A brief introduction of SPH

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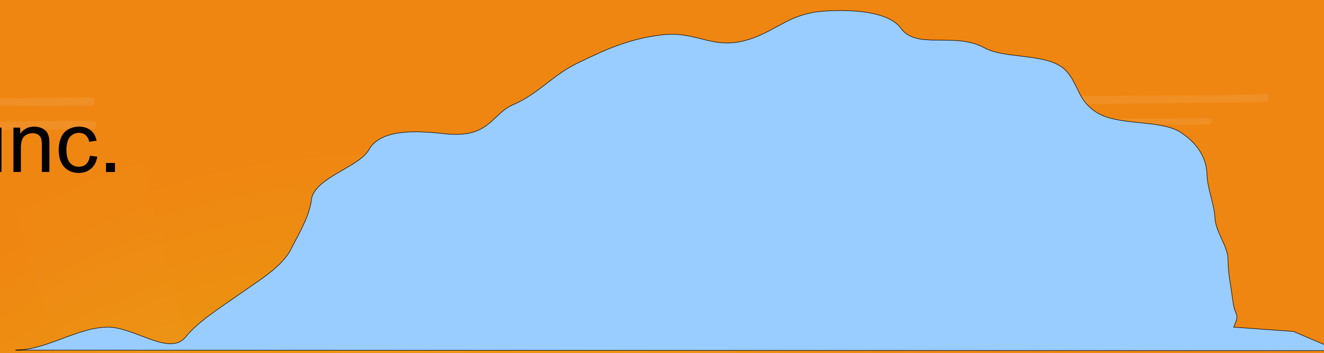
IATE - Conicet

Basics of SPH

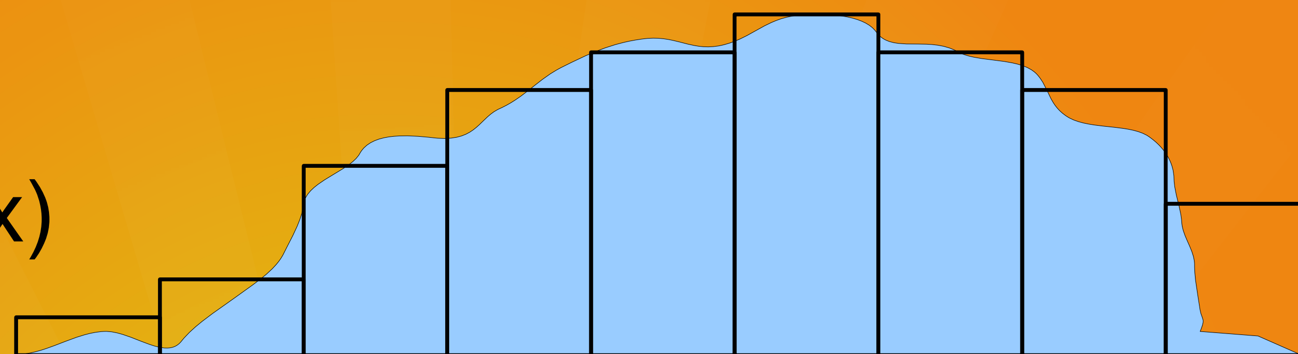
Origins.....

Continous func.



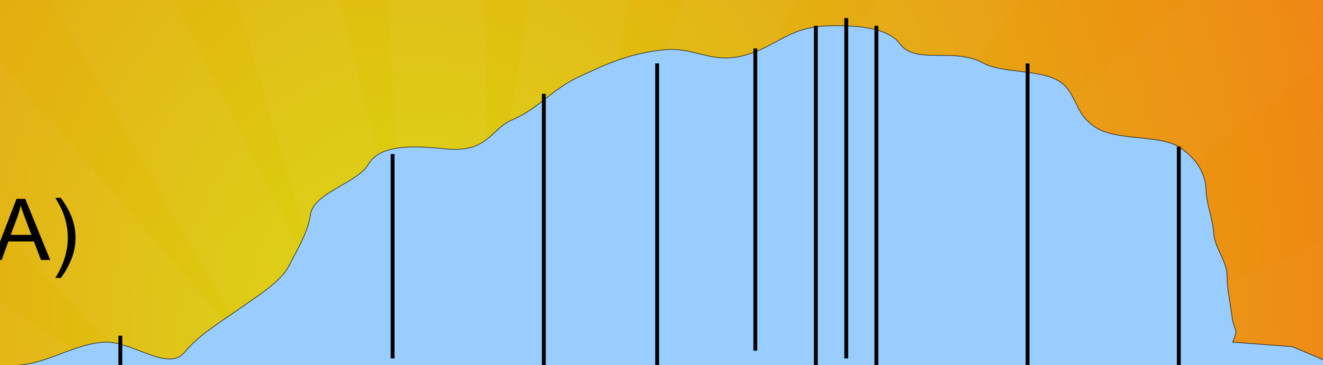
Space?:

Fix dx $\longrightarrow$ f(dx)



A?

Fix A $\longrightarrow$ dx(A)

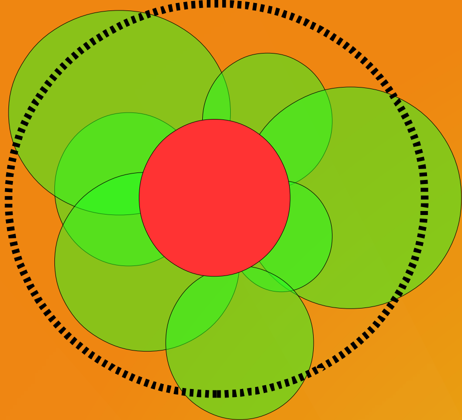


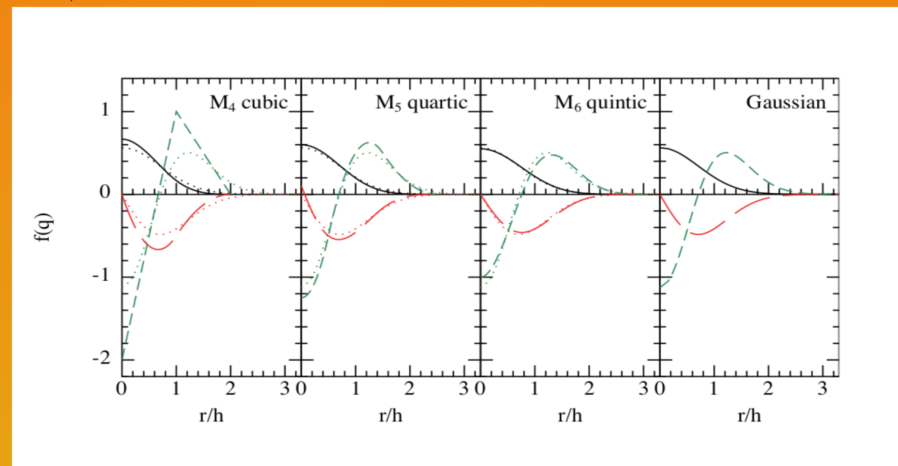
Basics.....

$$f(r)=\int f(r')\delta(r')dr'$$

$$\int_V W(\delta r, h(r))\,dV'=1$$

$$h(\rho)=\eta\left(\frac{m}{\rho}\right)^{1/d}$$





Kernel definition is important:

