Amplitudes and Logarithms in Chiral Perturbation Theory

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Overview

- Somewhat different audience than I usually have
- Usually I am "the high loop guy"
- Recent work: loops and lattice effects: finite volume, partial quenching, twisted boundary conditions, staggered ChPT,. . .
- This talk:
  - Introduction to Chiral Perturbation Theory/Effective field theory
  - Main part: leading logarithms to high orders
  - Some musings about recursion relations and ChPT

Chiral Perturbation Theory

Exploring the consequences of
the chiral symmetry of QCD
and its spontaneous breaking
using effective field theory techniques

Derivation from QCD:
H. Leutwyler,
On The Foundations Of Chiral Perturbation Theory,
Ann. Phys. 235 (1994) 165 [hep-ph/9311274]

For references to lectures see:
http://www.thep.lu.se/∼bijnens/chpt.html

Effective field theory

- Choose your degrees of freedom
- Construct the most general Lagrangian
- Too many (∞) parameters
- Need an ordering principle: power counting
- Can be combined with renormalizable interactions as well but then need to be very careful about what exactly you expand in
  (Discussion about EFT for Higgs)
- But still: new terms at every order in your expansion, polynomial in masses and momenta typically not determined
- Makes unitarity and amplitude methods a bit more tricky to use

Chiral Perturbation Theory

A general Effective Field Theory:

- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:

- Degrees of freedom: Goldstone Bosons from spontaneous breaking of chiral symmetry
- Powercounting: Dimensional counting in momenta/masses
- Breakdown scale: Resonances, so about M_ρ.

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Chiral Symmetry

QCD: N_f light quarks: equal mass: interchange: $SU(N_f)_V$

But $$\mathcal{L}_{QCD} = \sum_{q=u,d,s} \left[i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q \left(\bar{q}_R q_L + \bar{q}_L q_R \right) \right]$$

So if $m_q = 0$ then $SU(3)_L \times SU(3)_R$.

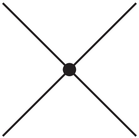
Spontaneous breakdown

- $\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
- $SU(3)_L \times SU(3)_R$ broken spontaneously to $SU(3)_V$
- 8 generators broken $\Longrightarrow$ 8 massless degrees of freedom and interaction vanishes at zero momentum

Goldstone Bosons

Power counting in momenta: Meson loops, Weinberg powercounting

rules

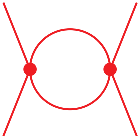


p^2

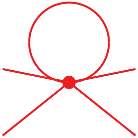
$1/p^2$

$\int d^4p$ $\quad p^4$

one loop example



$$(p^2)^2\,(1/p^2)^2\,p^4 = p^4$$



$$(p^2)\,(1/p^2)\,p^4 = p^4$$

Chiral Perturbation Theories

- Which chiral symmetry: $SU(N_f)_L \times SU(N_f)_R$, for $N_f = 2, 3, \ldots$ and extensions to (partially) quenched
- Or beyond QCD
- Space-time symmetry: Continuum or broken on the lattice: Wilson, staggered, mixed action
- Volume: Infinite, finite in space, finite T
- Which interactions to include beyond the strong one
- Which particles included as non Goldstone Bosons
- My general belief: if it involves soft pions (or soft K, η) some version of ChPT exists

Lagrangians: Lowest order

$U(\phi) = \exp(i\sqrt{2}\Phi/F_0)$ parametrizes Goldstone Bosons

$$\Phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix}.$$

LO Lagrangian: $\mathcal{L}_2 = \frac{F_0^2}{4}\{\langle D_\mu U^\dagger D^\mu U\rangle + \langle \chi^\dagger U + \chi U^\dagger\rangle\}\,,$

$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu\,,$
left and right external currents: $r(l)_\mu = v_\mu + (-)a_\mu$

Scalar and pseudoscalar external densities: $\chi = 2B_0(s + ip)$ quark masses via scalar density: $s = \mathcal{M} + \cdots$

$\langle A \rangle = Tr_F\,(A)$

Lagrangians: Lagrangian structure

	2 flavour		3 flavour		PQChPT/N_f flavour	
p^2	F, B	2	F_0, B_0	2	F_0, B_0	2
p^4	l_i^r, h_i^r	7+3	L_i^r, H_i^r	10+2	$\hat{L}_i^r, \hat{H}_i^r$	11+2
p^6	c_i^r	52+4	C_i^r	90+4	K_i^r	112+3

p^2: Weinberg 1966

p^4: Gasser, Leutwyler 84,85

p^6: JB, Colangelo, Ecker 99,00

- L_i LEC = Low Energy Constants = ChPT parameters
- H_i: contact terms: value depends on definition of currents/densities
- Finite volume: no new LECs
- Other effects: (many) new LECs

Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars
- Chiral logarithms
- includes Isospin and the eightfold way $(SU(3)_V)$
- Unitarity included perturbatively

$$m_\pi^2 = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2}\log\frac{(2B\hat{m})}{\mu^2} + 2l_3^r(\mu)\right] + \cdots$$

$$M^2 = 2B\hat{m}$$

Leading logarithms

- Main question: can we get information on high orders?
- This part:
  - Leading logarithms
  - Weinberg's argument
  - $O(N+1)/O(N)$
  - Anomaly
  - $SU(N) \times SU(N)/SU(N)$
  - Nucleon
- Collaborators: Lisa Carloni, Karol Kampf, Stefan Lanz, Alexey A. Vladimirov

References

- JB, L. Carloni, Nucl.Phys. B827 (2010) 237-255 [arXiv:0909.5086]
  Leading Logarithms in the Massive O(N) Nonlinear Sigma Model
- JB, L. Carloni, Nucl.Phys. B843 (2011) 55-83 [arXiv:1008.3499]
  The Massive O(N) Non-linear Sigma Model at High Orders
- JB, K. Kampf, S. Lanz, Nucl.Phys. B860 (2012) 245-266 [arXiv:1201.2608]
  Leading logarithms in the anomalous sector of two-flavour QCD
- JB, K. Kampf, S. Lanz, Nucl.Phys. B873 (2013) 137-164 [arXiv:1303.3125]
  Leading logarithms in N-flavour mesonic Chiral Perturbation Theory
- JB, A.A. Vladimirov, Nucl.Phys. B91 (2015) 700 [arXiv:1409.6127]
  Leading logarithms for the nucleon mass

Leading logarithms (LL)

BEWARE: leading logarithms can mean very different things

- Take a quantity with a single scale: $F(M)$
- Subtraction scale in QFT (in dim. reg.) is logarithmic
- $L = \log(\mu/M)$
- $F = F_0 + \left(F_1^1 L + F_0^1\right) + \left(F_2^2 L^2 + F_1^2 L + F_0^2\right) + \left(F_3^3 L^3 + \cdots\right) + \cdots$
- Leading Logarithms: The terms $F_m^m L^m$

The F_m^m can be more easily calculated than the full result

- $\mu\,(dF/d\mu) \equiv 0$
- Ultraviolet divergences in Quantum Field Theory are always local

