The many uses of form factors

Andi Brandhuber
Queen Mary University of London

with

Paul Heslop, Martyna Kostacinska, Brenda Penante, Gab Travaglini, Donovan Young

AB, Penante, Travaglini, Young 1412.1019, 1502.06626
AB, Heslop, Travaglini, Young, 1507.01504
AB, Kostacinska, Penante, Travaglini, Young 1606.08682

Beyond amplitudes

- Long-term goal: extend success of on-shell methods to "partially or fully off-shell" quantities
  - Partially off-shell: form factors (main focus today)
    - MHV diagrams, BCFW, generalised unitarity, computation of remainder functions using symbols… (AB, Penante, Spence, Travaglini, Wen, Yang, Young; Bork, Kazakov, Vartanov; Loebbert, Nandan, Sieg, Wilhelm, Yang; Gehrmann, Henn…)
  - Fully off-shell: correlation functions (Engelund-Roiban; AB, Penante, Travaglini, Young)

$$\langle 0|\, \mathcal{O}(x_1)\bar{\mathcal{O}}(x_2)\, |0\rangle \;\sim\; \frac{1}{((x_1 - x_2)^2)^{\Delta_0+\gamma}}$$

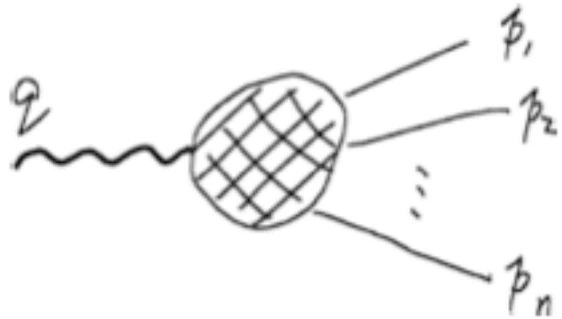
  - Anomalous dimensions γ = eigenvalues of hamiltonian $H^A{}_B$ (dilatation operator) of an integrable spin-chain!
  - Connect Yangian (<=> integrability) of H with Yangian symmetry of amplitudes (on-shell)

Form Factors in N=4

- more general objects than correlation functions, Wilson loops, amplitudes: e.g. Wilson loops with operator insertions, correlators of Wilson loops ...

- Form Factors: interpolate between correlators and amplitudes, partially off-shell

$$\int d^4x\, e^{-iqx}\, \langle 1 \cdots n | \mathcal{O}(x) | 0 \rangle \;=\; \delta^{(4)}(q - \sum_{i=1}^{n} p_i)\, \langle 1 \cdots n | \mathcal{O}(0) | 0 \rangle$$



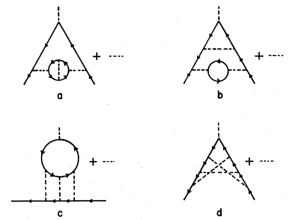
$$q = \sum_{i=1}^{n} p_i$$

- Simplest example in QCD:
  - Sudakov FF (n=2): exponentiation of IR divergences
    In N=4 2-loop Sudakov FF first studied by Van Neerven

Appears in many interesting contexts

- Three-loop correction to electron g–2

72 diagrams like



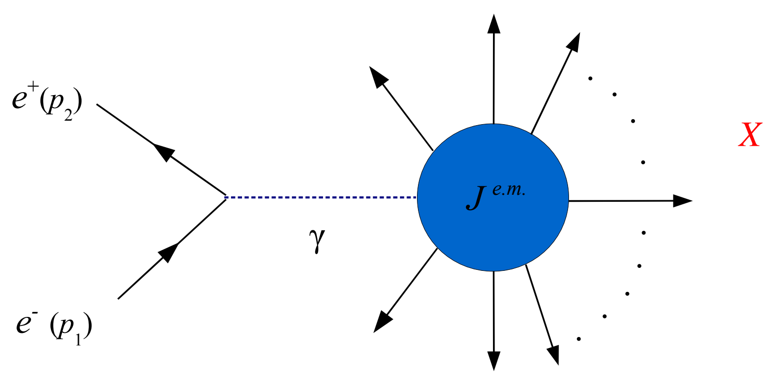
$$= (1.181241456\ldots)\,(\alpha_{\rm e.m.}/\pi)^3$$

(Cvitanovic & Kinoshita '74)

(Laporta & Remiddi '96)

  - wild oscillations between individual diagram
  - result is O(1) => mysterious cancellations

- $e^+ e^- \to$ hadrons (LEP):



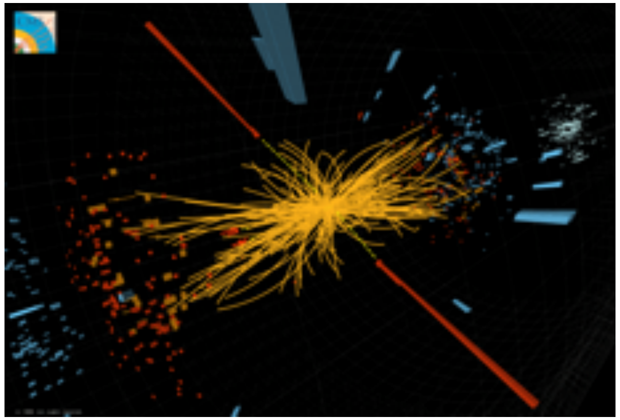
$$e\,\bar{v}(p_2)\gamma_\mu u(p_1)\ \frac{\eta^{\mu\nu}}{(p_1+p_2)^2}\ (-e)\langle X|\, J^{e.m.}_\nu(0)\,|0\rangle$$

$e^+ e^- \to$ hadrons (X)

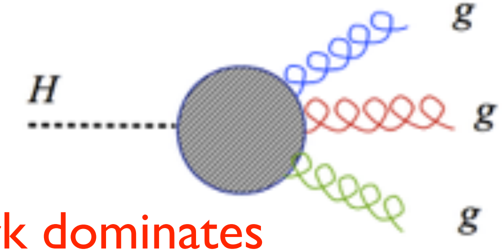
all orders in $\alpha_{\rm strong}$, first order in $\alpha_{\rm e.m.}$

hadronic electromagnetic current

Effective Lagrangians



- Higgs + multi-gluon amplitudes
  - at low M_H , dominant Higgs production at the LHC through gluon fusion
  - coupling to gluons through a fermion loop
    - proportional to the mass of the quark $\Rightarrow$ top quark dominates



  - for $M_H < 2\, m_t$ integrate out top quark
- Effective Lagrangian description: leading $\mathcal{L}_{\rm eff} \sim H\,{\rm Tr}\,F^2$
  - coupling $\dfrac{\alpha_S}{12\pi v}$, $v = 246 GeV$ independent of m_t
  - and subleading $\mathcal{L}_{\rm sub} \sim \dfrac{C_1}{v m_t^2} H {\rm tr} F^3 + \dfrac{C_2}{v m_t^2} H {\rm tr} DFDF + \ldots$

