String amplitudes in the high and low energy limits

Based on:
L. Magnea, S. Playle, R. R., S. Sciuto 1503.05182
G. D'Appollonio, P. Di Vecchia, R. R., G. Veneziano 1502.01254
E. D'Hoker, M. Green, B. Pioline, R. R. 1405.6185

See also:
G. D'Appollonio, P. Di Vecchia, R. R., G. Veneziano 1310.1254
G. D'Appollonio, P. Di Vecchia, R. R., G. Veneziano 1008.4773

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A new scale

- String amplitudes depends on the external state quantum numbers (as in QFT), but also on the string scale $\sqrt{\alpha'}$
- Even in simple examples, they may be rather different than their QFT counterparts
- The tree-level 4-gluon amplitude in bosonic string theory contains an α'-corrected version of the usual structure

$$A(--++) = A_{YM}(--++)\frac{\Gamma(1+s_{12})\Gamma(1+s_{14})}{\Gamma(1+s_{12}+s_{14})}\left[1-2\frac{s_{13}s_{14}}{1-s_{12}}\right]$$

  where $s_{ij} = \alpha'(p_i+p_j)^2$
- Plus new contributions

$$A(-+++) \sim \alpha' \frac{[23][34][24]^2}{[12][14]}\frac{\Gamma(1+s_{12})\Gamma(1+s_{14})}{\Gamma(1+s_{12}+s_{14})}$$

  and also $A(++++)$ is non trivial and proportional to α'

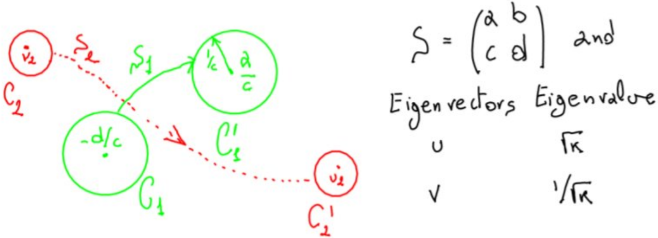
A Lego approach

- RNS strings are formulated in terms of super Riemann surfaces
- An example: $\mathbf{CP}^{1|1}$ can be parametrized by projective coordinates $(z_1, z_2|\theta) \sim (\lambda z_1, \lambda z_2|\lambda\theta)$
- The 3-punctured NS sphere contains a single fermionic modulus
- The 3-punctured R sphere has no moduli
- We can build super-Riemann surfaces by gluing these basic ingredients with a NS propagator (1 bosonic modulus) or a R propagator (1 bosonic + 1 fermionic modulus)
- The vacuum diagram at h loops is built with $v = 2(h-1)$ 3-punctured spheres and $e = 3(h-1)$ propagators
- Since the number of R propagators is always equal to the number v_R of R spheres, the dimension of the moduli space $\mathcal{M}_h$ is

$$(e|e_R + v_{NS}) = (3(h-1)|2(h-1))$$

The Schottky parametrization

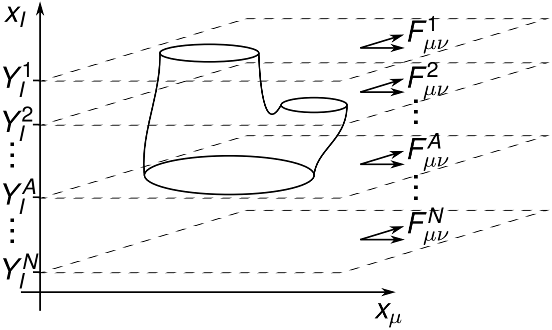
- Start from the sphere mapped to $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.
  A handle can be represented by a projective transformation $S \in SL(2, \mathbf{C})$ (generator) with two distinct eigenevectors
- This defines two isometric circles $\mathcal{C}$ and $\mathcal{C}'$ identified under S
- A genus g surface is described by g generators S_i with non-overlapping isometric circles



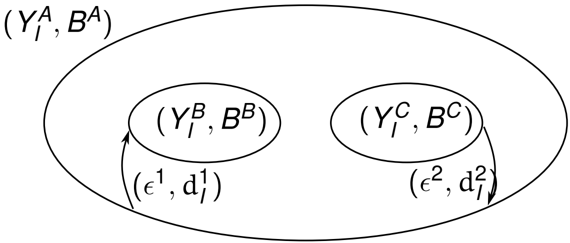
- The Schottky group is the set of all (finite) products of S_i and their inverses. Each element T_α is characterised by u_α, v_α and k_α.