Renormalizable theories

- Loop expansion $\equiv \alpha$ expansion
- $F = \alpha + \left(f_1^1\alpha^2 L + f_0^1\alpha^2\right) + \left(f_2^2\alpha^3 L^2 + f_1^2\alpha^3 L + f_0^2\alpha^3\right) + \left(f_3^3\alpha^4 L^3 + \cdots\right) + \cdots$
- f_i^j are pure numbers
- $\mu\frac{dF}{d\mu} \equiv F',\ \mu\frac{d\alpha}{d\mu} \equiv \alpha',\ \mu\frac{dL}{d\mu} = 1$
- $F' = \alpha' + \left(f_1^1\alpha^2 + f_1^1 2\alpha'\alpha L + f_0^1 2\alpha\alpha'\right) + \left(f_2^2\alpha^3 2L + f_2^2 3\alpha'\alpha^2 L^2 + f_1^2\alpha^3 + 3f_1^2\alpha'\alpha^2 L + 3f_0^2\alpha'\alpha^2\right) + \left(f_3^3\alpha^3 3L^2 + f_3^3 4\alpha'\alpha^3 L^3 + \cdots\right) + \cdots$
- $\alpha' = \beta_1\alpha^2 + \beta_2\alpha^3 + \cdots$
- $0 = F' = \left(\beta_1 + f_1^1\right)\alpha^2 + \left(2\beta_1 f_1^1 + 2f_2^2\right)\alpha^3 L + \left(\beta_2 + 2\beta_1 f_0^1 + f_1^2\right)\alpha^3 + \left(3\beta_1 f_2^2 + 3f_3^3\right)\alpha^4 L^2 + \cdots$
- $\boxed{f_1^1 = -\beta_1,\ f_2^2 = \beta_1^2,\ f_3^3 = -\beta_1^3, \ldots}$

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- $\alpha' = \beta_1 \alpha^2 + \beta_2 \alpha^3 + \cdots$
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- $\boxed{f_1^1 = -\beta_1,\ f_2^2 = \beta_1^2,\ f_3^3 = -\beta_1^3, \ldots}$

Renormalizable theories

- Leading logs known as soon as β_1 is known
- $F(M) = \alpha \left(1 - \alpha\beta_1 L + (\alpha\beta_1 L)^2 - (\alpha\beta_1 L)^3 + \cdots\right) + \cdots$
  $$= \frac{\alpha(\mu)}{1 + \alpha(\mu)\beta_1 \log(\mu/M)} + \cdots = \alpha(M) + \cdots$$
- running coupling constant
- Generalizes to lower leading logarithms as well
- Multiscale problems: many other terms possible

Renormalization Group

- Can be extended to other operators as well
- Underlying argument always $\mu \frac{dF}{d\mu} = 0$.
- Gell-Mann–Low, Callan–Symanzik, Weinberg–'t Hooft
- In great detail: J.C. Collins, *Renormalization*
- Relies on the α the same in all orders
- LL one-loop β_1
- NLL two-loop β_2 and one-loop f_0^1
- In effective field theories: different Lagrangian at each order
- The recursive argument does not work

Weinberg's argument for leading logarithms

- Weinberg, Physica A96 (1979) 327
- Two-loop leading logarithms can be calculated using only one-loop: Weinberg consistency conditions
- $\pi\pi$ at 2-loop: Colangelo, hep-ph/9502285
- General at 2 loop: JB, Colangelo, Ecker, hep-ph/9808421
- Proof at all orders
  - using β-functions: Büchler, Colangelo, hep-ph/0309049
  - with diagrams: JB, Carloni, arXiv:0909.5086
  - Extension to nucleons: JB, Vladimirov, arXiv:1409.6127
- First mesonic case where

  loop-order = power-counting (or $\hbar$) order
- Later nucleon case where

  loop-order $\neq$ power-counting (or $\hbar$) order

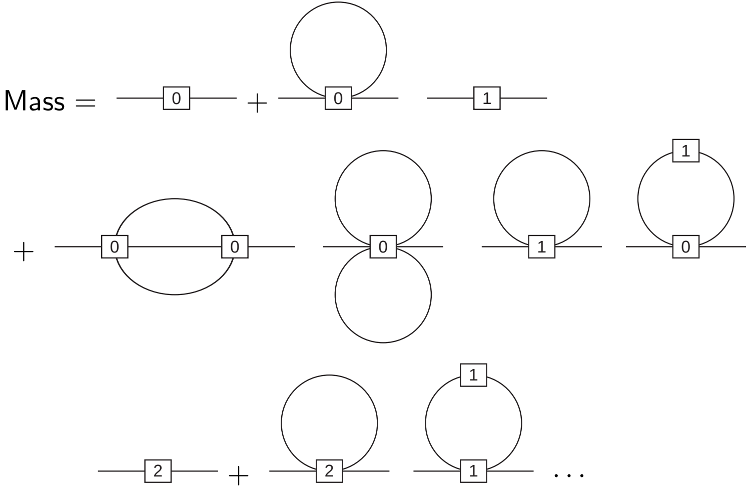
Weinberg's argument

- μ: dimensional regularization scale
- $d = 4 - w$
- loop-expansion $\equiv$ $\hbar$-expansion
- $\mathcal{L}^{\text{bare}} = \sum_{n\geq 0} \hbar^n \mu^{-nw} \mathcal{L}^{(n)}$
- $\mathcal{L}^{(n)} = \sum_i c_i^{(n)} \mathcal{O}_i$
- $c_i^{(n)} = \sum_{k=0,n} \frac{c_{ki}^{(n)}}{w^k}$
- $c_{0i}^{(n)}$ have a direct μ-dependence
- $c_{ki}^{(n)}$ $k \geq 1$ only depend on μ through $c_{0i}^{(m<n)}$

Weinberg's argument

- L^n_ℓ ℓ-loop contribution at order $\hbar^n$
- Expand in divergences from the loops (not from the counterterms) $L^n_\ell = \sum_{k=0,l} \frac{1}{w^k} L^n_{k\ell}$
- Neglected positive powers: not relevant here, but should be kept in general
- $\{c\}^n_\ell$ all combinations $c^{(m_1)}_{k_1 j_1} c^{(m_2)}_{k_2 j_2} \dots c^{(m_r)}_{k_r j_r}$ with $m_i \geq 1$, such that $\sum_{i=1,r} m_i = n$ and $\sum_{i=1,r} k_i = \ell$.
- $\{c^n_n\} \equiv \{c^{(n)}_{ni}\}$, $\{c\}^2_2 = \{c^{(2)}_{2i}, c^{(1)}_{1i} c^{(1)}_{1k}\}$
- $\mathcal{L}^{(n)} = \boxed{n}$

Weinberg's argument



Weinberg's argument

- $\hbar^0$: L_0^0
- $\hbar^1$: $\frac{1}{w}\left(\mu^{-w}L_{00}^1(\{c\}_1^1) + L_{11}^1\right) + \mu^{-w}L_{00}^1(\{c\}_0^1) + L_{10}^1$
- Expand $\mu^{-w} = 1 - w\log\mu + \frac{1}{2}w^2\log^2\mu + \cdots$
- $1/w$ must cancel: $L_{00}^1(\{c\}_1^1) + L_{11}^1 = 0$
  this determines the c_{1i}^1
- Explicit $\log\mu$: $-\log\mu\, L_{00}^1(\{c\}_1^1) \equiv \log\mu\, L_{11}^1$