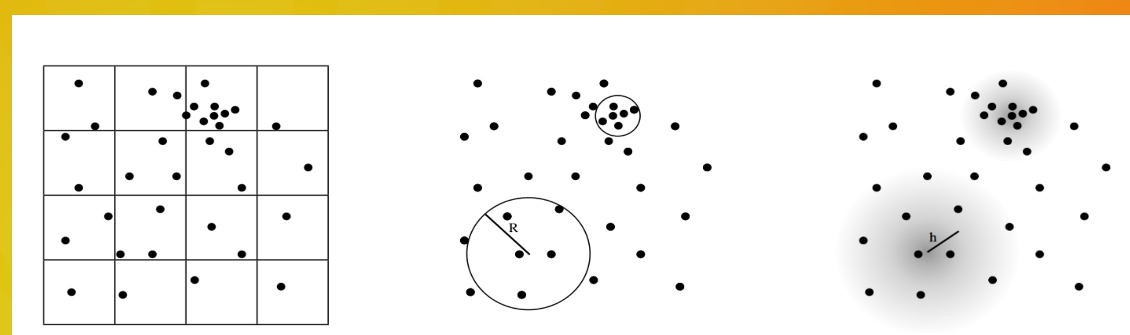
- -Gaussian
- -Cubic
- -Quintic
- - etc...

$$f(r)=\int f(r')W(\Delta r', h(\rho))\,dV'$$

$$f(r)=\int \frac{f(r')}{\rho(r')}W(\Delta r', h(\rho))\rho(r')\,dV'$$

Example: $\rho(r)=\sum_{j=1}^{N} m_j W(\delta r, h(\rho))$

$$f(r)=\sum_j^N \frac{f_b m_j}{\rho_b} W(\Delta r', h(\rho))$$

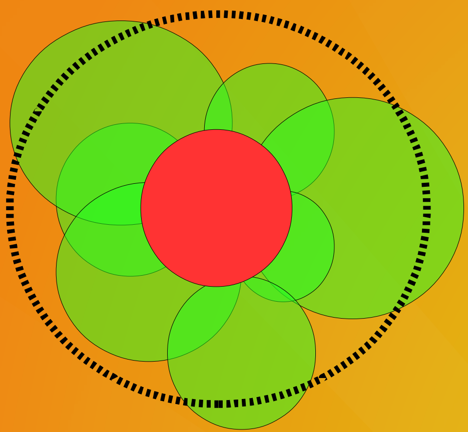


Why this is nice ???.....

$$\nabla f(r)=\sum_{i}^{N}\frac{f_b m_i}{\rho_b}\nabla W(\Delta r', h(\rho))$$

$$\nabla^2 f(r)=2\sum_{i}^{N}(f(r)-f_i)\frac{m_i}{\rho_b}w(r,i,h)$$

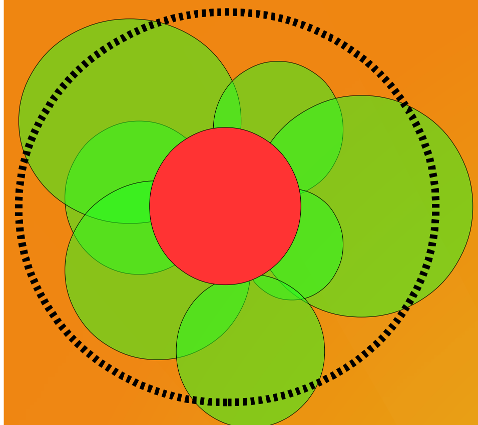
We can write the function derivatives as derivatives of the Kernel!



$$\nabla\cdot f(r)=\sum_{i}^{N}\frac{f_b m_i}{\rho_b}\cdot\nabla W(\Delta r', h(\rho))$$

$$\nabla\times f(r)=\sum_{i}^{N}\frac{f_b m_i}{\rho_b}\times\nabla W(\Delta r', h(\rho))$$

However, life is not easy...
The derivatives, in this straight forward way, are not the best choice.
Therefore there are a lot of games in the literature of writing the SPH eq.
And adding the physics that we need...



$$L_{mhd} = \int \left[\frac{1}{2}\rho v^2 - \rho u - \frac{B^2}{2\mu_0}\right] dV$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \vec{v}}\right) - \frac{\delta L}{\delta r} = 0$$

$$\frac{d\vec{v}_i}{dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \Pi_{\alpha\beta}\right) \vec{v}_{ij} \nabla_i W_{ij}$$

$$P = (\gamma - 1) u \rho$$

$$\frac{dv_i^\alpha}{dt} = \sum_j m_j \left[\frac{M_i^{\alpha\beta}}{\rho_i^2} \cdot \nabla_i^\beta W_{ij} + \frac{M_j^{\alpha\beta}}{\rho_j^2} \cdot \nabla_j^\beta W_{ij}\right]$$

$$\frac{du_i}{dt} = \sum_j m_j \frac{P_i}{\rho_i^2} \vec{v}_{ij} \cdot \nabla_i W_{ij}$$

$$\frac{d}{dt} = \frac{\delta}{\delta t} + \vec{v} \cdot \nabla$$

$$\rho_i = \sum_j m_j W_i$$

Reference: Price D.(PhD thesis)

Now..... what about MHD???

$$\frac{d}{dt}=\frac{\delta}{\delta t}+\vec{v}\cdot\nabla$$

MHD Eq.

$$\frac{d\rho}{dt}=-\rho(\nabla\cdot\vec{v})$$

$$\frac{d\vec{v}}{dt}=\frac{\nabla\cdot M}{\rho}=\frac{-\nabla P}{\rho}+\frac{(\nabla\times\vec{B})\times\vec{B}}{\mu_0\rho}$$

$$\frac{du}{dt}=\frac{-P}{\rho}(\nabla\cdot\vec{v})$$

$$\frac{d\vec{B}}{dt}=(\vec{B}\cdot\nabla)\vec{v}-\vec{B}(\nabla\cdot\vec{v})$$

$$M=-(P+\frac{B^2}{2\mu_0})I+\frac{\vec{B}\vec{B}}{\mu_0}$$

SPH form

$$\rho(r)=\sum_{i=1}^{N} m_i W_i$$

$$\frac{dv_i^\alpha}{dt}=\sum_j m_j[\frac{M_i^{\alpha\beta}}{\rho_i^2}+\frac{M_j^{\alpha\beta}}{\rho_j^2}+\Pi^{\alpha\beta}]\cdot\nabla_j^\beta W_{ij}$$

$$\frac{du_i}{dt}=\sum_j m_j[\frac{P_i}{\rho_i^2}v_j\cdot\nabla_i W_{ij}+\frac{P_j}{\rho_j^2}v_j\cdot\nabla_j W_{ij}]$$