Higgs + many gluon amplitudes

- Leading order
  - early applications of on-shell techniques to tree-level and one-loop amplitudes (Badger, Dixon, Glover, Khoze; Badger, Glover, Risager, Mastrolia, Williams)

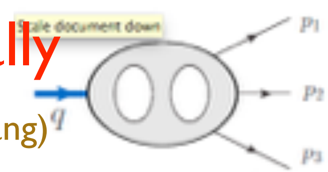
$$F^{\rm tree}_{{\rm tr}F^2}(1^-,2^-,3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \quad , \quad F^{\rm tree}_{{\rm tr}F^2}(1^+,2^+,3^+) = \frac{q^4}{[12][23][31]} \quad , \quad q^2 = m_H^2$$

  - This has been pushed in QCD to 2 & 3-loop order for 2 gluons (Anastasiou, Melnikov; Harlander, Kilgore; Anastasiou, Duhr, Buehler, Herzog, Dulat, Furlan, Mistlberger), and to 2 loops for 3 partons (Glover, Gehrmann, Jaquier & Koukoutsakis)
- Subleading, finite top-mass corrections have been studied as well (e.g. Neill; Dawson, Lewis, Zeng….)

- Also integrating out the top-quarks or stringy effects can induce new interaction terms such as: $\mathrm{tr}(F^3)$ or R^3
  (Dixon, Shadmi; Dixon, Glover, Khoze; Broedel, Dixon; Neill)
  - related to form factors via q → 0 limit
- In N=4 SYM these operators sit in multiplets of operators and hence form factors of different operators can be related by supersymmetric Ward identities
- (Chiral) stress tensor multiplet (protected, 1/2-BPS):

$$\mathrm{tr}(X^2) = \mathrm{tr}(\phi_{12}^2) \xrightarrow{Q^4} \mathcal{L}_{\text{on-shell}} \sim \mathrm{tr}(F_{\mathrm{SD}}^2) + \ldots$$

  - 2-loop n=3 Higgs amplitude in N=4 captures maximally transcendental part of full QCD amplitude (AB, Travaglini, Yang)



- Non-protected: $\mathrm{tr}(F^3)$, $\mathrm{tr}(DFDF)$, ...
  - Related to Konishi operator, $K \sim \mathrm{tr}(\bar{X}X + \bar{Y}Y + \bar{Z}Z)$
    Question: are there similarities between QCD & N=4?

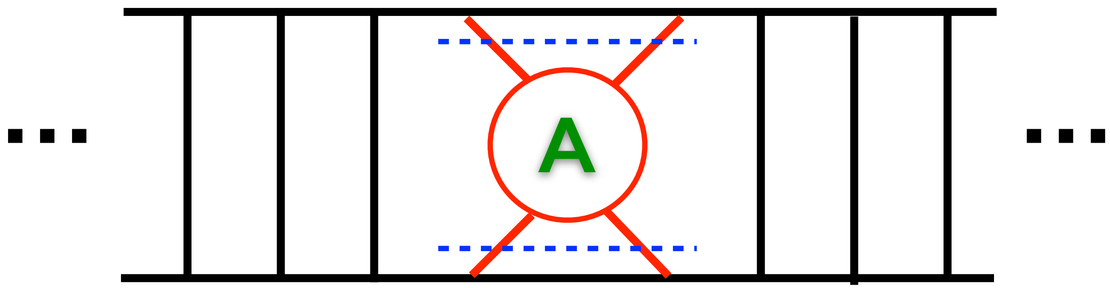
Dilatation operator and Yangian

- (At least) two ways to find anomalous dimensions γ
- 1.) 2-point functions: $\langle 0|\,\mathcal{O}(x_1)\bar{\mathcal{O}}(x_2)\,|0\rangle \sim \frac{1}{((x_1-x_2)^2)^{\Delta_0+\gamma}}$
- 2.) Form factors: $\langle 12\ldots n|\mathcal{O}(0)|0\rangle$
- Under renormalisation operators mix: $\mathcal{O}^A_{\text{ren}} = Z^A{}_B \mathcal{O}^B_{\text{bare}}$
- Find Z by demanding 1.) or 2.) are UV-finite!
  - FF linear in Z, but has IR-divergences
- Dilatation operator $H^A{}_B = -\mu_R \frac{d\log Z^A{}_B}{d\mu_R}$ note $\log Z \sim \frac{1}{\epsilon}$ while $Z^{(L)} \sim \frac{1}{\epsilon^L}$
- Next:
  - derive 1.) and 2.) using on-shell methods (one-loop)
  - Connect Yangian symmetry of H with that of amplitudes

2-point function vs. FF at one loop

- 2-point function in SO(6) and SU(2|3) sectors (AB, Penante Travaglini, Young)

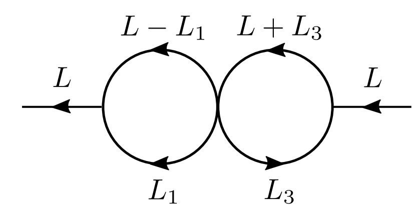
$$\mathrm{tr}(\ldots \Phi_{A_i} \;\; \Phi_{B_{i+1}} \ldots)$$



$$\to \sum H_{i,i+1}$$

$$\mathrm{tr}(\ldots \Phi_{A'_i} \;\; \Phi_{B'_{i+1}} \ldots)$$

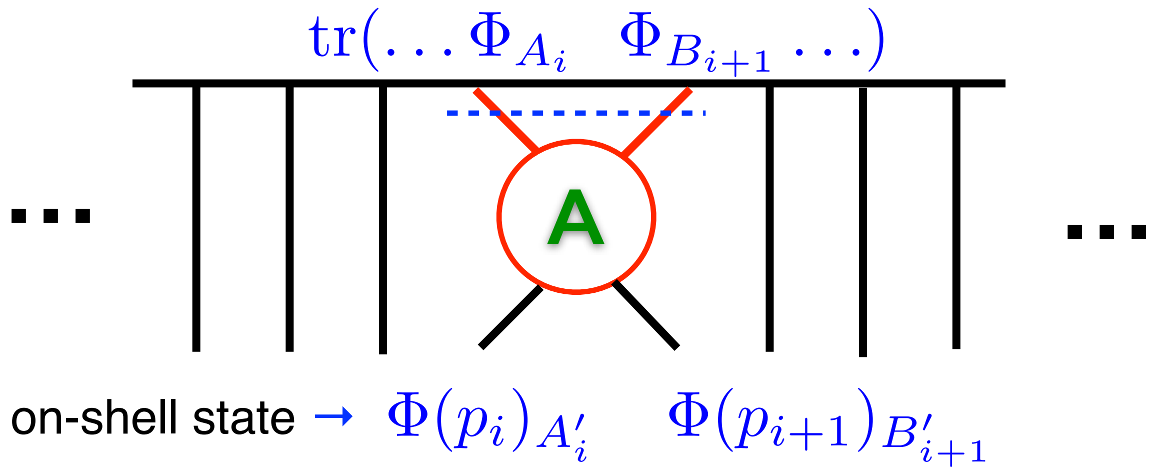
- Planar → only adjacent legs
- double-2-particle cuts, only divergence from