The amplitude integrands: an open string example

- String amplitudes are written as integrals over the supermoduli
- In flat space we can think of X^M, ψ^M (for each M) and the ghosts (b, c), (β, γ) as separate sectors
- As a concrete example, consider three stacks of Dp-branes



- In the open string channel we have



- Schematically, the partition function reads

$$\mathbf{Z}_h(\vec{\epsilon}, \vec{\mathrm{d}}\,) = \mathcal{N}_h^{(\vec{\epsilon})} \int d\mu_h \, \mathbf{F}_{\mathrm{gh}}(\mu) \, \mathbf{F}_{\mathrm{scal}}^{(\vec{\mathrm{d}}\,)}(\mu) \, \mathbf{F}_{\parallel}^{(\vec{\epsilon})}(\mu) \, \mathbf{F}_{\perp}(\mu)$$

- In the NS sector, the **F**'s are known as infinite series for instance around the ⚭ degeneration

The low-energy limit

- At low energies, the string result should match the field theory effective action in a constant $F_{\mu\nu}$ background
- The $\alpha' \to 0$ limit is taken by keeping fixed all the QFT parameters

$$\mathrm{d}^i_I = Y_I^{A_0} - Y_I^{A_i}\,, \quad m_i^2 = \frac{1}{(2\pi\alpha')^2} \sum_{I=d}^{9} \left(\mathrm{d}^i_I\right)^2$$

$$\tan\left(\pi\epsilon^i\right) = 2\pi\alpha' \left(B^{A_0} - B^{A_i}\right)$$

- The bosonic gluing parameters p_i in $\mathcal{M}_2$ ($k[S_1 S_2^{-1}] = p_1 p_2$ and $k_i = p_i p_3$) are related to the QFT Schwinger parameters

$$\ln p_i = -\frac{t_i}{\alpha'}$$

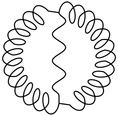
- In the $p_i \to 0$ expansion (with t_i fixed) the **F**'s reduce to few terms
- We can integrate over the Grassmann variables

A 1-2-1 map

- There is a 1-2-1 map between Feynman diagrams and terms in the p_i expansion of the string integrand
- This is a gauge dependent statement. The best guess is to use the Gervais-Neveu gauge

$$\mathcal{G}\,(\mathcal{A},\mathcal{Q}) \;=\; \mathfrak{D}_M \mathcal{Q}^M + \mathrm{i}\,\gamma\, g\, \mathcal{Q}_M \mathcal{Q}^M \;=\; 0$$

- For instance we have



$$= -\,(d-2)\,\frac{g^2}{(4\pi)^d}\int_0^\infty \frac{\prod_{i=1}^3 dt_i\, \mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1}\Delta_B}\,\frac{1}{\Delta_0}\left\{\left[2t_3 + \frac{1-\gamma^2}{2}(t_1+t_2)\right]\right.$$
$$\left.\times \cosh(2B_1 t_1 - 2B_2 t_2) + \text{cycl. perm.}\right\},$$

with $\Delta_B = \sinh[B_1 t_1]/B_1 \sinh[B_2 t_2]/B_2 \cosh[B_3 t_3] + \mathrm{cycl.\ perm.}$

- ...which is reproduced for $\gamma = 1$ by the first subleading terms of $\mathbf{F}_{\parallel}^{(\vec{\epsilon})}$ in p_i, p_j and $\mathbf{F}_{\perp}$ in p_k!

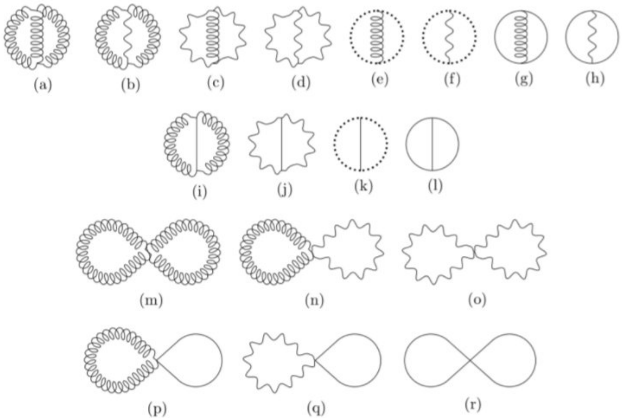


Figure 1: Two-loop 1PI vacuum Feynman graphs in Yang-Mills with adjoint scalars with VEVs. The dotted edges signify Faddeev-Popov ghosts, and the plain edges symbolize scalars, the helical edges denote gluons polarized *parallel* to the plane of the background magnetic field and the wavy edges indicate gluons polarized *perpendicular* to the background magnetic field.

The amplitude integrals: a closed string example

- We consider the 4-graviton amplitude in flat type II theories
- A compact 2-loop expression (after integrating the Grassmann moduli and summing the spin structures) D'Hoker and Phong

$$\int_{\mathcal{M}_{2,4}} \frac{d^2 z_i d^2\Omega_{ab}}{(\det \mathrm{Im}\Omega)^5} |\mathcal{Y}_{\mathcal{S}}|^2 \exp\left[-\frac{\alpha'}{2}\sum_{i<j} k_i k_j G(z_i, z_j)\right]$$

  where Ω is the period matrix, G the (bosonic) Green function and $\mathcal{Y}_{\mathcal{S}}(z_i) \sim k_j k_k$ is written in terms of Abelian differential
- At low energies ($\alpha' k_i k_j \to 0$)
  - The first term is $\sim (k_i k_j)^2$ and yields a 2-loop contribution to the $D^4 R^4$ term in the effective action
  - At order k^6 the integrand involves the Kawazumi-Zhang invariant
  - Susy & S-duality fix completely the 2-loop $D^6 R^4$ contribution!