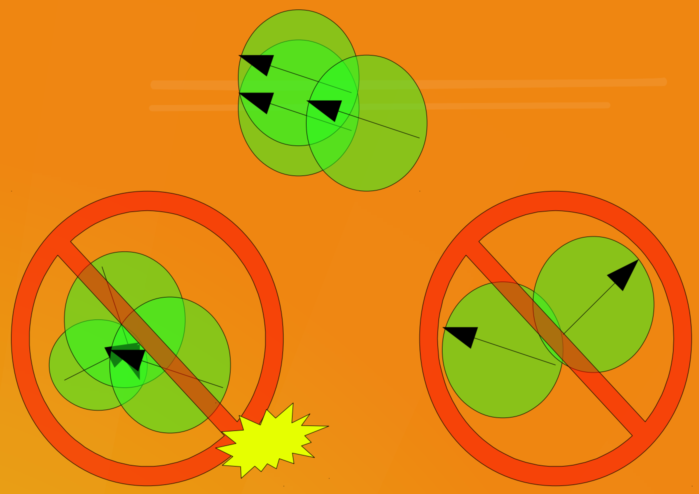
$$\frac{dB_i}{dt}=\frac{1}{\rho_i}\sum_j m_j[B_i(v_{ij}\cdot\nabla_i W_i)+v_{ij}(B_i\cdot\nabla_i W_i)]$$

$$M^{\alpha\beta}=-(P+\frac{B^2}{2\mu_0})\delta^{\alpha\beta}+\frac{B^\alpha B^\beta}{\mu_0}$$

$$P=(\gamma-1)u\rho$$

Ref: Price D. (2010), Springel, Rosswog (2009), Borve
Videos: Price D. (2009, Krakow), Springel (2009, Cambrigde)

- Winning Recipes against instabilities:
    - DivB Subtraction
    - Cleaning Schemes
    - Smoothing of the Field
    - Art. Dissipation
  - Euler Potentials
  - Vector Potential

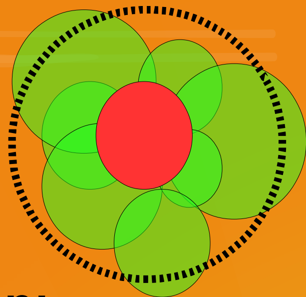


$$\frac{\delta B}{\delta t} = \nabla \times V \times B = V(\nabla \cdot B) - V(\nabla \cdot V) + (B \cdot \nabla) V - (V \cdot \nabla) B$$

DivB substraction:

$$\frac{d\vec{V}}{dt} = \frac{\nabla \cdot M}{\rho} = \frac{-\nabla P}{\rho} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{\mu_0 \rho} + \frac{\vec{B} \nabla \cdot \vec{B}}{\mu_0 \rho}$$

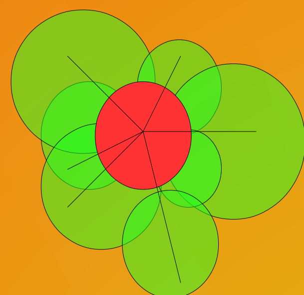
Smoothing:



$$\langle \vec{B} \rangle = \frac{\sum_i \vec{B}_i \frac{m_i}{\rho_b} W_i}{\sum_i \frac{m_i}{\rho_b} W_i}$$

Just Averaging

Art. Dissipation:

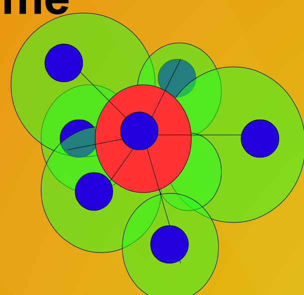


$$\left.\frac{d\vec{B}_i}{dt}\right|_{diss} = \eta_{art} \nabla^2 B_i$$

$$\eta_{art} = \alpha_B c_s |r_{ij}|$$

Diffuse gradients.
Eta "local"

Cleaning Scheme (Dedner):



$$\left.\frac{d\vec{B}_i}{dt}\right|_{Dedner} = -\nabla \phi$$

$$\frac{d\phi}{dt} = -c_s^2 \nabla \cdot B - \frac{\phi}{\tau} - \frac{\phi}{2} (\nabla \cdot \vec{v})$$

Clean Scheme local.
Error as a source,
Propagated and
Damped

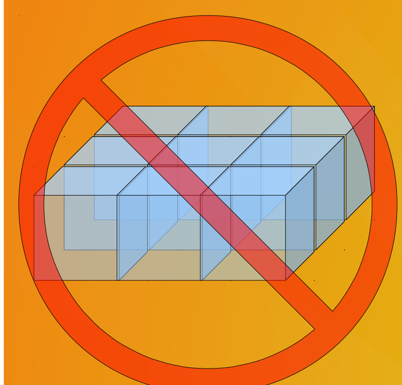
Euler Potentials:

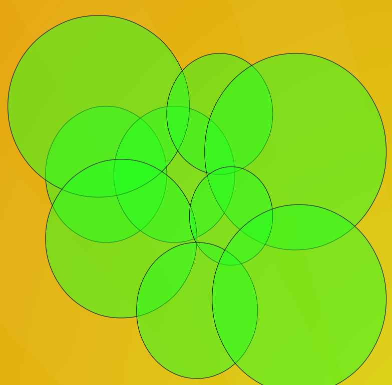
$$\vec{B}_i = \nabla \alpha \times \nabla \beta \qquad \frac{d\alpha}{dt} = \frac{d\beta}{dt} = 0$$

Divergenceless by Definition.

– Smoothed Particles Hydrodinamics:

- Natural Adaptativity and Huge Dynamical Range
- Gravity Coupling (Tree)
- Scalability
- Momentum Conservation
- Invariance
- ...





Highlight uses:

– Cosmology
– Galactic formation
– Star formation

Stability of the SPMHD codes are been reached a production stage (in general with constrains). To achieve this the necessary recipes are:

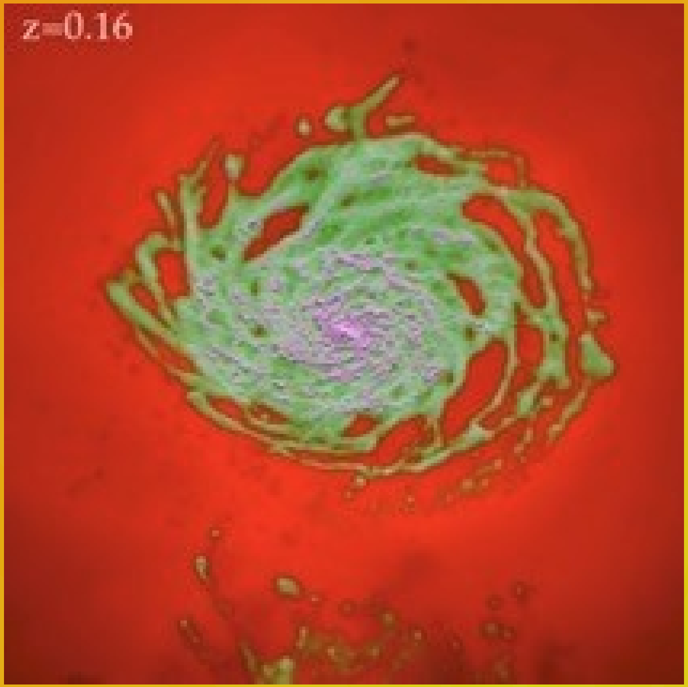
- Subtraction of DivB terms in the force
- Regularization of the field (always)
- Artificial Viscosity
- Thermal Conductivity

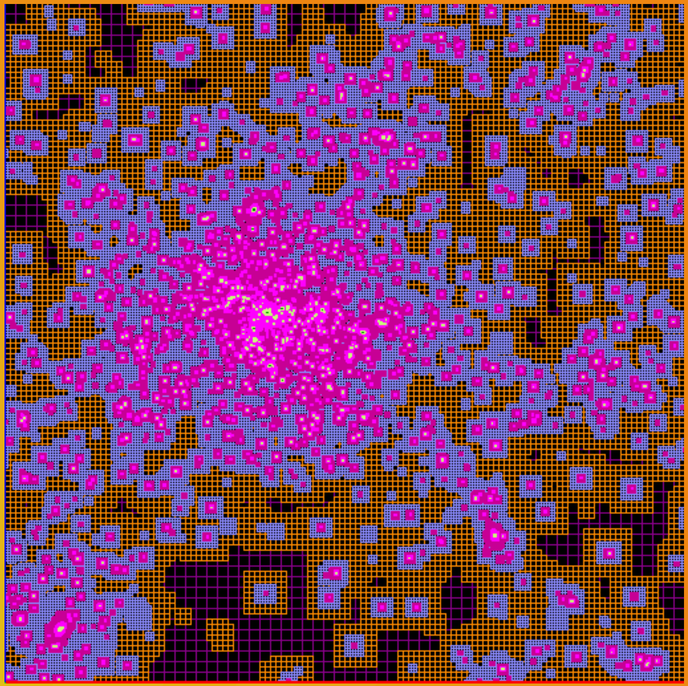
We are able to test several implementations, with different features (which does not to have to be forgotten). For example: Pressure SPH or a DivB cleaning scheme.