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Weinberg's argument

- $\hbar^2$:
$$\frac{1}{w^2}\left(\mu^{-2w}L^2_{00}(\{c\}^2_2)+\mu^{-w}L^2_{11}(\{c\}^1_1)+L^2_{22}\right)$$
$$+\frac{1}{w}\left(\mu^{-2w}L^2_{00}(\{c\}^2_1)+\mu^{-w}L^2_{11}(\{c\}^1_0)+\mu^{-w}L^2_{10}(\{c\}^1_1)+L^2_{21}\right)$$
$$+\left(\mu^{-2w}L^2_{00}(\{c\}^2_0)+\mu^{-w}L^2_{10}(\{c\}^1_0)+L^2_{20}\right)$$
- $1/w^2$ and $\log\mu/w$ must cancel
$$L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)+L^2_{22}=0$$
$$2L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)=0$$
- Solution: $L^2_{00}(\{c\}^2_2)=-\frac{1}{2}L^2_{11}(\{c\}^1_1)$
$$L^2_{11}(\{c\}^1_1)=-2L^2_{22}$$
- Explicit $\log\mu$ dependence (one-loop is enough)
$$\frac{1}{2}\log^2\mu\left(4L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)\right)=-\frac{1}{2}L^2_{11}(\{c\}^1_1)\log^2\mu\,.$$

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All orders

- $\hbar^n$:
$$\frac{1}{w^n}\left(\mu^{-nw}L^n_{00}(\{c\}^n_n)+\mu^{-(n-1)w}L^n_{11}(\{c\}^{n-1}_{n-1})+\cdots\right.$$
$$\left.+\mu^{-w}L^n_{n-1\ n-1}(\{c\}^1_1)+L^n_{nn}\right)+\frac{1}{w^{n-1}}\cdots$$
- $1/w^n, \log\mu/w^{n-1},\ldots,\log^{n-1}\mu/w$ cancel:
$$\sum_{i=0}^{n} i^j L^n_{n-i\ n-i}(\{c\}^i_i)=0 \qquad j=0,..,n-1.$$
- Solution: $L^n_{n-i\ n-i}(\{c\}^i_i)=(-1)^i\begin{pmatrix} n \\ i\end{pmatrix}L^n_{nn}$
- explicit leading $\log\mu$ dependence and divergence
$$\log^n\mu\frac{(-1)^{n-1}}{n}L^n_{11}(\{c\}^{n-1}_{n-1})$$
$$L^n_{00}(\{c\}^n_n)=-\frac{1}{n}L^n_{11}(\{c\}^{n-1}_{n-1})$$

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- $1/w^n, \log\mu/w^{n-1},\ldots,\log^{n-1}\mu/w$ cancel:

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- Solution: $L^n_{n-i\;n-i}(\{c\}^i_i)=(-1)^i\begin{pmatrix} n\\ i\end{pmatrix}L^n_{nn}$
- explicit leading $\log\mu$ dependence and divergence

$$\log^n\mu\frac{(-1)^{n-1}}{n}L^n_{11}(\{c\}^{n-1}_{n-1})$$
$$L^n_{00}(\{c\}^n_n)=-\frac{1}{n}L^n_{11}(\{c\}^{n-1}_{n-1})$$

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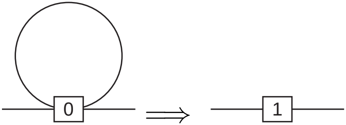
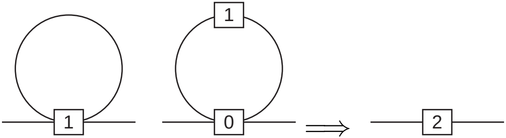
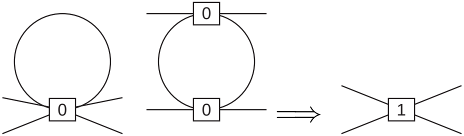
$$\left. + \mu^{-w} L^n_{n-1\ n-1}(\{c\}^1_1) + L^n_{nn}\right) + \frac{1}{w^{n-1}}\cdots$$
- $1/w^n, \log\mu/w^{n-1}, \ldots, \log^{n-1}\mu/w$ cancel:
$$\sum_{i=0}^{n} i^j L^n_{n-i\ n-i}(\{c\}^i_i) = 0 \qquad j = 0, .., n-1.$$
- Solution: $L^n_{n-i\ n-i}(\{c\}^i_i) = (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} L^n_{nn}$
- explicit leading $\log\mu$ dependence and divergence
$$\log^n\mu \frac{(-1)^{n-1}}{n} L^n_{11}(\{c\}^{n-1}_{n-1})$$
$$L^n_{00}(\{c\}^n_n) = -\frac{1}{n} L^n_{11}(\{c\}^{n-1}_{n-1})$$

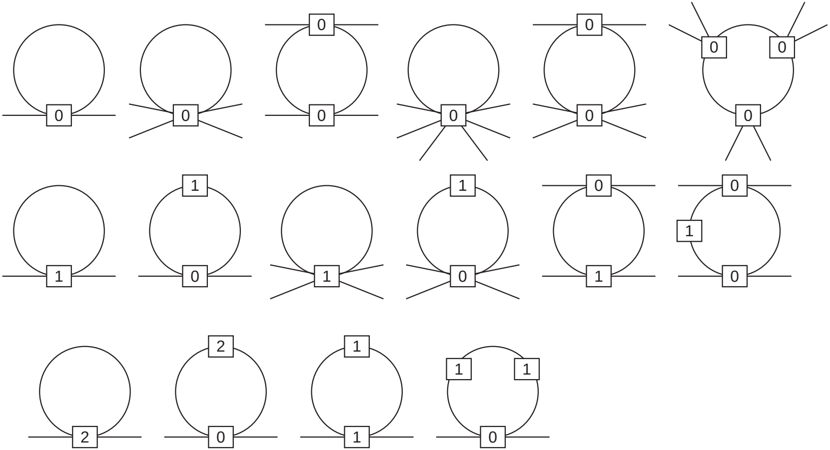
Mass to $\hbar^2$

- $\hbar^1$: 

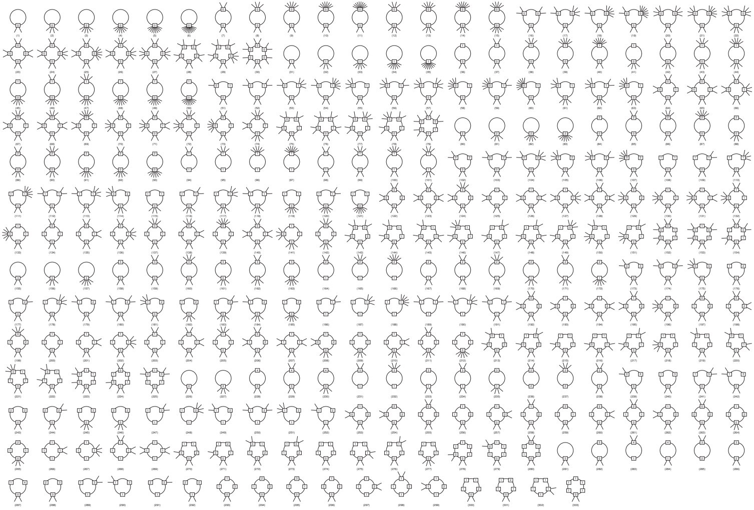
- $\hbar^2$: 

- but also needs $\hbar^1$: 