- in momentum space 2-loop integral $\frac{1}{\epsilon^2}$ pole
- in x-space, after FT: $\frac{1}{\epsilon}$ -pole // renormalisation const. Z
- Reproduce known dilatation operators $\sum H_{i,i+1}$
  (Minahan, Zarembo; Beisert)

- Alternatively calculate "minimal" Form Factors at one loop
(M.Wilhelm)

- simple 2-particle cuts



- Result: bubbles (UV divergent) and ~~triangles (IR divergent)~~

- Coefficients of bubbles give dilatation operator: $\sum H_{i,i+1}$

- Gives physical interpretation of Zwiebel's form of dilatation operator; more on that in a moment…

Amplitude Yangian = Dilatation Operator Yangian

(AB, Heslop, Travaglini, Young)

- N=4 super Yang-Mills thought to be integrable
- Two different manifestations of Yangian symmetry on
    - amplitudes
    - dilatation operator
- Goal: derive the action of the Yangian on the one-loop dilatation operator from the Yangian of amplitudes

Amplitude Yangian

- Fact 1: Tree-level super-amplitudes in N=4 SYM are Yangian invariant (Drummond, Henn, Plefka)
  - level-zero charges $J^A = \sum J_i^A \longrightarrow$ superconformal algebra
  - level-one generators $Q^A = \sum_{i<j} Q_{ij}^A$
    - $Q_{ij}^A := f_{CB}^A J_i^B J_j^C$ are non-local densities acting on particles i and j
  - level-one generators $\longrightarrow$ dual superconformal algebra
  - dual superconformal symmetry of amplitudes (Drummond, Henn, Korchemsky, Sokatchev; AB, Heslop, Travaglini)

Dilatation operator Yangian

- Fact 2: The complete one-loop dilatation operator is Yangian invariant up to boundary terms (Dolan, Nappi, Witten)

$$[Q^A\,,\,H] \sim J_1^A - J_L^A$$

- Equivalent to showing $[Q_{12}^A\,,\,H_{12}] \sim J_1^A - J_2^A$
  - $H = \sum_{i=1}^{L} H_{ii+1}$, where H_{12} acts on sites 1 and 2
- When acting on spin chains with periodic boundary conditions the boundary term vanishes

- Next: derive Fact 2 from Fact 1

- Main tool: form of one-loop dilatation operator by Zwiebel
- Building blocks of this formula:
  - tree-level four-point superamplitude (Yangian invariant)
  - tree-level minimal form factors $\langle \Phi_1 \ldots \Phi_L | \mathrm{tr}(\Phi_1 \ldots \Phi_L)(0) | 0 \rangle$
    - represent the states on which the dilatation operator acts
- Idea: use known action of Yangian generators on amplitudes to derive action on the dilatation operator

- States & single-trace operators
  - A "state" corresponds to a single-trace operator $\mathrm{Tr}\big(\Phi_1 \cdots \Phi_L\big)(x)$
  - The letters Φ_i : $F^{\alpha\beta},\ \psi^{\alpha ABC},\ \phi^{[AB]},\ \bar{\psi}^{\dot{\alpha}A},\ \bar{F}^{\dot{\alpha}\dot{\beta}}$
    (and symmetrised covariant derivatives D acting on them)
  - representation in terms of spinor helicity variables via the map

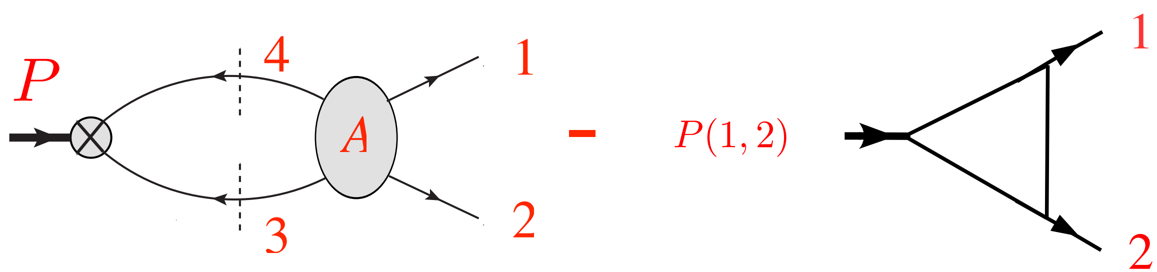
$$
\begin{aligned}
\overline{F} &\sim \tilde{\lambda}\tilde{\lambda} \\
\overline{\psi} &\sim \tilde{\lambda}\eta \\
\phi &\sim \eta\eta \\
\psi &\sim \lambda\eta\eta\eta \\
F &\sim \lambda\lambda\eta\eta\eta\eta \\
D &\sim \lambda\tilde{\lambda}
\end{aligned}
$$

- States in spinor-helicity language:
  - combine $\Lambda^a := (\lambda^\alpha, \tilde{\lambda}^{\dot{\alpha}}, \eta^A)$
  - a state is a polynomial $P(\Lambda_1, \ldots, \Lambda_L)$ in the Λ 's
- Examples:
  - half-BPS $\cdots \phi^{12}\phi^{12} \cdots \leftrightarrow (\eta_1^1 \eta_1^2)(\eta_2^1 \eta_2^2)$ ← R-symmetry ← position
  - Konishi $\cdots \epsilon_{ABCD}\phi^{AB}\phi^{CD} \cdots \leftrightarrow \epsilon_{ABCD}(\eta_1^A \eta_1^B)(\eta_2^C \eta_2^D)$
- $P(\Lambda_1, \ldots, \Lambda_L)$ = tree-level minimal form factor of the corresponding operator (Wilhelm)

- One-loop form factor phase space integral…

$$H_{12}|1,2\rangle = \int d\Lambda_3 d\Lambda_4\, A(1,2,3,4)\left[P(-4,-3) - \left(\frac{\langle 12\rangle}{\langle 34\rangle}\right)^2 P(1,2)\right]$$

"un-integrated form"