SPH vs GRID

Fight!

GRID:
- RAMSES
- FLASH
- ENZO
- AMIGA
- ZEUS
- PENCIL
- ….





SPH:
- GADGET
- GASOLINE
- GIZMO
- MAGMA
- PHANTOM
- ….

SPH vs GRID

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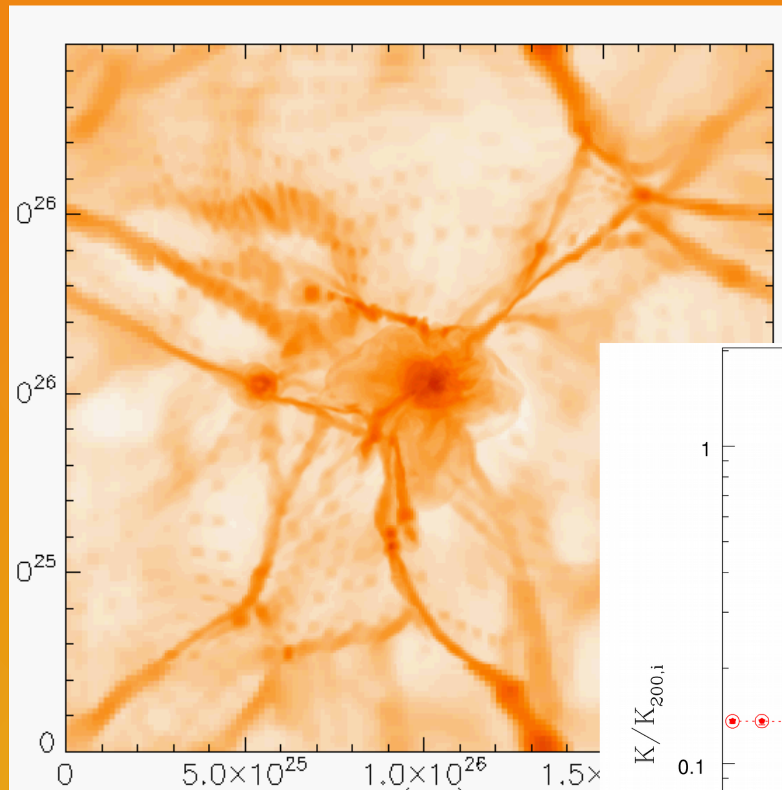

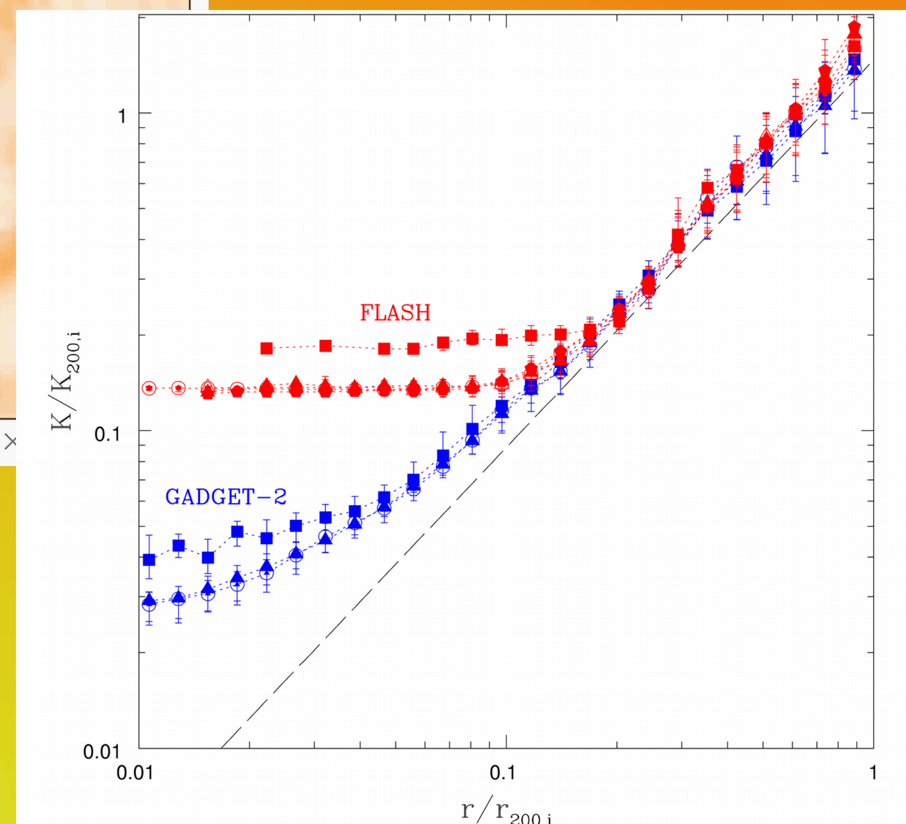

SPH:
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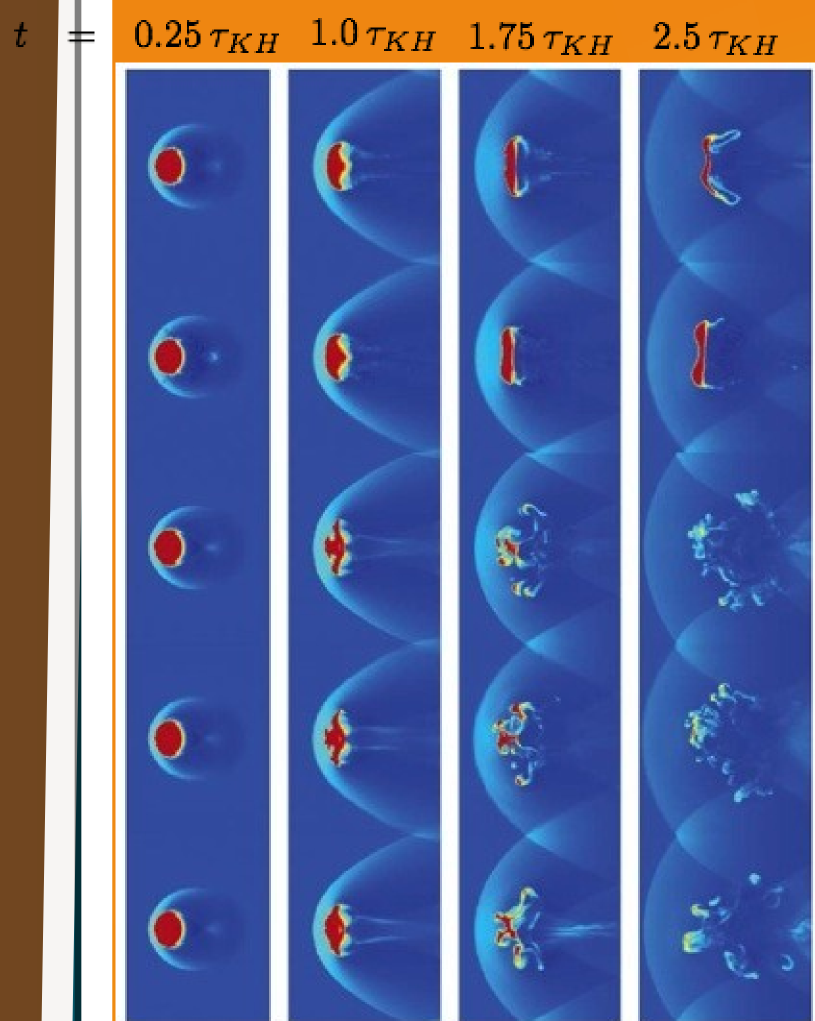
Agertz et al. (2007)