Mass to order $\hbar^3$



Mass to order $\hbar^6$



General

- Calculate the divergence
- rewrite it in terms of a local Lagrangian
  - Luckily: symmetry kept: we know result will be symmetrical, hence do not need to explicitly rewrite the Lagrangians in a nice form
  - Luckily: we do not need to go to a minimal Lagrangian
  - So everything can be computerized
  - Thank Jos Vermaseren for FORM
- We keep all terms to have all 1PI (one particle irreducible) diagrams finite

Massive $O(N)$ sigma model

- $O(N+1)/O(N)$ nonlinear sigma model
- $\mathcal{L}_{n\sigma} = \frac{F^2}{2}\partial_\mu \Phi^T \partial^\mu \Phi + F^2 \chi^T \Phi \,.$
  - Φ is a real $N+1$ vector; $\Phi \to O\Phi$; $\Phi^T\Phi = 1$.
  - Vacuum expectation value $\langle \Phi^T \rangle = (1\ 0 \ldots 0)$
  - Explicit symmetry breaking: $\chi^T = \left(M^2\ 0 \ldots 0\right)$
  - Both spontaneous and explicit symmetry breaking
  - N-vector ϕ
- N (pseudo-)Nambu-Goldstone Bosons
- $N = 3$ is two-flavour Chiral Perturbation Theory

Massive $O(N)$ sigma model: Φ vs ϕ

- $$\Phi_1 = \begin{pmatrix} \sqrt{1 - \frac{\phi^T\phi}{F^2}} \\ \frac{\phi^1}{F} \\ \vdots \\ \frac{\phi^N}{F} \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \frac{\phi^T\phi}{F^2}} \\ \frac{\phi}{F} \end{pmatrix}$$ Gasser, Leutwyler

- $$\Phi_5 = \frac{1}{1 + \frac{\phi^T\phi}{4F^2}} \begin{pmatrix} 1 - \frac{\phi^T\phi}{2F^2} \\ \frac{\phi}{F} \end{pmatrix} \qquad \Phi_3 = \begin{pmatrix} 1 - \frac{1}{2}\frac{\phi^T\phi}{F^2} \\ \sqrt{1 - \frac{1}{4}\frac{\phi^T\phi}{F^2}}\,\frac{\phi}{F} \end{pmatrix}$$
  Weinberg — only mass term

- $$\Phi_4 = \begin{pmatrix} \cos\sqrt{\frac{\phi^T\phi}{F^2}} \\ \sin\sqrt{\frac{\phi^T\phi}{F^2}}\,\frac{\phi}{\sqrt{\phi^T\phi}} \end{pmatrix}$$ CCWZ

Massive $O(N)$ sigma model: Checks

Need (many) checks:

- use many different parametrizations
- compare with known results:

$$M^2_{phys} = M^2 \left(1 - \frac{1}{2} L_M + \frac{17}{8} L_M^2 + \cdots \right) ,$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{\mathcal{M}^2}$$

  Usual choice $\mathcal{M} = M$.
- large N but massive results more hidden
  Coleman, Jackiw, Politzer 1974
- JB, Carloni, Kampf, Lanz: mass to 6 loops

Results

$$M^2_{\rm phys} = M^2(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots)$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2}$$

i	a_i, $N = 3$	a_i for general N
1	$-\frac{1}{2}$	$1 - \frac{N}{2}$
2	$\frac{17}{8}$	$\frac{7}{4} - \frac{7N}{4} + \frac{5\ N^2}{8}$
3	$-\frac{103}{24}$	$\frac{37}{12} - \frac{113N}{24} + \frac{15\ N^2}{4} - N^3$
4	$\frac{24367}{1152}$	$\frac{839}{144} - \frac{1601\ N}{144} + \frac{695\ N^2}{48} - \frac{135\ N^3}{16} + \frac{231\ N^4}{128}$
5	$-\frac{8821}{144}$	$\frac{33661}{2400} - \frac{1151407\ N}{43200} + \frac{197587\ N^2}{4320} - \frac{12709\ N^3}{300} + \frac{6271\ N^4}{320} - \frac{7\ N^5}{2}$
6	$\frac{1922964667}{6220800}$	$158393809/3888000 - 182792131/2592000\ N$ $+1046805817/7776000\ N^2 - 17241967/103680\ N^3$ $+70046633/576000\ N^4 - 23775/512\ N^5 + 7293/1024\ N^6$

Results

$$F_{\rm phys} = F(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots)$$

i	a_i for $N=3$	a_i for general N
1	1	$-1/2 + 1/2\,N$
2	$-5/4$	$-1/2 + 7/8\,N - 3/8\,N^2$
3	$83/24$	$-7/24 + 21/16\,N - 73/48\,N^2 + 1/2\,N^3$
4	$-3013/288$	$47/576 + 1345/864\,N - 14077/3456\,N^2 + 625/192\,N^3$ $-105/128\,N^4$
5	$\frac{2060147}{51840}$	$-23087/64800 + 459413/172800\,N - 189875/20736\,N^2$ $+546941/43200\,N^3 - 1169/160\,N^4 + 3/2\,N^5$
6	$-\frac{69228787}{466560}$	$-277079063/93312000 + 1680071029/186624000\,N$ $-686641633/31104000\,N^2 + 813791909/20736000\,N^3$ $-128643359/3456000\,N^4 + 260399/15360\,N^5 - 3003/1024\,N^6$

Results

$$\langle \bar{q}_i q_i \rangle = -BF^2(1 + c_1 L_M + c_2 L_M^2 + c_3 L_M^3 + \ldots)$$

$M^2 = 2B\hat{m} \qquad \chi^T = 2B(s\ 0\ \ldots 0)$

s corresponds to $\bar{u}u + \bar{d}d$ current

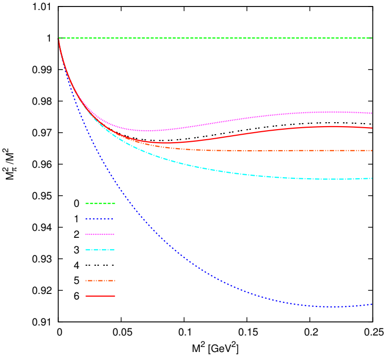
i	c_i for $N=3$	c_i for general N
1	$\frac{3}{2}$	$\frac{N}{2}$
2	$-\frac{9}{8}$	$\frac{3N}{4} - \frac{3N^2}{8}$
3	$\frac{9}{2}$	$\frac{3N}{2} - \frac{3N^2}{2} + \frac{N^3}{2}$
4	$-\frac{1285}{128}$	$\frac{145N}{48} - \frac{55N^2}{12} + \frac{105N^3}{32} - \frac{105N^4}{128}$
5	46	$\frac{3007N}{480} - \frac{1471N^2}{120} + \frac{557\ N^3}{40} - \frac{1191\ N^4}{160} + \frac{3\ N^5}{2}$

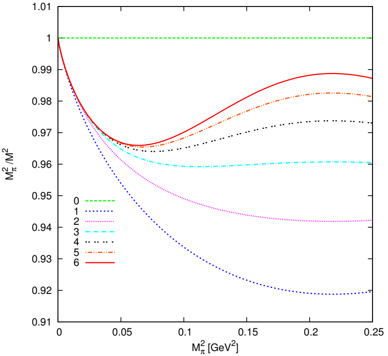
Anyone recognize any funny functions?