- phase-space measure $d\Lambda_i := d^2\lambda_i d^2\tilde{\lambda}_i d^4\eta_i$
- superamplitude $A(1,2,3,4) = \frac{\delta^{(4)}(\sum_i \lambda_i\tilde{\lambda}_i)\delta^{(8)}(\sum_i \lambda_i\eta_i)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$
- $P(1,2)$ represents the operator/state $|\cdots 1,2,\cdots\rangle$
- 

- … gives Zwiebel's dilatation operator

$$H_{12}|1,2\rangle = -\frac{1}{\pi}\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta\, \left[e^{2i\phi}P(1',2') - P(1,2)\right]$$

"integrated form"

- $\lambda_1' = \lambda_1\cos\theta - e^{i\phi}\lambda_2\sin\theta\,, \quad \lambda_2' = \lambda_1\sin\theta + e^{i\phi}\lambda_2\cos\theta$ (similarly for $\tilde{\lambda}', \eta'$)
- one-loop phase space is a two-sphere! $\theta\,,\ \phi$

- Summarising:
  - Unintegrated form:

$$H_{12}|1,2\rangle \;=\; \int d\Lambda_3 d\Lambda_4\, A(1,2,3,4)\Big[P(-4,-3) \;-\; \Big(\frac{\langle 12\rangle}{\langle 34\rangle}\Big)^2 P(1,2)\Big]$$

  - integrated form (Zwiebel):

$$H_{12}|1,2\rangle = -\frac{1}{\pi}\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \,\Big[e^{2i\phi}P(1',2') - P(1,2)\Big]$$

- It is not at all obvious to see how the relation

$$[Q^A_{12}\,,\,H_{12}] \sim J^A_1 - J^A_2$$

is realised when acting on the integrated form

Amplitudes to the rescue!

- Act with level-one generator $p^{(1)}$ on un-integrated form:
  - from DHP (dual K)

$$Q_{ij} \;=\; \Big(m_j{}^{\gamma}{}_{\alpha}\delta^{\dot\gamma}_{\dot\alpha} + \bar m_j{}^{\dot\gamma}{}_{\dot\alpha}\delta^{\gamma}_{\alpha} - d_j\delta^{\gamma}_{\alpha}\delta^{\dot\gamma}_{\dot\alpha}\Big)p_{i\,\gamma\dot\gamma} \;+\; \bar q_{j\dot\alpha C}q^{C}_{i\alpha} - (i \leftrightarrow j)$$

$$[Q_{12}, H_{12}]|1,2\rangle = Q_{12}\int d\Lambda_3 d\Lambda_4\ A(1,2,3,4)\big[P(-4,-3) - r\,P(1,2)\big]$$
$$- \int d\Lambda_3 d\Lambda_4\ A(1,2,3,4)\big[Q_{-4,-3}P(-4,-3) - r\,Q_{12}P(1,2)\big]$$

$$r := \left(\frac{\langle 12\rangle}{\langle 34\rangle}\right)^2$$

- Ingredients of the general proof (for arbitrary states):
  - after integration by parts (IBP), combination of generators acting on amplitude is

$$Q_{12} + Q_{34} = \sum_{i<j} Q_{ij} - (Q_{13} + Q_{14} + Q_{23} + Q_{24})$$

  - $\sum_{i<j} Q_{ij}$ = dual conformal K, which annihilates amplitude!
  - $(Q_{13} + Q_{14} + Q_{23} + Q_{24})\, A = 0$ since action of Yangian generators on amplitudes is compatible with cyclicity of amplitudes!
  - the left-over terms combine after phase space integration into

$$[Q_{12}, H_{12}]|1,2\rangle \; = \; 2(p_1 - p_2)|1,2\rangle$$

- Comments:

1. can check other commutators: e.g. if Q is the level-one generator associated to supersymmetry q:

$$[Q_{12}, H_{12}]|1,2\rangle \; = \; 2(q_1 - q_2)|1,2\rangle$$

2. not obvious to see this result on the "integrated form" of Zwiebel's formula (without amplitudes)!

3. Direct link between Yangian symmetry of amplitudes and Yangian (almost)-invariance of dilatation operator H !!

2-loop FF's of unprotected operators

- Interesting to calculate form factors of $\mathrm{tr}(F^3)$, $\mathrm{tr}(DF\,DF)$,.. in QCD
- N=4 SYM captures "most complicated part", e.g. $\mathrm{tr}(F^2)$
- In N=4 such operators are related to simpler operators without derivates e.g. $\mathrm{tr}(X^2)$, $\mathrm{tr}(X[Y,Z])$, $\mathrm{tr}([X,Y]^2)$
- Goals:
  - extract universal building blocks, identify regularities
  - compare with QCD (currently only known at 1-loop)

Form factors in SU(2|3) sector

- In N=4 SYM local operators built from the following letters form the largest sector closed under renormalisation

$$X = \phi_{12},\; Y = \phi_{23},\; Z = \phi_{31},\; \psi_\alpha = \psi_{123,\alpha}$$

- Dilatation operator known up to 3 loops, length changing interactions $XYZ \sim \psi\psi$ from 2 loops on (Beisert dynamic spin chain)

- Focus on operators of (classical) dimension = 3:
  - protected $\mathcal{O}_{\rm BPS} = {\rm tr}(X\{Y,Z\})$ same FF as ${\rm tr}(X^3)$
  - unprotected $\mathcal{O}_{\rm B} = {\rm tr}(X[Y,Z])$ ⟷ $\mathcal{O}_{\rm F} = \frac{1}{2}{\rm tr}(\psi^\alpha\psi_\alpha)$

operator mixing

some comments on the operators

- starting with chiral primary operator $\mathrm{tr}(X^2)$

$$\mathrm{tr}(X^2) \xrightarrow{Q_\alpha} \mathrm{tr}(X\psi_\alpha) \xrightarrow{Q^\alpha} \mathcal{O}_F + g\mathcal{O}_B = \frac{1}{2}\mathrm{tr}(\psi\psi) + g(X[Y,Z])$$

- part of (chiral) stress tensor multiplet

- Konishi $K \sim \mathrm{tr}(\bar{X}X + \bar{Y}Y + \bar{Z}Z)$

$$\downarrow (Q)^2$$

$$g\mathrm{tr}(X[Y,Z]) - \frac{g^2 N}{16\pi^2}\mathrm{tr}(\psi\psi)$$ → Konishi Anomaly

- in the process of renormalisation/diagonalisation we will recover these combinations