SPH vs GRID

Fight!



Gasoline (SPH)

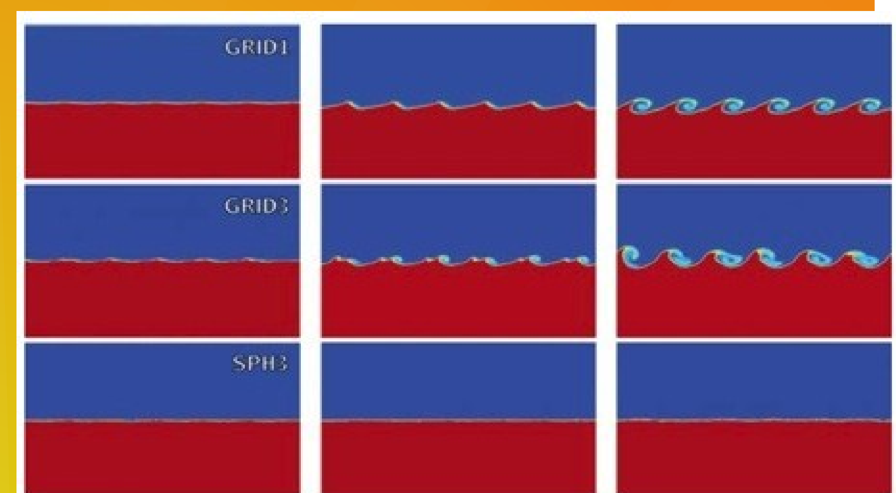
Gadget-2 (SPH)

Enzo (Grid)

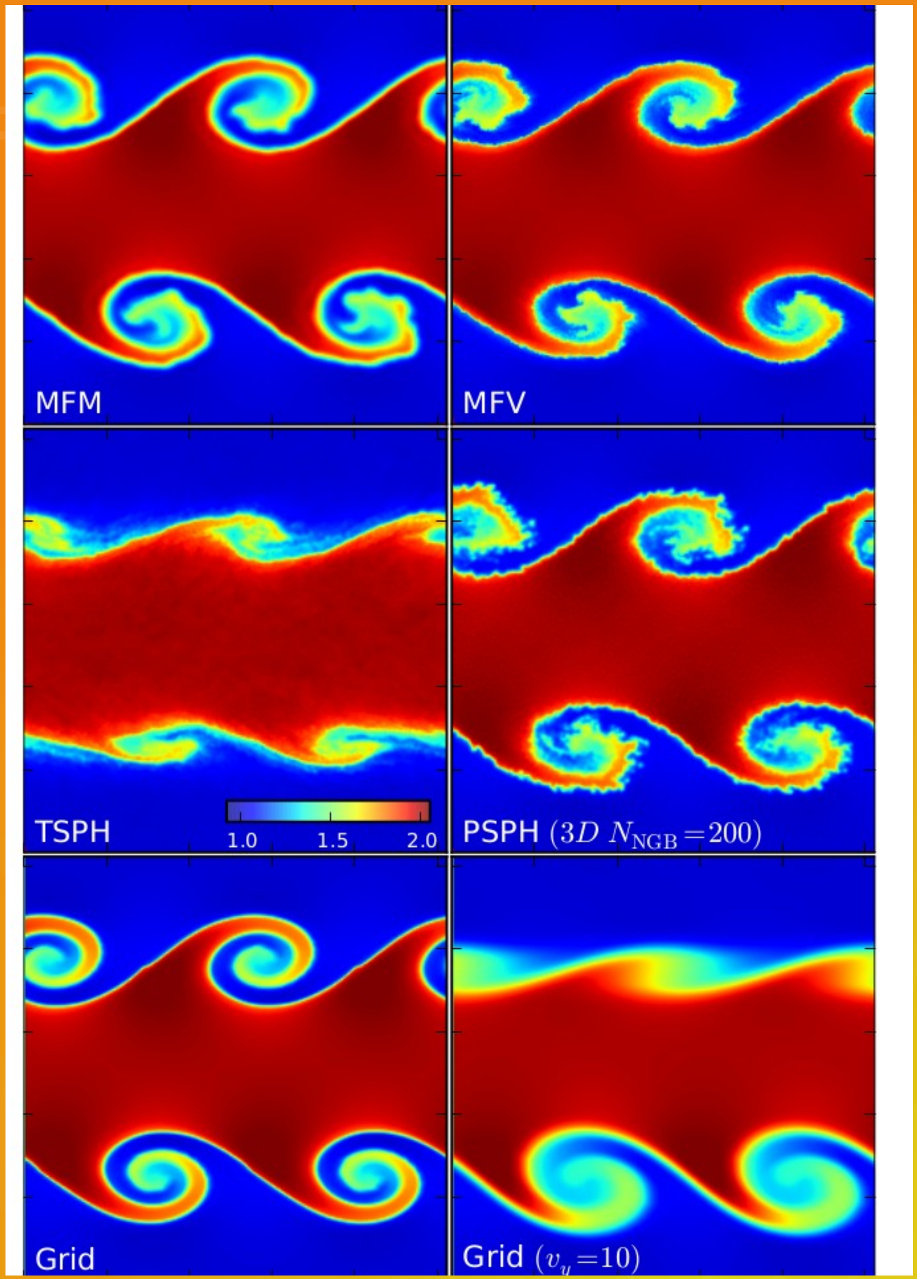
Flash (Grid)

Art-hydro (Grid)

T=1/3 2/3 1 t_hk



Agertz et al (2007)