Alternative expansions

- $L_M = \dfrac{M^2}{16\pi^2 F^2} \log \dfrac{\mu^2}{M^2}$
- $\tilde{L}_M = \dfrac{M^2_{\mathrm{phys}}}{16\pi^2 F^2} \log \dfrac{\mu^2}{M^2_{\mathrm{phys}}}$
- $L_{\mathrm{phys}} = \dfrac{M^2_{\mathrm{phys}}}{16\pi^2 F^2_{\mathrm{phys}}} \log \dfrac{\mu^2}{M^2_{\mathrm{phys}}}$
- For masses expansion in $\tilde{L}_M$ best, but no general obvious choice

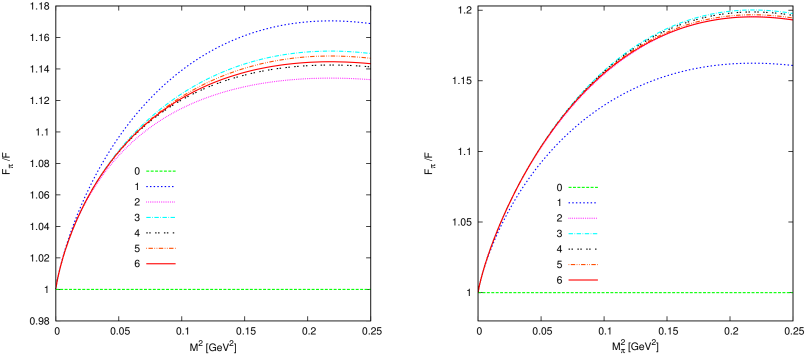
Numerical results





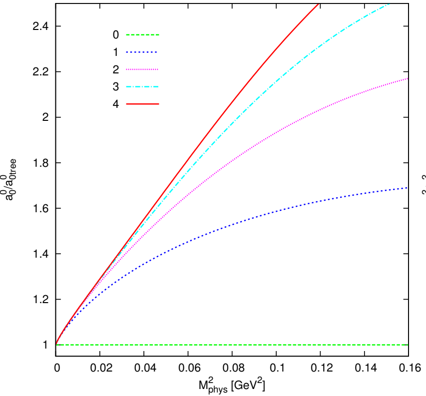
- Left: $\frac{M^2_{\text{phys}}}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots$
  $F = 90$ MeV, $\mu = 0.77$ GeV
- Right: $\frac{M^2_{\text{phys}}}{M^2} = 1 + c_1 L_{\text{phys}} + c_2 L_{\text{phys}}^2 + c_3 L_{\text{phys}}^3 + \cdots$
  $F_\pi = 92$ MeV, $\mu = 0.77$ GeV

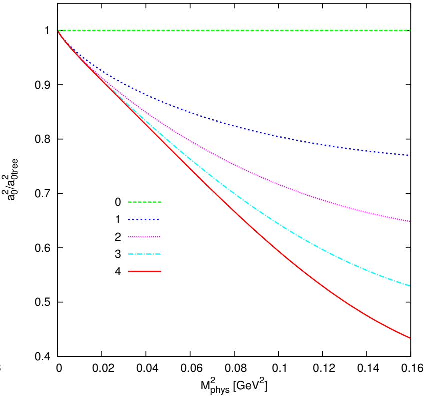
Numerical results



- Left: $\frac{F_{\rm phys}}{F} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots$
  $F = 90$ MeV, $\mu = 0.77$ GeV
- Right: $\frac{F_{\rm phys}}{F} = 1 + c_1 L_{\rm phys} + c_2 L_{\rm phys}^2 + c_3 L_{\rm phys}^3 + \cdots$
  $F_\pi = 92$ MeV, $\mu = 0.77$ GeV

Numerics: π-π scattering





Other results

- We also did: vector and scalar form-facors
- Bissegger, Fuhrer, hep-ph/0612096 Dispersive methods, massless Π_S to five loops
- Kivel, Polyakov, Vladimirov, 0809.3236, 0904.3008, 1004.2197, 1012.4205, 1105.4990
  - In the massless case tadpoles vanish
  - hence the number of external legs needed does not grow
  - All 4-meson vertices via Legendre polynomials
  - can do divergence of all one-loop diagrams analytically
  - algebraic (but quadratic) recursion relations
  - massless $\pi\pi$, F_V and F_S to arbitrarily high order
  - large N agrees with Coleman, Wess, Zumino
  - large N is not a good approximation

Other results

- Leading chiral logarithms of $K_S \to \gamma\gamma$ and $K_S \to \gamma l^+ l^-$ at two loops, K. Ghorbani, 1403.6791
- $K \to \pi\pi$ at two-loop order and $K \to n\pi$,
  M. Büchler, hep-ph/0504180, hep-ph/0511087

Anomaly for $O(4)/O(3)$

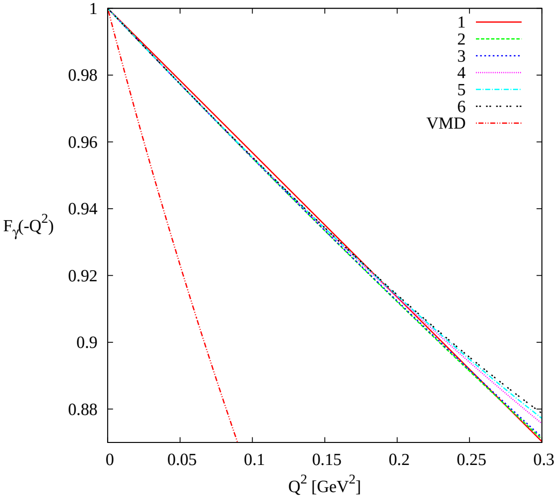
JB, Kampf, Lanz, arXiv:1201.2608

- $$\mathcal{L}_{WZW} = -\frac{N_c}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\left\{\epsilon^{abc}\left(\frac{1}{3}\Phi^0\partial_\mu\Phi^a\partial_\nu\Phi^b\partial_\rho\Phi^c - \partial_\mu\Phi^0\partial_\nu\Phi^a\partial_\rho\Phi^b\Phi^c\right)v_\sigma^0 + (\partial_\mu\Phi^0\Phi^a - \Phi^0\partial_\mu\Phi^a)v_\nu^a\partial_\rho v_\sigma^0 + \frac{1}{2}\epsilon^{abc}\Phi^0\Phi^a v_\mu^b v_\nu^c\partial_\rho v_\sigma^0\right\}.$$
- $A(\pi^0 \to \gamma(k_1)\gamma(k_2)) = \epsilon_{\mu\nu\alpha\beta}\,\varepsilon_1^{*\mu}(k_1)\varepsilon_2^{*\nu}(k_2)\,k_1^\alpha k_2^\beta\,F_{\pi\gamma\gamma}(k_1^2,k_2^2)$
- $F_{\pi\gamma\gamma}(k_1^2,k_2^2) = \frac{e^2}{4\pi^2 F_\pi}\hat{F}F_\gamma(k_1^2)F_\gamma(k_2^2)F_{\gamma\gamma}(k_1^2,k_2^2)$
- $\hat{F}$: on-shell photon; $F_\gamma(k^2)$: formfactor; $F_{\gamma\gamma}$ nonfactorizable part