Matrix of form factors

- In the following we will consider form factors of the bare operators $\mathcal{O}_{\rm B} = \mathrm{tr}(X[Y,Z])$ and $\mathcal{O}_{\rm F} = \frac{1}{2}\mathrm{tr}(\psi^\alpha \psi_\alpha)$ with external states: $\langle \bar{X}(1)\bar{Y}(2)\bar{Z}(3)|$ $\langle \bar{\psi}(1)\bar{\psi}(2)|$
- It is natural to package them into a matrix of form factors

$$\mathcal{F} := \begin{pmatrix} \langle \bar{\psi}\bar{\psi}|\mathcal{O}_F|0\rangle & \langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_F|0\rangle \\ \langle \bar{\psi}\bar{\psi}|\mathcal{O}_B|0\rangle & \langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle \end{pmatrix}$$

- Goal: calculate the form factor matrix to 2-loop order
- Find the renormalisation constants $\mathcal{Z}$ and dilatation operator H from:

$$\mathcal{Z}\mathcal{F} = \mathcal{F}^{ren} \qquad H = -\mu_R \frac{d}{d\mu_R} \log \mathcal{Z}$$

Computation of form factors

- Uses on-shell recursion relations to find non-minimal tree-level form factors needed in cuts and generalised unitarity
- Some expressions are simplified using the symbol of transcendental functions
  - Form factors have IR and/or UV divergences
  - degree of transcendentality ranges from 4…0
- Start with most complicated FF: $\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle$
  - at tree level $\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle^{(0)} = 1$

Minimal $\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle$ at one loop

- A useful decomposition

$$\mathcal{O}_B = \mathcal{O}_{\rm BPS} + \mathcal{O}_{\rm offset}$$

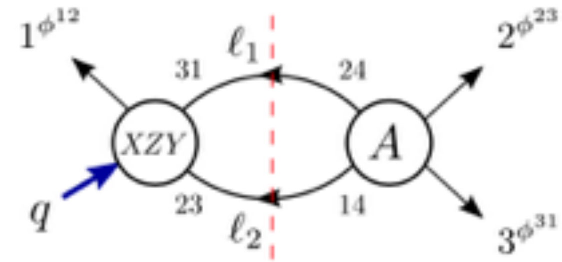
$$\mathcal{O}_{\rm BPS} = {\rm tr}(X\{Y,Z\})$$

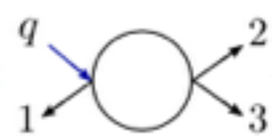
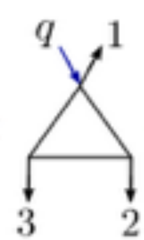
$$\mathcal{O}_{\rm offset} = -2{\rm tr}(XZY)$$

- contribution from BPS is known to 2 loops

- offset contribution particularly simple: contains UV divergences and terms of strictly less than maximal degree of transcendentality

- 1-loop cut



- gives $F^{(1)}_{\mathcal{O}_B}(1^{\phi^{12}}, 2^{\phi^{23}}, 3^{\phi^{31}}; q) = 2i \times$  $+ i\, s_{23} \times$  $+ \text{cyclic}(1,2,3)$

UV divergent

One-loop anomalous dimension of $\mathcal{O}_B$

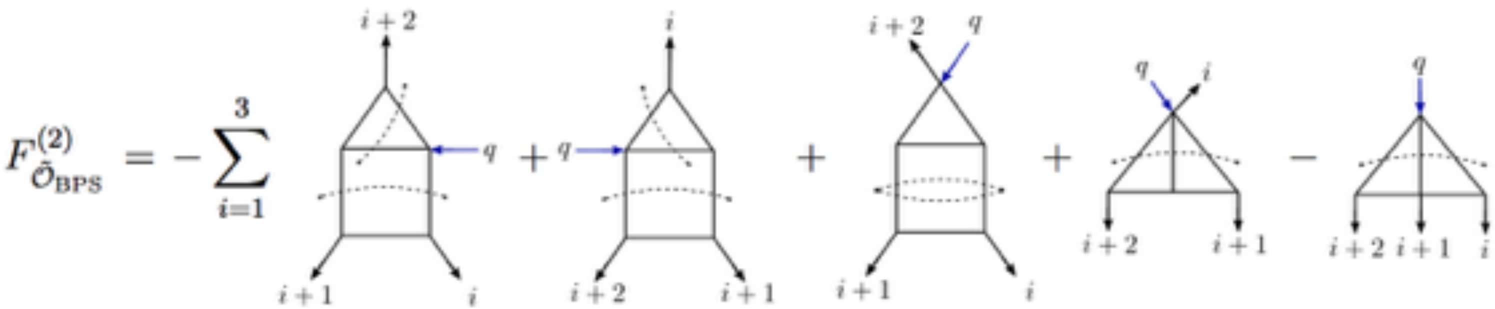
- UV divergence is $F^{(1)}_{\mathcal{O}_B}\Big|_{\mu_R,\mathrm{UV}} = -\frac{6}{\epsilon}\, a(\mu_R)$ with $a(\mu_R) := \frac{g^2 N e^{-\epsilon\gamma_{\mathrm{E}}}}{(4\pi)^{2-\epsilon}} \left(\frac{\mu_R}{\mu}\right)^{-2\epsilon}$
- Hence we find the 1-loop counterterm $\mathcal{Z}^{(1)}_{\mathcal{O}_B} = \frac{6}{\epsilon}\, a(\mu_R)$
- with this we find

$$\gamma_{\mathcal{O}_B} = -\mu_R \frac{\partial}{\partial \mu_R} \log(1 + \mathcal{Z}^{(1)}_{\mathcal{O}_B} + \cdots)\Big|_{\epsilon\to 0} = 12a$$

- This is in agreement with the known one-loop anomalous dimension of the Konishi multiplet
  - at this order inclusion of $\mathrm{tr}(\psi\psi)$-term not needed

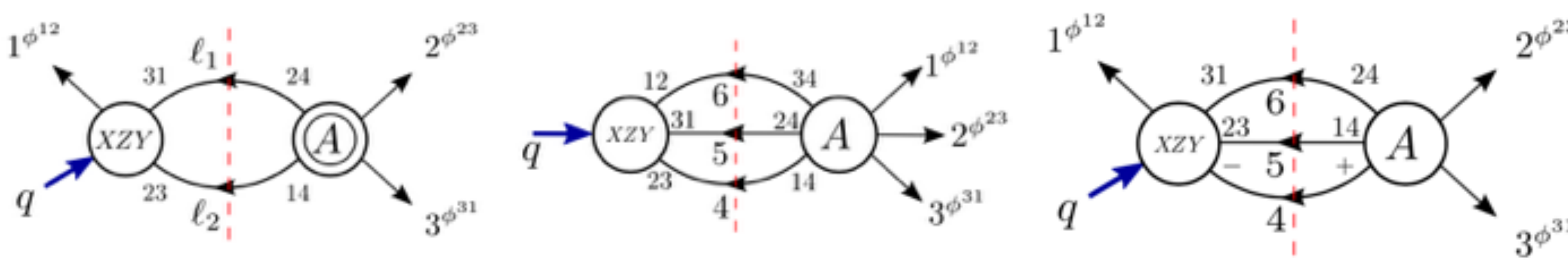
Minimal $\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle$ at two loops