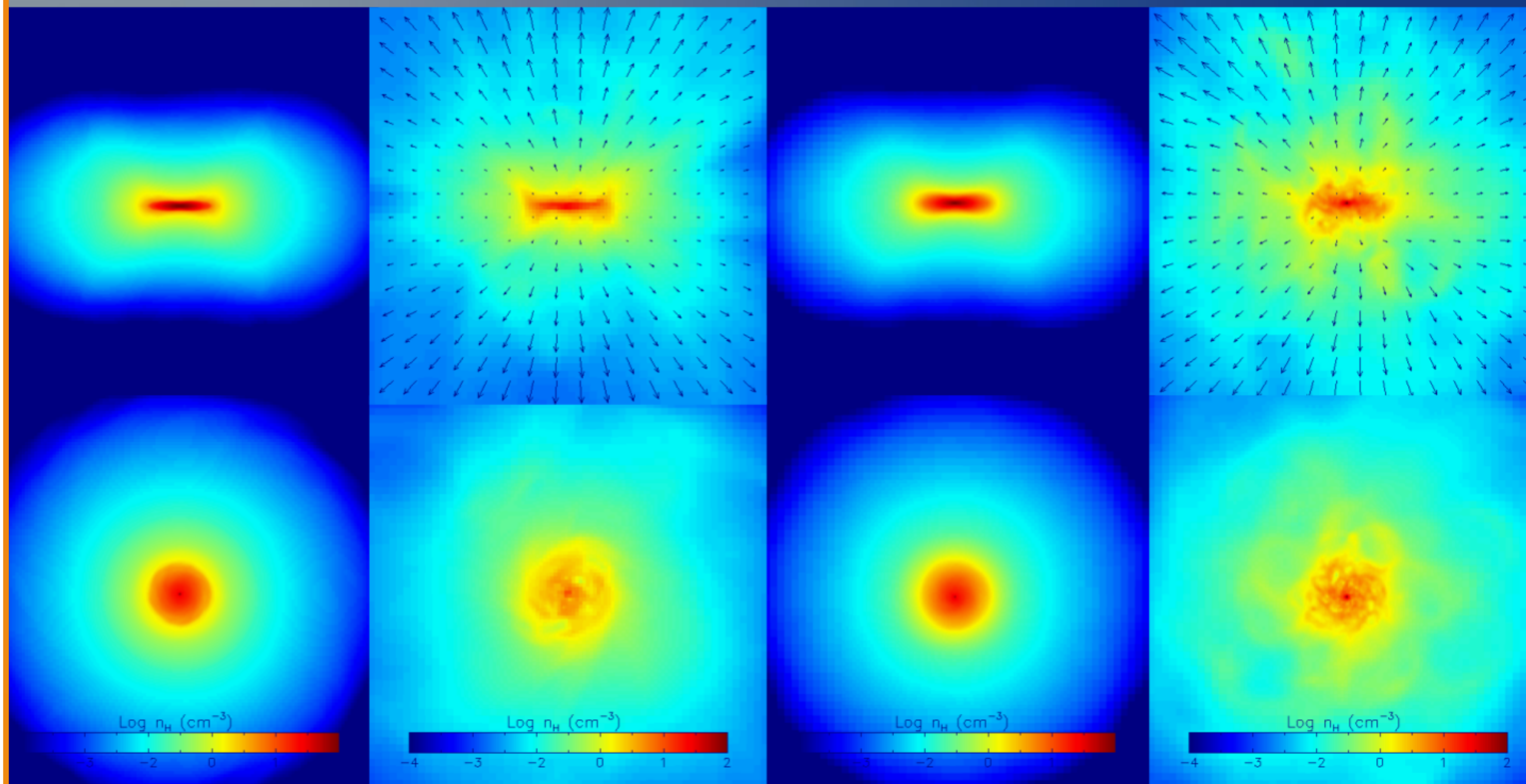


But, this issues have been solved changing the algorithms. From 2014 towards…

So finnally, the real problem is which sub-grid physics you are gonna use….

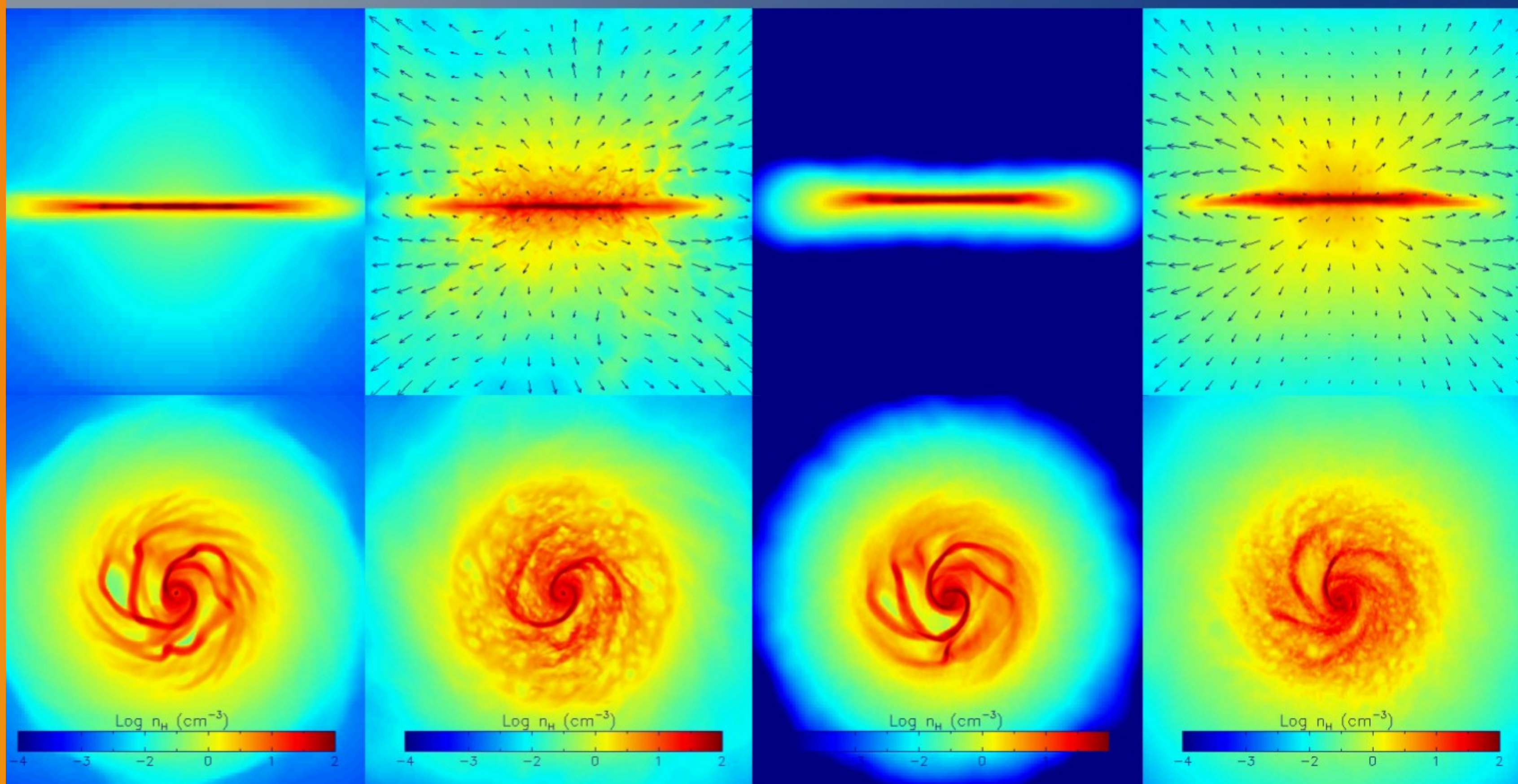
10^{10} Msolar Disk Galaxy

FLASH – No Feedback	FLASH – With Feedback	Gadget – No Feedback	Gadget – With Feedback



10^{12} Msolar Disk Galaxy

FLASH – No Feedback	FLASH – With Feedback	Gadget – No Feedback	Gadget – With Feedback



Mitchell (2013)

Here a story Begins...