Anomaly for $O(4)/O(3)$

- Done to six-loops
- $\hat{F} = 1 + 0 - 0.000372 + 0.000088 + 0.000036 + 0.000009 + 0.0000002 + \dots$
- Really good convergence
- $F_{\gamma\gamma}$ only starts at three-loop order (could have been two)
- $F_{\gamma\gamma}$ in the chiral limit only starts at four-loops.
- The leading logarithms thus predict this part to be fairly small.
- $F_\gamma(k^2)$: plot

Anomaly for $O(4)/O(3)$



Leading logs small, converge fast

$\gamma 3\pi$

- Experiment 1: $\bar{F}^{3\pi}_{\text{exp}} = 12.9 \pm 0.9 \pm 0.5\ \text{GeV}^{-3}$
- Experiment 2: $F^{3\pi}_{0,\text{exp}} = 9.9 \pm 1.1\ \text{GeV}^{-3}$
- Theory lowest order: $F^{3\pi}_0 = 9.8\ \text{GeV}^{-3}$
- Theory (LL only)
  $F^{3\pi LL}_0 = (9.8 - 0.3 + 0.04 + 0.02 + 0.006 + 0.001 + \ldots)\ \text{GeV}^{-3}$
- good convergence

$SU(N) \times SU(N)/SU(N)$

- $SU(N) \times SU(N)/SU(N)$ (vector and scalar)
- Mass, Decay constants, Form-factors
- Meson-Meson, $\gamma\gamma \to \pi\pi$
- No luck with guess for general N-dependence either
- Four different parametrizations of a unitary matrix used

Mass

i	a_i for $N=2$	a_i for $N=3$	a_i for general N
1	$-1/2$	$-1/3$	$-N^{-1}$
2	$17/8$	$27/8$	$9/2\,N^{-2} - 1/2 + 3/8\,N^2$
3	$-103/24$	$-3799/648$	$-89/3\,N^{-3} + 19/3\,N^{-1} - 37/24\,N - 1/12\,N^3$
4	$24367/1152$	$146657/2592$	$2015/8\,N^{-4} - 773/12\,N^{-2} + 193/18 + 121/288\,N^2$ $+41/72\,N^4$
5	$-8821/144$	$-\frac{27470059}{186624}$	$-38684/15\,N^{-5} + 6633/10\,N^{-3} - 59303/1080\,N^{-1}$ $-5077/1440\,N - 11327/4320\,N^3 - 8743/34560\,N^5$
6*	$\frac{1922964667}{6220800}$	$\frac{12902773163}{9331200}$	$7329919/240\,N^{-6} - 1652293/240\,N^{-4}$ $-4910303/15552\,N^{-2} + 205365409/972000$ $-69368761/7776000\,N^2 + 14222209/2592000\,N^4$ $+3778133/3110400\,N^6$

Meson-meson scattering

$$
\begin{aligned}
M(s,t,u) = & \left[\mathrm{tr}\, T^a T^b T^c T^d + \mathrm{tr}\, T^a T^d T^c T^b\right] B(s,t,u) \\
& + \left[\mathrm{tr}\, T^a T^c T^d T^b + \mathrm{tr}\, T^a T^b T^d T^c\right] B(t,u,s) \\
& + \left[\mathrm{tr}\, T^a T^d T^b T^c + \mathrm{tr}\, T^a T^c T^b T^d\right] B(u,s,t) \\
& + \delta^{ab}\delta^{cd} C(s,t,u) + \delta^{ac}\delta^{bd} C(t,u,s) + \delta^{ad}\delta^{bc} C(u,s,t)\,.
\end{aligned}
$$

- Two functions needed
- Two-loops known exactly JB, J. Lu, arXiv:1102.0172, JHEP 03 (2011) 028
- Leading logs done to five loops
- 7 different channels ($\pi\pi$ has I=0,1,2)
- No obvious pattern, not even large N

Nucleon

- In this sector very little done at two-loop and none higher (here N^3LO)
- Mass:
  p^5 Birse, McGovern, hep-ph/9807384, hep-lat/0608002
  p^6 Schindler, Djukanovic, Gegelia, Scherer, nucl-th/0611083, 0707.4296
- g_A to p^5 via RGE Bernard, Meissner, hep-lat/0605010

Nucleon Lagrangian

- We use the heavy-baryon approach: explicit powercounting
- IR-regularization plus relativistic Lagrangian used as a check
- LO Lagrangian is order p (mesons p^2):
  $\mathcal{L}^{(0)}_{N\pi} = \bar{N}\left(iv^\mu D_\mu + g_A S^\mu u_\mu\right) N$
- Propagator is order $1/p$ (mesons $1/p^2$)
- Loops add p^2 just as for mesons
- Massless case: vertices with more legs needed (mesonic argument fails)

- Different parametrizations for mesons
- Two different p^2 Lagrangians:
  Bernard-Kaiser-Meissner:

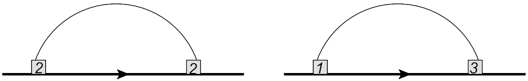
$$\mathcal{L}^{(1)}_{\pi N} = \bar{N}_v \Big[\frac{(v\cdot D)^2 - D\cdot D - i g_A \{S\cdot D, v\cdot u\}}{2M} + c_1 \mathrm{tr}\left(\chi_+\right)$$
$$+ \left(c_2 - \frac{g_A^2}{8M}\right)(v\cdot u)^2 + c_3 u\cdot u +$$
$$\left(c_4 + \frac{1}{4M}\right) i \epsilon^{\mu\nu\rho\sigma} u_\mu u_\nu v_\rho S_\sigma \Big] N_v$$

Ecker-Mojzis:

$$\mathcal{L}^{(1)}_{N\pi} = \frac{1}{M}\bar{N}\Big[-\frac{1}{2}\left(D_\mu D^\mu + i g_A \{S_\mu D^\mu, v_\nu u^\nu\}\right) + A_1 \mathrm{tr}\left(u_\mu u^\mu\right)$$
$$+ A_2 \mathrm{tr}\left((v_\mu u^\mu)^2\right) + A_3 \mathrm{tr}\left(\chi_+\right) + A_5 i \epsilon^{\mu\nu\rho\sigma} v_\mu S_\nu u_\rho u_\sigma \Big] N$$

Nucleon loops

- set $\hbar^n \sim p^{n+1}$ for meson-nucleon
- set $\hbar^n \sim p^{n+2}$ for mesons
- Introduce a RGO renormalization group order $\approx$ max power of $1/w$
- same p-order can be different RGO, e.g.



both p^5, left RGO 1, right RGO 2

- Note: same equations, if no tree level contribution next-to-leading log also calculable
- For nucleon can have fractional powers of quark masses

Nucleon loops

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both p^5, left RGO 1, right RGO 2

- Note: same equations, if no tree level contribution next-to-leading log also calculable
- For nucleon can have fractional powers of quark masses

Results

M: nucleon mass, m: pion mass, $L = \frac{m^2}{(4\pi F)^2} \log \frac{\mu^2}{m^2}$

$$
\begin{aligned}
M_{\text{phys}} &= M + k_2 \frac{m^2}{M} + k_3 \frac{\pi m^3}{(4\pi F)^2} + k_4 \frac{m^4}{(4\pi F)^2 M} \ln \frac{\mu^2}{m^2} \\
&\quad + k_5 \frac{\pi m^5}{(4\pi F)^4} \ln \frac{\mu^2}{m^2} + \dots \\
&= M + \frac{m^2}{M} \sum_{n=1}^{\infty} k_{2n} L^{n-1} + \pi m \frac{m^2}{(4\pi F)^2} \sum_{n=1}^{\infty} k_{2n+1} L^{n-1},
\end{aligned}
$$