- Use (iterated) 2-particle and 3-particle cuts
- 2-loop form factor of $\mathcal{O}_{\rm BPS} = {\rm tr}(X\{Y,Z\})$ is equal to known FF of ${\rm tr}(X^3)$ which is given by (AB, Penante, Travaglini, Wen)

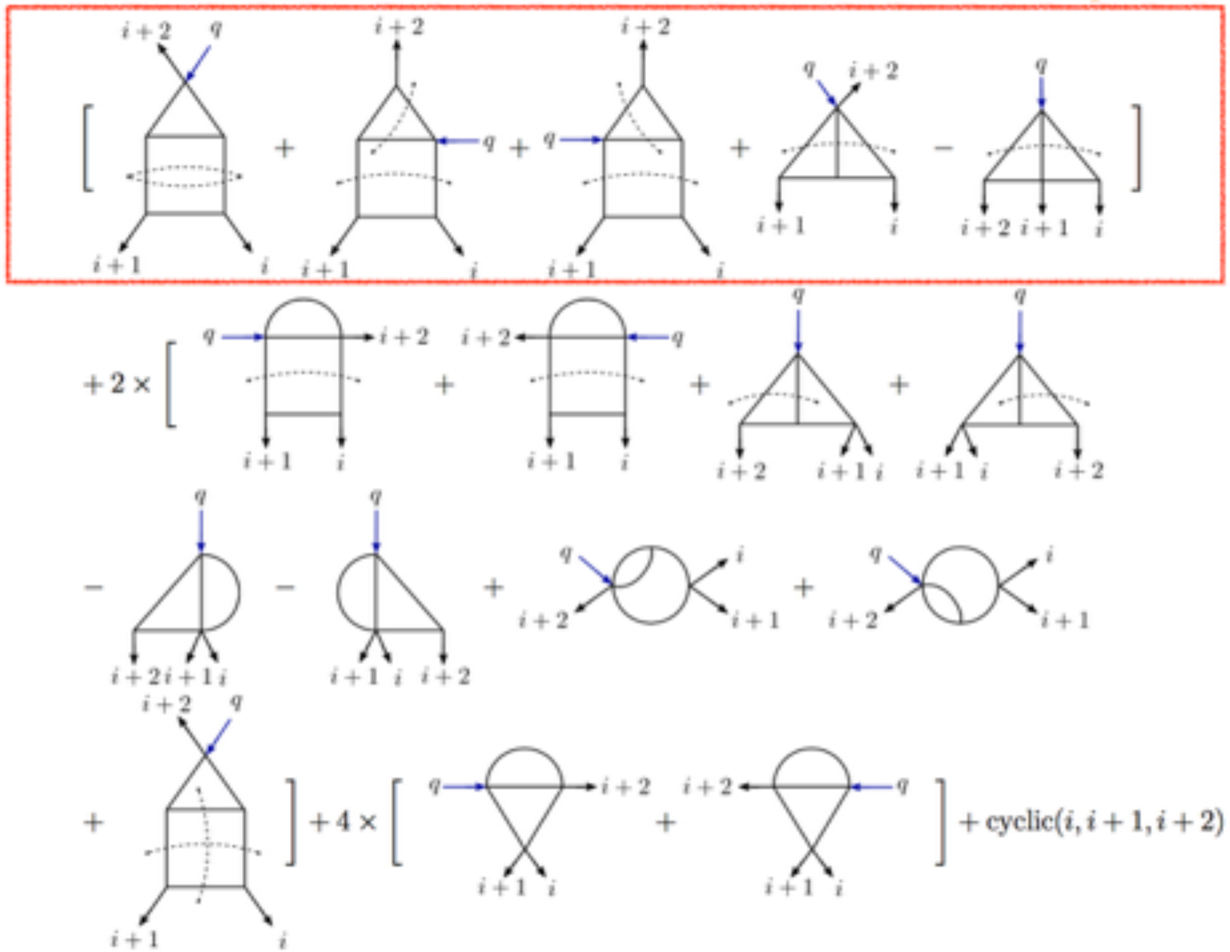


maximal degree of transcen-dentality

- Use (iterated) 2- and 3-particle cuts for $\mathcal{O}_{\rm offset} = -2{\rm tr}(XZY)$



- the combined result in terms of integral functions



- BPS form factor given by first line
- numerators indicated by dotted lines
- remaining integrals: UV divergent, transcendentality < 4
- all integrals known (Gehrmann-Remiddi)

- BDS-style remainder: $\mathcal{R}^{(2)}_{\mathcal{O}_{X[Y,Z]}} = \mathcal{R}^{(2)}_{\rm BPS} + \mathcal{R}^{(2)}_{\rm offset}$

$$\mathcal{R}^{(2)}_{\rm BPS} = F^{(2)}_{\mathcal{O}_{\rm BPS}}(\epsilon) - \frac{1}{2}\big(F^{(1)}_{\mathcal{O}_{\rm BPS}}(\epsilon)\big)^2 - f^{(2)}(\epsilon)\, F^{(1)}_{\mathcal{O}_{\rm BPS}}(2\epsilon) - C^{(2)}$$

$$\mathcal{R}^{(2)}_{\rm offset} = F^{(2)}_{\mathcal{O}_{\rm offset}}(\epsilon) - F^{(1)}_{\mathcal{O}_{\rm offset}}\Big(\frac{1}{2}F^{(1)}_{\mathcal{O}_{\rm offset}} + F^{(1)}_{\mathcal{O}_{\rm BPS}}\Big)(\epsilon) - f^{(2)}(\epsilon)\, F^{(1)}_{\mathcal{O}_{\rm offset}}(2\epsilon)$$

- Mixed UV/IR divergences cancel!
- All IR divergences have cancelled
- Left over: (log of) 2-loop UV divergences: $\frac{18}{\epsilon} - \frac{\pi^2}{\epsilon}$
  - only first term enters Z, second term is "spurious"
- BPS contribution, transcendentality=4, classical polylogs

$$\mathcal{R}^{(2)}_{\rm BPS} := \frac{3}{2}\,{\rm Li}_4(u) - \frac{3}{4}\,{\rm Li}_4\Big(-\frac{uv}{w}\Big) + \frac{3}{2}\log(w)\,{\rm Li}_3\Big(-\frac{u}{v}\Big) - \frac{1}{16}\log^2(u)\log^2(v)$$
$$- \frac{\log^2(u)}{32}\Big[\log^2(u) - 4\log(v)\log(w)\Big] - \frac{\zeta_2}{8}\log(u)[5\log(u) - 2\log(v)]$$
$$- \frac{\zeta_3}{2}\log(u) - \frac{7}{16}\zeta_4 + {\rm perms}\,(u,v,w)$$

$$u = \frac{s_{12}}{q^2},\, v = \frac{s_{23}}{q^2},\, w = \frac{s_{31}}{q^2},\, u+v+w = 1$$