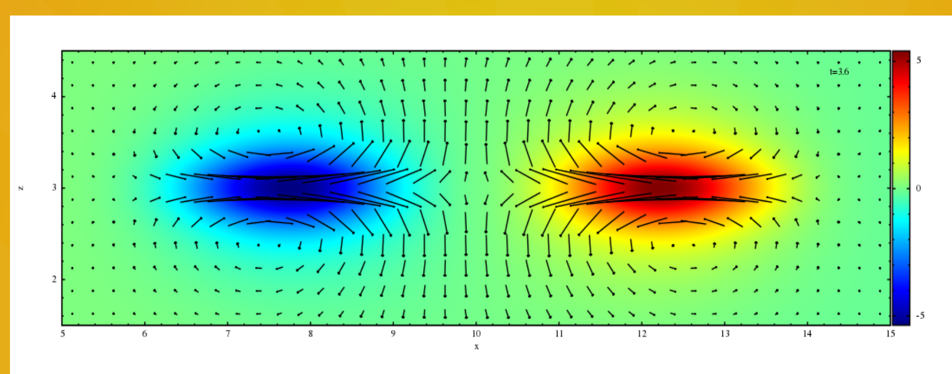
While I was studying dynamos in Postdam, I need to change the code to use the Vector Potential

$$\vec{B} = \nabla \times \vec{A}$$

$$\frac{\delta \vec{A}}{\delta t} = \vec{v} \times \vec{B} + \nabla \phi$$

$$\frac{d\phi}{dt} = -\left(c_h^2 \nabla \cdot \vec{A} + c_h \frac{\phi}{h}\right)$$

Pseudo-Lorenz Gauge



Roberts Flow Dynamo

- Definition:

$$\vec{v} = U_0 [\sin(ky)\cos(kx)\hat{x}, \sin(kx)\cos(ky)\hat{y}, \frac{1}{\sqrt{(2)}}\cos(ky)\cos(kx)\hat{z}]$$

Colors: Vz

2.5 The Roberts Cell Dynamo 73

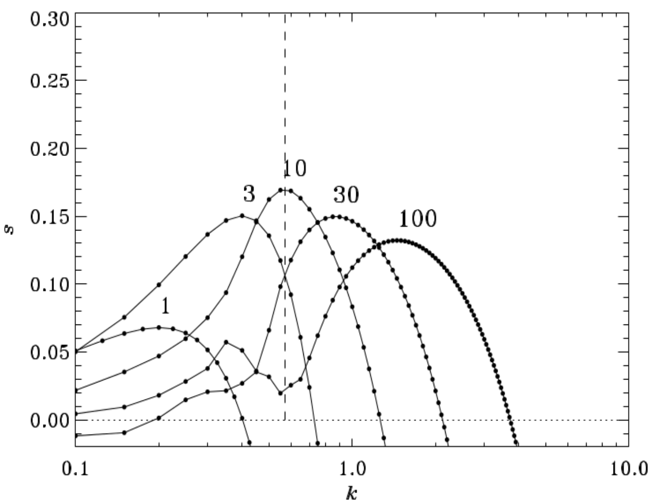
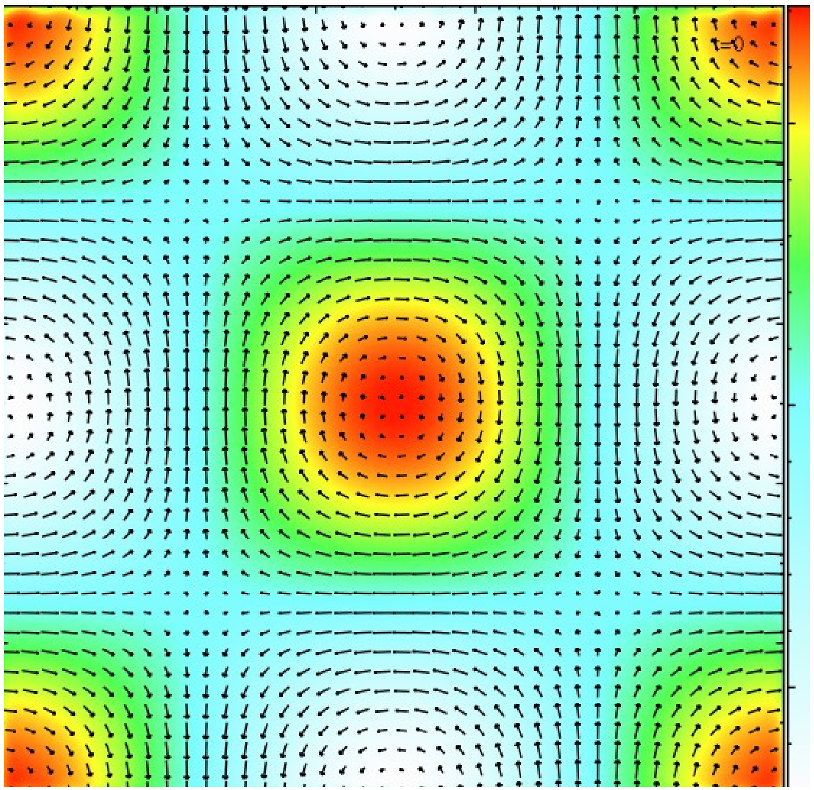


Fig. 2.16 Growth rates of the magnetic energy in the Roberts cell, for sequences of solutions with increasing k and various values of R_m, as labeled near the maxima of the various curves. Growth typically occurs for a restricted range in k, and peaks at a value k_{max} that increases slowly with increasing R_m. Note however how the corresponding maximum growth rate decreases with increasing R_m. The small "dip" left of the main peaks for the high-R_m solutions is a real feature, although here it is not very well resolved in k.

Charboneau (2012)

