantum) fluctuation. Thus we can take the **background field method** and use Wilson's idea to integrate them out.

- Another point of view is to use OPE method to replace the product of two quantum fields by a **boundary operator**,

$$\mathcal{O}_I(k_L + zq)\mathcal{O}_J(k_R - zq) = \sum_K C_{IJ}^K(k_L + zq)\mathcal{O}_K(k_L + k_R)$$

Expanding the coefficient around $z = \infty$

$$C_{IJ}^K(k_L + zq) = \sum_i C_{IJ,i}^K z^i$$

we get the boundary operator

$$\mathcal{F} = \sum_K C_{IJ,0}^K \mathcal{O}_K(k_1 + k_n)$$

To track the large z behavior, we just need to focus on the hard line connecting two deformed particles. Along the hard line, there are two key facts:

- Each vertex has two and only two hard fields;
- Two hard particles can be contracted to become (hard) inner propagator.
- Using the observation, we can split $\Phi \to \Phi + \Phi^{\Lambda}$ and expand

$$S[\Phi + \Phi^{\Lambda}] = S[\Phi] + S_1^{\Lambda}[\Phi^{\Lambda}, \Phi] + S_2^{\Lambda}[\Phi^{\Lambda}, \Phi] + \cdots$$

- Now the correlation function can be evaluated as

$$A(z) = \int D\Phi D\Phi^{\Lambda} \exp\left(iS[\Phi] + iS_2^{\Lambda}[\Phi^{\Lambda}, \Phi]\right) \Phi_1^{\Lambda}\Phi_n^{\Lambda}\Phi_2 \cdots \Phi_{n-1}$$
$$= \int D\Phi \exp\left(iS[\Phi]\right) \mathcal{Z}(z)\Phi_2 \cdots \Phi_{n-1}$$

with $\mathcal{Z}(z) = \left[\int D\Phi^{\Lambda} \exp\left(iS_2^{\Lambda}[\Phi^{\Lambda}, \Phi]\right) \Phi_1^{\Lambda}\Phi_n^{\Lambda}\right]$.

- Writing the quadratic term as

$$L_2^{\Lambda} = \frac{1}{2} H_{\alpha}^{\dagger} \mathcal{D}^{\alpha}{}_{\beta} H^{\beta},$$

where H for hard fields, we have $\mathcal{Z}(z) \sim \mathcal{D}^{-1}$.

- If we decompose $\mathcal{D}^{\alpha}{}_{\beta}(\Phi) = (D_0)^{\alpha}{}_{\beta} + V^{\alpha}{}_{\beta}(\Phi)$, after the LSZ reduction to amplitude, we will have

$$\epsilon^1_{\alpha_1} \epsilon^n_{\alpha_n} \left[V^{\alpha_1 \alpha_n} - V^{\alpha_1 \beta_1} (D_0^{-1})_{\beta_1 \beta_2} V^{\beta_2 \alpha_2} + \cdots \right]^{\alpha_1 \alpha_n}$$

which has clear Feynman diagram picture.

- For example, for $L = -\frac{1}{2}(\partial\phi)^2 + \frac{g}{m!}\phi^m$, it is

$$\frac{g}{(m-2)!}\phi^{m-2} - \frac{g}{(m-2)!}\phi^{m-2}\frac{1}{\partial^2}\frac{g}{(m-2)!}\phi^{m-2} + \cdots$$

Using $\partial_\mu \to \partial_\mu - izq_\mu$, $\partial^2 \to \partial^2 - 2iz\partial \cdot q$, so $\frac{1}{\partial^2} \to \mathcal{O}(\frac{1}{z})$, we are left with $\frac{g}{(m-2)!}\phi^{m-2}$ as boundary operator.

- Having boundary operator, we can write down boundary Lagrangian

$$L_{\mathcal{B}\langle 1|n]} = -\frac{1}{2}(\partial\phi)^2 + \frac{g}{m!}\phi^m + \overline{\mathcal{F}}^{\langle 1|n]}\frac{g}{(m-2)!}\phi^{m-2} - \overline{\mathcal{F}}^{\langle 1|n]}\mathcal{F}^{\langle 1|n]}$$

where Field $\overline{\mathcal{F}}^{\langle 1|n]}$ will be a Lagrangian multiplier.

Good points for defining boundary Lagrangian:

- Now the boundary terms can be calculated using Feynman diagrams based on boundary Lagrangian
- It is easier to calculate boundary of boundary. For example, for the second deformation $\langle 2|n]$, the path-integration of $\int D\phi_2^\Lambda$ is

$$-\frac{1}{2}\phi_2^\Lambda\left\{\partial^2+\frac{g}{(m-2)!}\phi^{m-2}\right\}\phi_2^\Lambda$$
$$-\overline{\mathcal{F}}^{\langle 1|n]}\frac{g}{(m-3)!}\phi^{m-3}\phi_2^\Lambda+J\phi_2^\Lambda$$

- It is same integration as did for deformation $\langle 1|N]$ with source shifted $\widetilde{J} = J - \overline{\mathcal{F}}^{\langle 1|n]} \frac{g}{(m-3)!} \phi^{m-3}$.
- After taking the $\frac{\delta^2 \mathcal{Z}}{\delta J_\alpha \delta \overline{\mathcal{F}}}|_{J,\overline{\mathcal{F}} \to 0}$ we get

$$\frac{1}{2}(\mathcal{D}^{-1})^{\alpha\sigma} \frac{\delta \mathcal{O}}{\delta \Phi^\sigma} + \frac{1}{2} \frac{\delta \mathcal{O}}{\delta \Phi^\sigma} (\mathcal{D}^{-1})^{\sigma\alpha}$$

- Doing LSZ reduction and simplify further, we will get

$$\epsilon \left(1 + V D_0^{-1}\right)^{-1} \frac{\delta \mathcal{O}}{\delta \Phi_\rho^\dagger}$$

Applying to form factors:

- The form factor is given by

$$\left\langle 0|\widehat{\mathcal{O}}|1,2,...,n\right\rangle$$

- As we have seen, the boundary part can be represented as

$$\left\langle 0|\widehat{\mathcal{B}}|1,2,...,n\right\rangle$$

  where $\widehat{\mathcal{B}}$ is the boundary operator.
- If we can find a theory and the deformation, such that $\widehat{\mathcal{B}}=\widehat{\mathcal{O}}$, the calculation of form factor becomes the calculation of boundary!

- To calculate the boundary, we can take different deformation which does not have boundary, thus it is easy to get the full amplitude, and then easy to get the boundary of the particular deformation
- For this approach, the most tricky part is to find the right theory!

Thanks a lot for listening!!!