- **Novel part:** $\mathcal{R}^{(2)}_{\text{offset}} = \frac{18-\pi^2}{\epsilon} + \sum_{i=0}^{3} \mathcal{R}^{(2)}_{\text{offset,i}}$
- where we have ordered terms by transcendentality:

$$\mathcal{R}^{(2)}_{\text{offset};3} = 2\Big[\text{Li}_3(u) + \text{Li}_3(1-u)\Big] - \frac{1}{2}\log^2(u)\log\frac{vw}{(1-u)^2} + \frac{2}{3}\log(u)\log(v)\log(w) + \frac{2}{3}\zeta_3 + 2\zeta_2\log(-q^2) + \text{perms}(u,v,w)$$

$$\mathcal{R}^{(2)}_{\text{offset};2} = -12\Big[\text{Li}_2(1-u) + \text{Li}_2(1-v) + \text{Li}_2(1-w)\Big] - 2\log^2(uvw) + 36\zeta_2$$

$$\mathcal{R}^{(2)}_{\text{offset};1} = -12\log(uvw) - 36\log(-q^2)$$

$$\mathcal{R}^{(2)}_{\text{offset};0} = 126$$

- Since transcendentality < 4 only classical polylogs
- Next: Intriguing relation to FF densities in SU(2) sector

Unexpected relation with SU(2) sector

- 2-loop form factor remainders of tr(XXXYX….YXY) are sums of 3 independent remainder densities (Loebbert, Nandan, Sieg, Wilhelm, Yang)

$$(R_i^{(2)})^{XXX}_{XXX}, \quad (R_i^{(2)})^{XYX}_{XXY}, \quad (R_i^{(2)})^{YXX}_{XXY}$$

"zero shuffle", transcendentality = 4, BPS

"single shuffle", transcendentality up to 3

"double shuffle", transcendentality up to 2

- each depends on 3 adjacent momenta

$$u_i = \frac{s_{ii+1}}{s_{ii+1i+2}}, v_i = \frac{s_{i+1i+2}}{s_{ii+1i+2}}, w_i = \frac{s_{ii+2}}{s_{ii+1i+2}}$$

- intriguing relation between transcendentality and "shuffling"

- We found the following relations to the form factor in the SU(2|3) sector

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};3} = -\sum_{S_3} (R_i^{(2)})^{XYX}_{XXY}\Big|_3 + 6\,\zeta_3\,,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};2} = -\sum_{S_3} \left[(R_i^{(2)})^{XYX}_{XXY} - (R_i^{(2)})^{YXX}_{XXY}\right]\Big|_2 + 5\pi^2\,,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};1} = -\sum_{S_3} \left[(R_i^{(2)})^{XYX}_{XXY} - (R_i^{(2)})^{YXX}_{XXY}\right]\Big|_1\,,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};0} = -\sum_{S_3} \left[(R_i^{(2)})^{XYX}_{XXY} - (R_i^{(2)})^{YXX}_{XXY}\right]\Big|_0$$

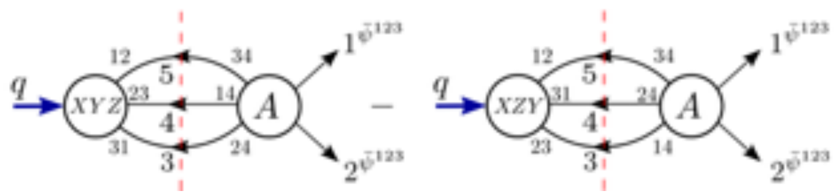
  - Sum over permutations of (u,v,w)

- Universality of form factors across different sectors?

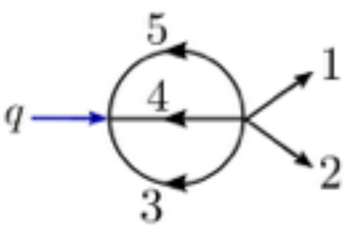
- No obvious explanation like Ward identities

Subminimal FF $\langle \bar{\psi}\bar{\psi} | \mathcal{O}_B | 0 \rangle$ at two loops

- Recall $\mathcal{O}_B = \mathrm{tr}(X[Y, Z])$
- "state shorter than operator"
- Zero at tree-level & 1-loop; at 2 loops: IR finite / UV divergent
- Induces "length changing interaction" in operator mixing
- 3-particle cut only!!



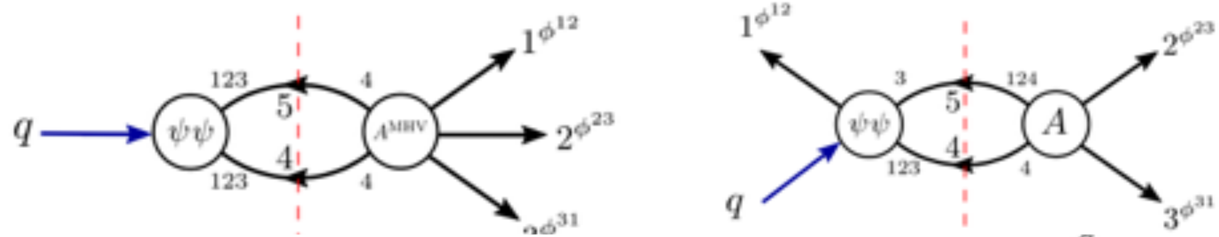
- Result $F^{(2)}_{\mathcal{O}_B}(1^{\bar{\psi}^{123}}, 2^{\bar{\psi}^{123}}; q) = \frac{1}{[12]} \frac{2(3\epsilon - 2)}{2\epsilon - 1} \times q$