$$R_m = \frac{U_0}{\eta k}$$



Roberts Flow

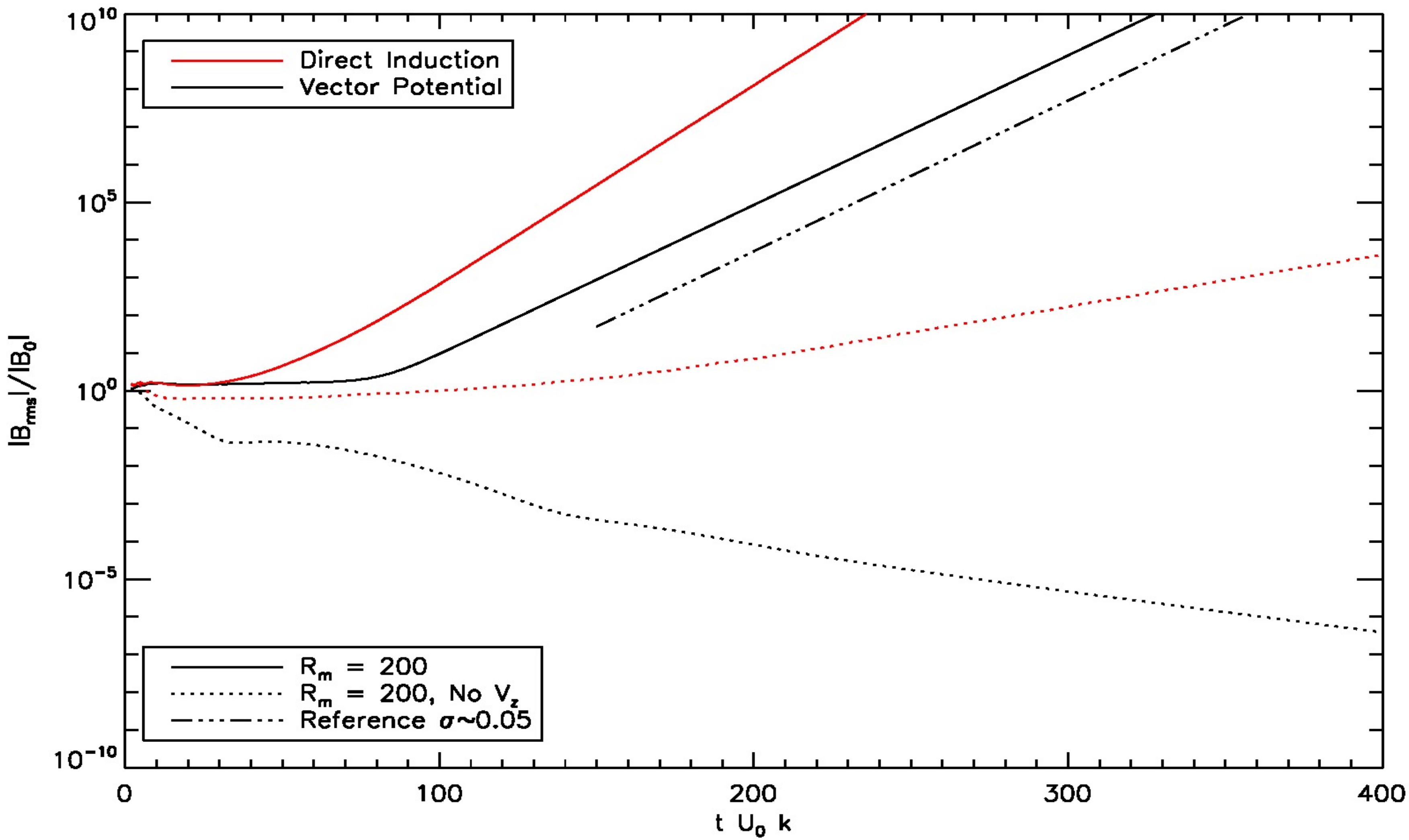


Table 1. Summary of Some Popular Numerical Hydrodynamics Methods

Method Name	Consistency /Order	Conservative? (Mass/Energy /Momentum)	Conserves Angular Momentum	Numerical Dissipation	Long-Time Integration Stability?	Number of Neighbors	Known Difficulties
Smoothed-Particle Hydro. (SPH)							
"Traditional" SPH (GADGET, TSPH)	0	✓	up to AV	artificial viscosity (AV)	✓	~ 32	fluid mixing, noise, E0 errors
"Modern" SPH (P-SPH, SPHS, PHANTOM, SPHGal)	0	✓	up to AV	AV+conduction +switches	✓	$\sim 128 - 442$	excess diffusion, E0 errors
"Corrected" SPH (rpSPH, Integral-SPH, Morris96 SPH, Moving-Least-Squares SPH)	0-1	×	×	artificial viscosity	×	~ 32	errors grow non-linearly, "self-acceleration"
"Godunov" SPH (GSPH, GSPH-I02, Cha03 SPH)	0	✓	up to gradient errors	Riemann solver + slope-limiter	✓	~ 300	instability, expense, E0 errors remain
Finite-Difference Methods							
Gridded/Lattice Finite Difference (ZEUS [some versions], Pencil code)	2-3	×	×	artificial viscosity	×	$\sim 8 - 128$	instability, lack of conservation, advection errors
Lagrangian Finite Difference (PHURBAS, FPM)						~ 60	
Finite-Volume Godunov Methods							
Static Grids (ATHENA, PLUTO)	2-3	✓	×	Riemann solver + slope-limiter	✓	~ 8 (geometric) $\sim 8 - 125$ (stencil)	over-mixing, ang. mom., velocity-dependent errors (VDE)
Adaptive-Mesh Refinement (AMR) (ENZO, RAMSES, FLASH)	2-3 (1)	✓	×	Riemann solver + slope-limiter	✓	$\sim 8 - 48$ $\sim 24 - 216$	over-mixing, ang. mom., VDE, refinement criteria
Moving-Mesh Methods (AREPO, TESS, FVMHD3D)	2	✓	×	Riemann solver + slope-limiter	✓	$\sim 13 - 30$	mesh deformation, ang. mom. (?), "mesh noise"
New Methods In This Paper							
Meshless Finite-Mass & Meshless Finite-Volume (MFM, MFV)	2	✓	up to gradient errors	Riemann solver + slope-limiter	✓	~ 32	partition noise ? (TBD)

A crude description of various numerical methods which are referenced throughout the text. Note that this list is necessarily incomplete, and specific sub-versions of many codes listed have been developed which do not match the exact descriptions listed. They are only meant to broadly categorize methods and outline certain basic properties.
(1) Method Name: Methods are grouped into broad categories. For each we give more specific sub-categories, with a few examples of commonly-used codes this category is intended to describe.
(2) Order: Order of consistency of the method, for smooth flows (zero means the method cannot reproduce a constant). "Corrected" SPH is first-order in the pressure force equation, but zeroth-order otherwise. Those with 2-3 listed depend on whether PPM methods are used for reconstruction (they are not 3rd order in all respects). Note that all the high-order methods become 1st-order at discontinuities (this includes refinement boundaries in AMR).
(3) Conservative: States whether the method manifestly conserves mass, energy, and linear momentum (✓), or is only conservative up to integration accuracy (×).
(4) Angular Momentum: Describes the *local* angular momentum (AM) conservation properties, *when the AM vector is unknown or not fixed in the simulation*. In this regime, no method which is numerically stable exactly conserves local AM (even if *global* AM is conserved). Either the method has no AM conservation (×), or conserves AM up to certain errors, such as the artificial viscosity and gradient errors in SPH. If the AM vector is known and fixed (e.g. for test masses around a single non-moving point-mass), it is always possible to construct a method (using cylindrical coordinates, explicitly advecting AM, etc.) which perfectly conserves it.
(5) Numerical Dissipation: Source of numerical dissipation in e.g. shocks. Either this comes from an up-wind/Riemann solver type scheme (where diffusion comes primarily from the slope-limiting scheme; Toro et al. 2009), or artificial viscosity/conductivity/hyperdiffusion terms.
(6) Integration Stability: States whether the method has long-term integration stability (i.e. errors do not grow unstably).
(7) Number of Neighbors: Typical number of neighbors between which hydrodynamic interactions must be computed. For meshless methods this is the number in the kernel. For mesh methods this can be either the number of faces (geometric) when a low-order method is used or a larger number representing the stencil for higher-order methods.
(8) Known Difficulties: Short summary of some known problems/errors common to the method. An incomplete and non-representative list! These are described in actual detail in the text. "Velocity-dependence" (as well as comments about noise and lack of conservation) here refers to the property of the *errors*, not the converged solutions. Any well-behaved code is conservative (of mass/energy/momentum/angular momentum), Galilean-invariant, noise-free, and captures the correct level of fluid mixing instabilities in the fully-converged (infinite-resolution) limit.

Doing simulations in nowadays common and even a requirement in some cases. However, the interpretation of those results are in general not rigorous enough.

The number of parameters involved, models and sub-grid recipes, require an extreme detailed analysis.