Locality Bound for Dissipative Transport

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Motivation

Lieb-Robinson Bound

Gorini-Kossakowski-Sudarshan-Lindblad Equation

Dissipative Lieb-Robinson, Operator Decay and Diffusion

Sketch of the Proof

Results and Future Work

References

Complexity of Quantum Many-body Physics

QMA-complete to determine the ground state energy (up to polynomial accuracy in system size) for

- general 2-local Hamiltonians
- the 2D Heisenberg model with local magnetic fields
- the 2D Hubbard model with local magnetic fields

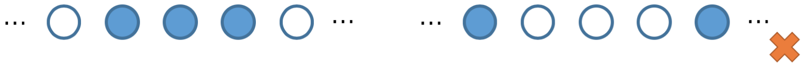
Non-perturbative Solutions:

- Large N/Holographic theories
- Numerics
- *Bounds from first principles*
- ...

Review of Transport Bounds

Name	Bounded Quantity	Principle
Mazur-Suzuki	Drude Weight	QM
Mott-Ioffe-Regel	Resistivity	Uncertainty
Lieb-Robinson	Ballistic Velocities	Locality
Kovtun-Starinets-Son	Viscosity	Causality
Hartman-Hartnoll-Mahajan	Diffusivity	Causality
...		

- How to describe locality more precisely?
- k-locality, spatial locality, ...



Locality Bound for Diffusivity

Diffusion: for $t \gtrapprox \tau_{\mathrm{th}}$ and $|x| \gtrapprox l_{\mathrm{th}}$,

$$\langle [n(t,x), n(0,0)] \rangle \propto \nabla^2 \frac{e^{-x^2/(4Dt)}}{t^{d/2}}.$$

Diffusivity bound:

$$D \lesssim v^2 \tau_{\mathrm{th}}$$

A naive derivation (assuming $|\langle [J(t), J(0)] \rangle| \le A \|J\|^2 e^{-t/\tau}$)

$$\sigma = \chi D = \int_0^\infty dt \, \langle [J(t), J(0)] \rangle \le A \|J\|^2 \tau$$

Interesting to generalize to dissipative case

- Physics out of thermal equilibrium
- Decoherence
- Model for strongly correlated systems
- Dissipative Lieb-Robinson bound exists

Lieb-Robinson Bound

Theorem (Lieb and Robinson, 1972)

Consider a lattice of spins with a translationally invariant local Hamiltonian. There exists $A, v, a > 0$, as functions of lattice geometry and the microscopic Hamiltonian, such that for any local operators O_1 and O_2 and spacetime positions (x, t),

$$\|[O_1(x,t), O_2(0,0)]\| \leq A\|O_1\|\|O_2\|e^{(vt-x)/a}.$$

- Unitary dynamics with locally f.d. Hilbert spaces.
- At least *exponential* spatial locality of the Hamiltonian.
- $A \sim 1$, $v \sim Ja$, $a \sim$ lattice spacing.

Lieb-Robinson Bound

$$\|[O_1(x,t), O_2(0,0)]\| \leq A\|O_1\|\|O_2\|e^{(vt-x)/a}$$

The Heisenberg chain as an example,

$$H = -J\sum_x (X_x X_{x+1} + Y_x Y_{x+1} + Z_x Z_{x+1}).$$

Expand

$$Z_0(t) = e^{iHt} Z_0 e^{-iHt} = Z_0 + it[H, Z_0] - \frac{1}{2}t^2[H,[H,Z_0]] + \ldots,$$

so that the operator growth is manifest,

$$\begin{aligned} i[H, Z_0] &= -2JX_{-1}Y_0 - 2JY_0X_1 + 2JY_{-1}X_0 + 2JX_0Y_1 \\ -[H,[H,Z_0] &= -4J^2Z_0 + 4J^2Y_{-2}Z_{-1}Y_0 - 4J^2Z_{-2}Y_{-1}Y_0 + \ldots \\ \ldots \end{aligned}$$

Lieb-Robinson Bound

$$\|[O_1(x,t), O_2(0,0)]\| \le A\|O_1\|\|O_2\|e^{(vt-x)/a}$$

- Exponential clustering of correlations;
- Area law of entanglement (1d gapped system);
- Bound of sound velocities.

Lindblad Equation

Theorem (Choi-Kraus)

A map $\mathcal{N}_{A\to B}$ between finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ is linear, completely positive, and trace-preserving if and only if it has a Choi-Kraus decomposition as follows:

$$\mathcal{N}_{A\to B}(X_A) = \sum_l V_l X_A V_l^\dagger,$$

where $X_A : \mathcal{H}_A \to \mathcal{H}_A$, $V_l : \mathcal{H}_A \to \mathcal{H}_B$ and $\sum_l V_l^\dagger V_l = I_A$.