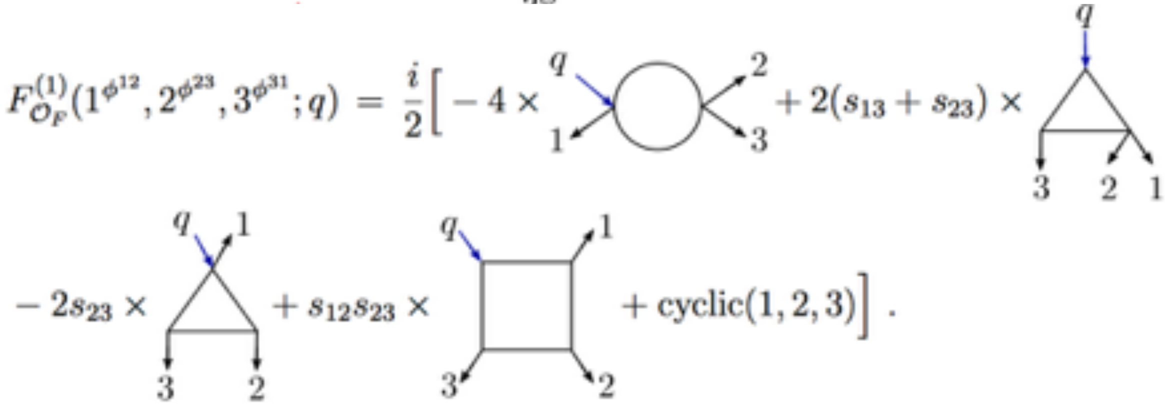
- UV-divergence: $\langle 21 \rangle \frac{6}{\epsilon} \frac{a^2(\mu_R)}{g}$

Non-minimal FF $\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_F|0\rangle$ at one loop

- Recall $\mathcal{O}_F = \frac{1}{2}\mathrm{tr}(\psi\psi)$
- vanishes at tree level. At 1-loop IR finite/UV divergent
- cuts



- result



$$= 2\,\frac{(-s_{12})^{-\epsilon}}{\epsilon(1-2\epsilon)} - \Big[2\mathrm{Li}_2(1-u) + \log u \log v\Big] + \zeta_2 + \mathrm{cyclic}(1,2,3)$$

- UV-divergence: $\frac{6}{\epsilon} g a(\mu_R)$

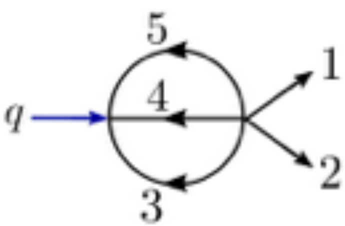
Minimal FF $\langle \bar{\psi}\bar{\psi}|\mathcal{O}_F|0\rangle$ at two loops

- Write $\mathcal{O}_F = \frac{1}{2}\mathrm{tr}(\psi\psi) + g\mathrm{tr}(X[Y,Z]) - g\mathrm{tr}(X[Y,Z]) = \mathcal{O}_{\mathrm{BPS}'} - g\mathcal{O}_B$
- IR divergent at 1 & 2 loops/UV divergent at 2 loops
- $F^{(2)}_{\mathcal{O}_{\mathrm{BPS}'}}(1^{\bar{\psi}^{123}}, 2^{\bar{\psi}^{123}}) = \langle 21\rangle \times$ 4 s_{12}^2 

+ 1 s_{12}^2 

  - part of stress tensor multiplet

(van Neerven)

- $-g\mathcal{O}_B$ contribution:(-g)x $F^{(2)}_{\mathcal{O}_B}(1^{\bar{\psi}^{123}}, 2^{\bar{\psi}^{123}}; q) = \frac{1}{[12]}\frac{2(3\epsilon-2)}{2\epsilon-1} \times q$ 

- clean separation between UV/IR divergences

- UV-divergence: $-\langle 21\rangle \frac{6}{\epsilon} a^2(\mu_R)$

The two-loop dilatation operator

$$\begin{pmatrix} \langle\bar\psi\bar\psi|\mathcal{O}_F|0\rangle & \langle\bar X\bar Y\bar Z|\mathcal{O}_F|0\rangle \\ \langle\bar\psi\bar\psi|\mathcal{O}_B|0\rangle & \langle\bar X\bar Y\bar Z|\mathcal{O}_B|0\rangle \end{pmatrix}_{\rm ren} = \underbrace{\begin{pmatrix} \mathcal{Z}_F{}^F & \mathcal{Z}_F{}^B \\ \mathcal{Z}_B{}^F & \mathcal{Z}_B{}^B \end{pmatrix}}_{\mathcal{Z}} \underbrace{\begin{pmatrix} \langle\bar\psi\bar\psi|\mathcal{O}_F|0\rangle & \langle\bar X\bar Y\bar Z|\mathcal{O}_F|0\rangle \\ \langle\bar\psi\bar\psi|\mathcal{O}_B|0\rangle & \langle\bar X\bar Y\bar Z|\mathcal{O}_B|0\rangle \end{pmatrix}}_{\mathcal{F}}$$

- demanding that the left-hand side is UV finite, and after removing universal IR divergences we find

$$\log(\mathcal{Z}) = \begin{pmatrix} a^2(\mu_R)\,\frac{6}{\epsilon} & -a(\mu_R)\,g\,\frac{6}{\epsilon} \\ -\frac{a^2(\mu_R)}{g}\,\frac{6}{\epsilon} & a(\mu_R)\frac{6}{\epsilon} - a^2(\mu_R)\frac{18}{\epsilon} \end{pmatrix} + \mathcal{O}\big(a(\mu_R)^3\big)$$

- from which we get the dilatation operator up to 2 loops

$$H = lim_{\epsilon\to 0}\Big[-\mu_R\frac{\partial}{\partial\mu_R}\log(\mathcal{Z})\Big] = 12\times\begin{pmatrix} 2a^2 & -a\,g \\ -2\,\frac{a^2}{g} & a - 6\,a^2 \end{pmatrix}$$

- Eigenvalues of H (anomalous dimension) up to 2 loops are:

$$\gamma_{\mathrm{BPS}'} = 0 \quad , \quad \gamma_K = 12a - 48a^2 + \mathcal{O}(a^3)$$

- the corresponding diagonal operators are

$$\mathcal{O}_{\mathrm{BPS}'} = \mathcal{O}_F + g\mathcal{O}_B \quad , \quad \mathcal{O}_K = \mathcal{O}_B - \frac{gN}{8\pi^2}\mathcal{O}_F$$

- in agreement with known results, where $\mathcal{O}_K$ is a descendant of the Konishi operator (which has the same anomalous dimension)

- 2-loop FF of $\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_{\mathrm{BPS}'}|0\rangle$ proportional to $\langle \bar{X}\bar{X}g^+|\mathrm{tr}(X^2)|0\rangle$

- 2-loop remainders of $\mathcal{O}_B$ and $\mathcal{O}_K$ differ by terms with transcendentally < 3

Conclusions

- Calculation of one- and two-loop dilatation operator in N=4 SYM using on-shell methods
  - 2-point functions vs. form factors
  - Relation between amplitude and dilatation operator Yangians. What about higher loops? Zwiebel at 2 loops?
- 2-loop Form Factors in SU(2|3) sector
  - transcendentally 0…4, operator mixing, unexpected similarities with SU(2) sector (universal building blocks?), 2-loop dilatation operator
  - To do: longer operators, other operators like $tr(F^3)$ comparison with QCD, …