Results

k_2	$-4c_1 M$
k_3	$-\frac{3}{2}g_A^2$
k_4	$\frac{3}{4}\left(g_A^2+(c_2+4c_3-4c_1)M\right)-3c_1 M$
k_5	$\frac{3g_A^2}{8}\left(3-16g_A^2\right)$
k_6	$-\frac{3}{4}\left(g_A^2+(c_2+4c_3-4c_1)M\right)+\frac{3}{2}c_1 M$
k_7	$g_A^2\left(-18g_A^4+\frac{35g_A^2}{4}-\frac{443}{64}\right)$
k_8	$\frac{27}{8}\left(g_A^2+(c_2+4c_3-4c_1)M\right)-\frac{9}{2}c_1 M$
k_9	$\frac{g_A^2}{3}\left(-116g_A^6+\frac{2537g_A^4}{20}-\frac{3569g_A^2}{24}+\frac{55609}{1280}\right)$
k_{10}	$-\frac{257}{32}\left(g_A^2+(c_2+4c_3-4c_1)M\right)+\frac{257}{32}c_1 M$
k_{11}	$\frac{g_A^2}{2}\left(-95g_A^8+\frac{5187407g_A^6}{20160}-\frac{449039g_A^4}{945}+\frac{16733923g_A^2}{60480}-\frac{298785521}{1935360}\right)$

- $g_A \leftrightarrow -g_A$: only even powers
- k_{2n} peculiar structure
- Drop g_A^3 then can calculate k_{12}

Results

r_2	$-4c_1 M$
r_3	$-\frac{3}{2}g_A^2$
r_4	$\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - 5c_1 M$
r_5	$-6g_A^2$
r_6	$5c_1 M$
r_7	$\frac{g_A^2}{4}\left(-8 + 5g_A^2 - 72g_A^4\right)$
r_8	$\frac{25}{3}c_1 M$
r_9	$\frac{g_A^2}{3}\left(-116g_A^6 + \frac{647g_A^4}{20} - \frac{457g_A^2}{12} + \frac{17}{40}\right)$
r_{10}	$\frac{725}{36}c_1 M$
r_{11}	$\frac{g_A^2}{2}\left(95g_A^8 - \frac{1679567g_A^6}{20160} + \frac{451799g_A^4}{3780} - \frac{320557g_A^2}{15120} + \frac{896467}{60480}\right)$
r_{12}	$\frac{175}{4}c_1 M$

- everything rewritten in terms of physical pion mass
- Simpler expression

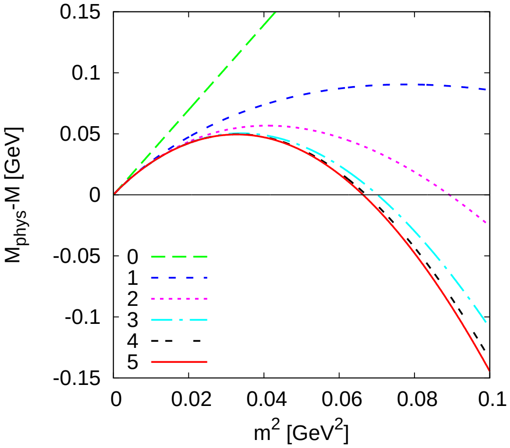
Results

- Conjecture:

$$M = M_{\text{phys}} + \frac{3}{4} m_{\text{phys}}^4 \frac{\log \frac{\mu^2}{m_{\text{phys}}^2}}{(4\pi F)^2} \left(\frac{g_A^2}{M_{\text{phys}}} - 4c_1 + c_2 + 4c_3 \right)$$
$$- \frac{3c_1}{(4\pi F)^2} \int_{m_{\text{phys}}^2}^{\mu^2} m_{\text{phys}}^4(\mu') \, \frac{d\mu'^2}{\mu'^2} .$$

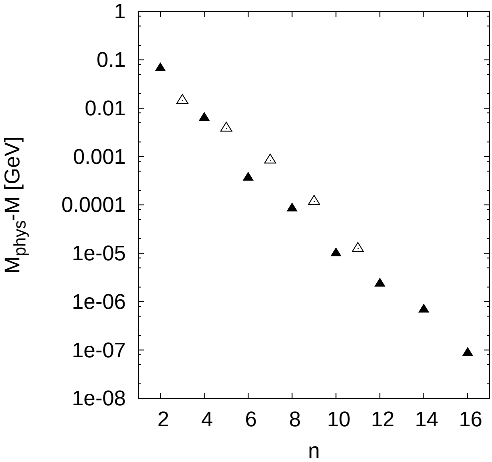
- Take now known result for pion mass, k_{14} and k_{16} calculable

Numerics



$$
\begin{aligned}
M &= 938\ \text{MeV}, \\
c_1 &= -0.87\ \text{GeV}^{-1} \\
c_2 &= 3.34\ \text{GeV}^{-1} \\
c_3 &= -5.25\ \text{GeV}^{-1} \\
g_A &= 1.25 \\
\mu &= 0.77\ \text{GeV} \\
F &= 92.4\ \text{MeV}
\end{aligned}
$$

Numerics



1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
M_phys-M [GeV]
2
4
6
8
10
12
14
16
n

Conclusions Leading Logarithms

- Leading logarithms can be calculated using only one-loop diagrams
- Results for a large number of quantities for mesons
- Look at the (very) many tables, we would be very interested in all-order conjectures
- Nucleonmass as the first result in the nucleon sector
- Work in progress for other nucleon quantities
- Exploratory steps done for gravity (Relefors)

Nonlinear sigma model

The nonlinear $U(N)$ sigma model recursion relations are derived in

- Kampf, Novotny, Trnka, 1212.5224
- Kampf, Novotny, Trnka, 1304.3048
- Cheung, Kampf, Novotny, Shen, Trnka, 1509.03309

Can this be generalized to the massive $U(N)$ nonlinear sigma model?

Underlying assumptions

- The amplitude can be written in an ordered form (colour-ordered, flavour-ordered,...)

$$\mathcal{M}_n^{a_1 a_1 \dots a_n} = \sum_{\sigma \in S_n/Z_n} \langle t^{a_{\sigma(1)}} \dots t^{a_{\sigma(n)}} \rangle \, \mathcal{M}_\sigma(p_1, \dots, p_n)$$

$$\mathcal{M}_\sigma(p_1, \dots, p_n) = \mathcal{M}(p_{\sigma(1)}, \dots, p_{\sigma(n)})$$

- The ordered amplitude consists of planar graphs
- Scalar particles (no spin)
- Interactions involve only an even number of particles
- All particles have the same mass

Consequences

- $\mathcal{M}(p_1, \ldots, p_n) = f(m^2, \{p_{ij} \cdot p_{kl}\})$
- $p_{ij} = \sum_{k=i,j} p_k, \qquad p_{ii} = p_i$
- The tree level amplitude is thus independent of the dimension
- Every internal propagator has momentum p_{ij} as a sum of an odd number of consecutive external momenta

