Topological solitons in confined systems

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INTRODUCTION: BELAVIN-POLYAKOV SOLITONS

The two-component fields are represented by vector

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \tag{1}$$

The components $\psi^{1,2}$ may be written as $\psi^{1,2} = \sqrt(N^{1,2}) z^{1,2}$, where $N^{1,2} = \psi^{1,2\dagger}\psi^{1,2}$and $z^{1,2} = z^{1,2}(x, y, t)$ are complex numbers. In what follows we consider $N^{1,2} = N = \text{const}$. Then we have $\psi = \sqrt{N}\zeta$, where

$$\zeta = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \tag{2}$$

with the normalization condition $z^1 z^1 + z^2 z^2 = 1$.

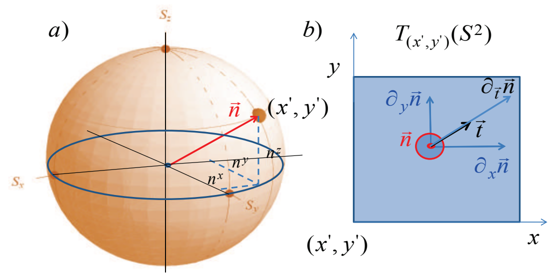

FIG. 1: a) Poincaré sphere with $\vec{n} = (n_x, n_y, n_z)$ in the point (x', y') representing the quantities defined in eq. (3); b) Tangent plane $T_{(x',y')}S^2$ from which the condition eq. (8) between a vector $\vec{n}$ and their derivatives may be read off.

We next make transformation

$$n^{s_i} = n^{s_i}(x, y) = \zeta^\dagger \tau^{s_i} \zeta, \tag{3}$$

where τ_{s_i}, with $s_i = x, y, z$, are the usual Pauli matrices. For the components of the vector $\vec{n} = (n^x, n^y, n^z)$ hold $\sum_{s_i}(n^{s_i})^2 = 1$. The transformation is a mapping from a complex plane to a 2-sphere, that is, $\vec{n} : \mathbb{R}^2 \to S^2$. The sphere described by the tip of the vector $\vec{n}$ can be identified with the Poincaré (or Bloch) sphere. The points in the Poincaré sphere may be interpreted as pseudo-spins [8]. Figure 1 describes the above statements.

Now, we follow the discussion in [9] and take following boundary condition

$$\lim_{\rho\to\infty} n(x, y) = (0, 0, 1). \tag{4}$$

for $\rho = \sqrt{x^2 + y^2}$. Direct consequence of (4) is possibility of compactification from $\mathbb{R}^2$ to S^2 (we consider only one chart of the atlas)

$$\vec{n}_i : S^2 \to S^2. \tag{5}$$

All mappings defined by 5 form configuration space: $Q = \{\vec{n}_i\}$. As homotopy group of configuration space $\pi_1(Q)$ is isomorphic to homotophy group $\pi_2(S^2)$

$$\pi_1(Q) = \pi_2(S^2) = \mathbb{Z} \tag{6}$$

it is itself union of homotopy classes Q_{N_i}

$$Q = \bigcup_{N_i} Q_{N_i}. \tag{7}$$

A class is determined by $N_i \in \mathbb{Z}$ which is degree of the mapping $\vec{n}_i$: $N_i = \deg(\vec{n}_i)$.

At a fixed point on the Poincaré sphere (x', y') ([1]) (see figure 1b and expression (9))

$$\partial_\alpha n^{s1}(x, y) = \varepsilon_{\alpha\beta}\varepsilon_{s1s2s3} n^{s2}(x, y)\partial_\beta n^{s3}(x, y) \tag{8}$$

which means fixing covariant derivative $\mathcal{D}_\alpha$ ($\alpha = x, y$ are now coordinates of tangent plane), i.e. demanding parallel transport on a curve on the sphere

$$\mathcal{D}_\alpha \vec{n}(x, y) \equiv \partial_\alpha \vec{n}(x, y) - \varepsilon_{\alpha\beta}\partial_\beta \vec{n}(x, y) \times \vec{n}(x, y) = 0. \tag{9}$$

The field ζ in terms of n^{s_i} reads ([6]) in circular polarization basis

$$\zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + n^z} \\ \frac{n^x + i n^y}{\sqrt{1+n^z}} \end{pmatrix}. \tag{10}$$

Comparing last vector with (1) one sees that $z^1 = \sqrt{1+n^z}$ and $z^1 z^2 = (n^x + in^y)$.

In what follows we chose polar coordinates for local field $\zeta = \zeta(x,y) = \zeta(r\cos\phi, r\sin\phi)$. From the other side $\vec{n} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)$ where $\Theta = \Theta(r,\phi)$ and $\Phi = \Phi(r,\phi)$ are global coordinates corresponding to polar and azimuthal angles of Poincaré sphere. Hence

$$\zeta = \begin{pmatrix} \cos\frac{\Theta}{2} \\ \sin\frac{\Theta}{2} e^{i\Phi} \end{pmatrix}. \tag{11}$$

Belavin and Polyakov showed that equality (8) implies Cauchy-Riemann conditions on complex function $u = z^2/z^1$

$$\partial_r u = -\frac{i}{r}\partial_\phi u. \tag{12}$$

In global coordinates $u = \tan\frac{\Theta}{2} e^{i\Phi}$ giving

$$\frac{d\Theta}{\sin\Theta} = -(\frac{N_i}{r} + i\frac{d\Phi(r)}{dr})dr \tag{13}$$

where we used

$$\Phi(r,\phi) = N_i\phi + \Phi(r). \tag{14}$$

M being topological charge and $\Theta = \Theta(r)$.

1)Standard 2D Belavin-Polyakov solitons ([9]) are then solution of (13) for $\Phi(r) = \Phi_0$

$$\Theta(r) = 2\arctan\left(\frac{r^{N_i}}{R}\right). \tag{15}$$

where R is constant describing soliton radius. One can check that $\lim_{r\to 0}\Theta(r) = 0$ and $\lim_{r\to\infty}\Theta(r) = \pi$ (a meridian on Poincaré sphere) which are necessary conditions for Belavin-Polyakov soliton ([12]).

2)Another kind of solution exists if $\Phi(r)$ varies with r.

$$\Phi(r,\phi) = N_i\phi + \Phi(r). \tag{16}$$

and this topic will be considered in the next section.

SOLITONS ON TORUS: NLSE(GP) EQUATION

The fundamental group of the torus $S^1 \times S^1$ is $\mathbb{Z}\bigoplus\mathbb{Z}$, which is abelian, so its covering spaces are in 1 - 1 correspondence with the subgroups of this group, namely with $p\mathbb{Z}\bigoplus q\mathbb{Z}$, p, q = 0, 1, 2, 3, ... These covering spaces are tori, if p, q > 0, cylinders, if p = 0 or q = 0 but not both, or $\mathbb{R}^2$ if p = q = 0. The solutions of the form (16) will be appropriate for the second case.

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