A perturbative check

- To check the Susy & S-duality prediction we need

$$\mathcal{E}^{(2)}_{(0,1)} = \pi \int\limits_{\mathcal{M}_2} \frac{d^2\Omega_{ij}}{(\det \mathrm{Im}\Omega)^3} \phi(\Omega) = \frac{2\pi^2}{45}$$

  where ϕ is Kawazumi-Zhang invariant
- In the interior of $\mathcal{M}_2$ we have $(\Delta - 5)\phi = 0$
- This is suggested by the form of ϕ in the complete degeneration limit (derived as above) and after compactification by the $\mathcal{E}^{(2)}_{(0,1)}$ dependence on torus moduli
- We can apply Gauss' theorem and reduce the integral to a boundary contribution
- Only the separating degeneration boundary contributes and the integral reduces to the square of a genus one structure
- The equation at the top of the page is verified!

The high energy limit

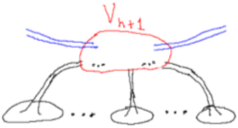
- It is very interesting also to consider the high energy regime. In particular much progress has been obtained in the Regge limit

  Amati, Ciafaloni and Veneziano

- Here we consider the $1 \to 1$ scattering of a closed string off a stack of Dp-branes ("fixed target" experiment)
- The Regge limit is $\alpha' s \gg 1$, $t/s \to 0$ with

$$s = E^2 = |k_\perp|^2 \,, \quad -t = (k_1 + k_2)^2 = \mathbf{q}^2 = 4E^2 \sin^2 \frac{\Theta}{2}$$

- The leading contribution is given by the "half-ladder" diagrams



The eikonal operator

- We can resum the leading contributions – more easily done in the impact parameter space (i.e. Fourier transform $e^{i\mathbf{qb}}$)
- In the leading Regge limit the S-matrix reads $e^{2i\hat{\delta}}$, with

$$2\hat{\delta}(s,\mathbf{b}) = \int_0^{2\pi} \frac{d\sigma}{2\pi} \int \frac{d^{24-p}\mathbf{q}}{(2\pi)^{24-p}} \frac{A_1(s,\mathbf{q})}{2E} : \mathrm{e}^{\mathrm{i}\mathbf{q}(\mathbf{b}+\hat{X}(\sigma))} :$$

$$A_1(s,\mathbf{q}) \sim \mathrm{e}^{-\mathrm{i}\pi\frac{\alpha' t}{4}} (\alpha' s)^{1+\frac{\alpha' t}{4}} \frac{\Gamma\left(-\frac{\alpha' t}{4}\right)}{1+\frac{\alpha' t}{4}}$$

- We can calculate any $1 \to 1$ transition amplitude simply by

$$\langle s'|e^{2i\hat{\delta}}|s\rangle \ ,$$

  where $|s\rangle$, $|s'\rangle$ are states in the light-cone gauge (fixed by the large components of k)
- The inelastic terms are due to tidal forces

Surprises without supersymmetry

- By approximating A_1 as

$$A_1(s,\mathbf{q}) \sim \mathrm{e}^{-\mathrm{i}\pi\frac{\alpha' t}{4}} (\alpha' s)^{1+\frac{\alpha' t}{4}} \frac{\Gamma\left(-\frac{\alpha' t}{4}\right)}{1+\frac{\alpha' t}{4}} \to -\frac{4s}{t}$$

  one obtains the results for a gravity theory with higher derivative terms (the scale of these corrections is $\sqrt{\alpha'} \to \ell$)

- In this case $\hat{\delta}$ is an operator even when restricted to the massless sector (this does not happen in the maximally susy case)
- Some eigenvectors of $\hat{\delta}$ develop a negative eigenvalue for $b \sim \ell$
- This yields a negative Shapiro's time delay (causality violations?)

  Camanho, Edelstein, Maldacena, Zhiboedov

- In string theory the Regge behavior (i.e. $s^{\alpha' t/4}$) avoids this issue

Conclusions

- The common theme of these examples: the degenerations of the full string amplitude often capture all relevant information for a particular physical process
- This is expected for the case of the low-energy limit (though explicit checks are non-trivial!) but applies also to other situations
  - $D^6 R^4$ correction at 2-loop
  - Regge high energy limit for closed strings
- Many interesting directions
  - Include the fermion loops in the open string effective action **Z**
  - Study the higher loop string amplitudes on T^d (more info on the low energy couplings such as $D^6 R^4$)
  - Subleading terms in the Regge limit
- Many connections to mathematically interesting problems