- *Key Assumption: Markovian*
- $V_0 = I - (iH + K)\delta t + \ldots$ and $V_{l>0} = L_x\sqrt{2\delta t} + \ldots$,

$$\dot{\rho} = -i[H,\rho] + \sum_x (2L_x \rho L_x^\dagger - L_x^\dagger L_x \rho - \rho L_x^\dagger L_x)$$

$$\dot{O} = i[H,O] + \sum_x (2L_x^\dagger O L_x - L_x^\dagger L_x O - O L_x^\dagger L_x)$$

Diffusion and Operator Decay

$$\dot{O} = i[H, O] + \gamma \sum_x (2L_x^\dagger O L_x - L_x^\dagger L_x O - O L_x^\dagger L_x) \equiv \mathcal{L}[O]$$

To zeroth order in H ($\gamma \gg 1$), spectrum of the superoperator $\mathcal{L}$ is $\{0, -\gamma, -2\gamma, \ldots\}$:

$$L_x = Z_x, \quad \dot{O} = -\gamma \sum [Z_x, [Z_x, O]]$$

For a product of Pauli operators, the decay rate is $n\gamma$ where n counts the number of Pauli X or Y's:

$$\dot{Z}_0 = 0, \quad \dot{X}_0 = -4\gamma X_0, \quad \dot{\overline{X_0 Y_1}} = -8\gamma X_0 Y_1, \quad \ldots$$

In perturbation theory (of $1/\gamma$), change of the spectrum is

$$\delta E_n = \operatorname{tr} O_n i[H, O_n] + \sum_{m \neq n} \frac{|\operatorname{tr} O_m [H, O_n]|^2}{E_n - E_m} + \ldots$$

Diffusion and Operator Decay

Let $O_k \equiv \sum_x O_x e^{ikx}$ and C_0 conserved ($E_0 = 0$). Hydrodynamical expansion in k reads

$$\mathcal{L}[O_k] \equiv \dot{O}_k = \sum_x \mathcal{L}[O_x] e^{ikx} \equiv \sum_x \sum_{i=0}^{\infty} k^i \mathcal{L}^{(i)}[O]_x e^{ikx}$$

Example (Heisenberg chain)

$$\begin{aligned}\mathcal{L}[Z_k] &= -2J \sum_x (X_{x-1}Y_x + Y_x X_{x+1} - Y_{x-1}X_x - X_x Y_{x+1}) e^{ikx} \\ &= -2J \sum_x [(e^{ik} - 1) X_x Y_{x+1} + (1 - e^{ik}) Y_x X_{x+1}] e^{ikx}\end{aligned}$$

$$\mathcal{L}^{(0)}[Z] = 0, \quad \mathcal{L}^{(1)}[Z] = -2iJ(X \otimes Y - Y \otimes X),$$
$$\mathcal{L}^{(2)}[Z] = J(X \otimes Y - Y \otimes X), \quad \dots$$

Diffusion and Operator Decay

$$\dot{O}_k \equiv \mathcal{L}[O_k] = \sum_x \sum_{i=0}^{\infty} k^i \mathcal{L}^{(i)}[O]_x e^{ikx}$$

$$\mathcal{L}^{(1)}[Z] = -2iJ(X \otimes Y - Y \otimes X), \quad \mathcal{L}^{(2)}[Z] = J(X \otimes Y - Y \otimes X)$$

Perturbation theory in small k (to the second order),

$$E_0(k) = (Z|(k\mathcal{L}^{(1)} + k^2\mathcal{L}^{(2)})|Z) - k^2 \sum_{n \neq 0} \frac{(Z|\mathcal{L}^{(1)}|O_n)(O_n|\mathcal{L}^{(1)}|Z)}{E_n}$$

$$= -\frac{J^2 k^2}{\gamma} + o(k^2, \gamma^{-1})$$

$$\tilde{Z}_k(t) = e^{-J^2 k^2 t/\gamma} \tilde{Z}_k, \quad k \to 0, \gamma \to \infty$$

$D = J^2/\gamma$; *diffusion is explicit and generic!*

Idea of the Proof

Assume no ballistic modes $(Z|\mathcal{L}^{(1)}|Z) = 0$,

$$\begin{aligned} D &= -(Z|\mathcal{L}^{(2)}|Z) + \sum_{n\neq 0} \frac{(Z|\mathcal{L}^{(1)}|O_n)(O_n|\mathcal{L}^{(1)}|Z)}{E_n} \\ &= D_0 - \int_0^\infty dt\, (Z|\mathcal{L}^{(1)} e^{t\mathcal{L}^{(0)}} \mathcal{L}^{(1)}|Z) \\ &= D_0 - \int_0^\infty dt\, (Z| \lim_{k\to 0} \partial_k |\dot{J}_k(t)) \end{aligned}$$

If $\dot{J}_k(t) = \sum O_x^\alpha(t) e^{ikx}$,

$$\lim_{k\to 0} \partial_k \dot{J}_k(t) = \sum_{\alpha,x} ix O_x^\alpha(t)$$

$$\int_0^\infty dt\, (Z| \lim_{k\to 0} \partial_k |\dot{J}_k(t)) \sim \|\dot{J}\|/\|Z\| \int_0^\infty dt\, v_{\mathrm{LR}} t e^{-t/\tau} \sim v_{\mathrm{C}} v_{\mathrm{LR}} \tau$$

Results and Future Work

Final result:

$$D \leq D_0 + (\alpha v_{\mathrm{LR}} \tau + \beta \xi) v_{\mathrm{C}},$$

where D_0, v_{LR}, v_{C} and ξ can be easily derived from the microscopic Hamiltonian, α and β are some well-defined constants and $1/\tau > 0$ is the main dynamical input — minimal decay rate of non-conserved operators.

- $D \neq 0$ even if $\tau = 0$. Hints for resistivity saturation?
- Can be easily generalized to bounds for sub-diffusive coefficients or attenuation rate of sound modes.
- Goal: understand hydrodynamic coefficients from microscopic couplings. Quantum information theory may lead to a more precise understanding of applicability of hydrodynamics as well.

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