Massive to massless

- p_i with $p_i^2 = m^2$, propagators are $1/(p_{ij}^2 - m^2)$ and $\sum_i p_i = 0$
- go to 5 dimensions
- set $q = (0,0,0,0,m)$ and $p_i = (p_i, 0)$ so $p_i^2 = m^2$, $q^2 = -m^2$ and $q \cdot p_i = 0$
- $q_{2i+1} = p_{2i+1} + q$ and $q_{2i} = p_{2i} - q$
- q_i with $q_i^2 = 0$, propagators are $1/q_{ij}^2$ and $\sum_i q_i = 0$
- Caveat 1: do not use simplifications in the amplitude in 5 dimensions since in the numerator when going back $q_{ij} \cdot q_{kl} \to p_{ij} \cdot p_{kl}$ since it came from $\partial_\mu \phi \partial^\mu \phi$ and in the denominator $q_{ij}^2 \to p_{ij}^2 - m^2$
- Caveat 2: It's not clear whether that is compatible with the recursion relations in general

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Massive $U(N)$ nonlinear sigma model

- The previous method allows to remove masses from propagators and external momenta
- What about mass-dependent interaction terms?
- Massive $U(N)$ nonlinear sigma model:

$$\mathcal{L} = \frac{F^2}{4}\left\langle \partial_\mu U \partial^\mu U^\dagger \right\rangle + \frac{F^2 M^2}{4}\left\langle U + U^\dagger \right\rangle$$

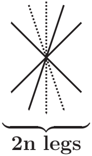
- Usual parametrization $U = \exp(i\Phi)$
- Choose: $U = 1 - \Phi^2/2 + i\Phi\sqrt{1 - \Phi^2/4}$ then the mass term has no interaction vertices
- $(\Phi = \sqrt{2}\phi/F)$

Conclusions

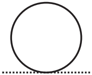
- Very short intro to EFT and ChPT
- Leading logarithms for a number of EFT known to a high number of loops
- First steps towards recursion relations for the massive $U(N)$ nonlinear sigma model

Large N

Power counting: pick $\mathcal{L}$ extensive in $N \Rightarrow F^2 \sim N$, $M^2 \sim 1$

- 

  $\Leftrightarrow F^{2-2n} \sim \frac{1}{N^{n-1}}$

  $\Leftrightarrow N$

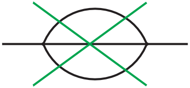
- 1PI diagrams:

$$\left.\begin{array}{l} N_L = N_I - \sum_n N_{2n} + 1 \\ 2N_I + N_E = \sum_n 2nN_{2n} \end{array}\right\} \Rightarrow N_L = \sum_n (n-1)N_{2n} - \frac{1}{2}N_E + 1$$

- diagram suppression factor: $\frac{N^{N_L}}{N^{N_E/2-1}}$

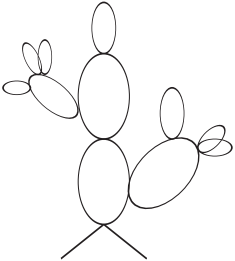
Large N

- diagrams with shared lines are suppressed



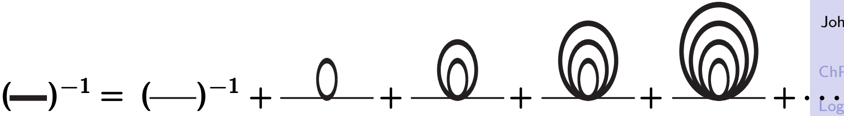
each new loop needs also a new flavour loop

- in the large N limit only "*cactus*" diagrams survive:



large N: propagator

Generate recursively via a Gap equation



$\Rightarrow$ resume the series and look for the pole

$$M^2 = M^2_{\text{phys}}\sqrt{1+\frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

$$\overline{A}(m^2) = \frac{m^2}{16\pi^2}\log\frac{\mu^2}{m^2}.$$

Solve recursively, agrees with other result

Note: can be done for all parametrizations

large N: Decay constant

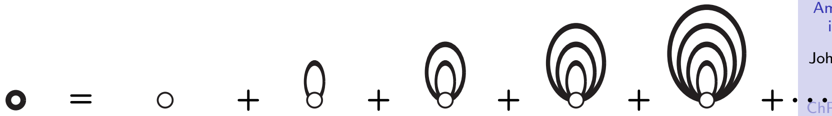
$\Rightarrow$ and include wave-function renormalization

$$F_{\mathrm{phys}} = F\sqrt{1+\frac{N}{F^2}\overline{A}(M^2_{\mathrm{phys}})}$$

Solve recursively, agrees with other result

Note: can be done for all parametrizations

large N: Vacuum Expectation Value



$$\langle \bar{q}q \rangle_{\mathrm{phys}} = \langle \bar{q}q \rangle_0 \sqrt{1 + \frac{N}{F^2}\overline{A}(M^2_{\mathrm{phys}})}$$

Comments:

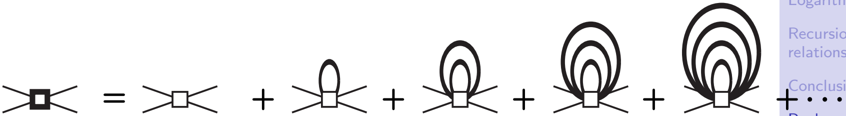
- These are the full* leading N results, not just leading log
- But depends on the choice of N-dependence of higher order coefficients
- Assumes higher LECs zero ($< N^{n+1}$ for $\hbar^n$)
- Large N as in $O(N)$ not large N_c

Large N: $\pi\pi$-scattering

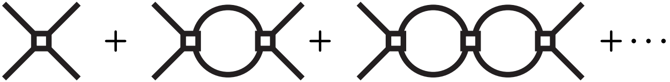
- Semiclassical methods Coleman, Jackiw, Politzer 1974
- Diagram resummation Dobado, Pelaez 1992
- $A(\phi^i\phi^j \to \phi^k\phi^l) =$
  $A(s,t,u)\delta^{ij}\delta^{kl} + A(t,u,s)\delta^{ik}\delta^{jl} + A(u,s,t)\delta^{il}\delta^{jk}$
- $A(s,t,u) = A(s,u,t)$
- Proof same as Weinberg's for $O(4)/O(3)$, group theory and crossing

Large N: $\pi\pi$-scattering

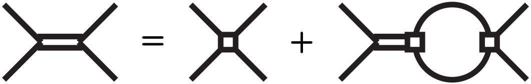
- Cactus diagrams for $A(s, t, u)$
- Branch with no momentum: resummed by ——
- Branch starting at vertex: resum by



- The full result is then



- Can be summarized by a recursive equation



Large N: $\pi\pi$ scattering

$y = \frac{N}{F^2}\overline{A}(M^2_{\rm phys})$

$$A(s,t,u) = \frac{\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}}{1 - \frac{1}{2}\left(\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}\right)\overline{B}(M^2_{\rm phys}, M^2_{\rm phys}, s)}$$

or

$$A(s,t,u) = \frac{\frac{s-M^2_{\rm phys}}{F_{\rm phys}}}{1 - \frac{1}{2}\frac{s-M^2_{\rm phys}}{F^2_{\rm phys}}\overline{B}(M^2_{\rm phys}, M^2_{\rm phys}, s)}$$

- $M^2 \to 0$ agrees with the known results
- Agrees with our 5-loop results