Bounding Transport and Chaos in Condensed Matter and Holography
@ Nordita, August 21st, 2018

# Space-Time in the SYK Model

Kenta Suzuki

S. Das, A. Ghosh, A. Jevicki, & KS; 1712.02725 [hep-th]

# MOTIVATION & INTRODUCTION

• Simple and Solvable examples of AdS/CFT are needed for better understanding of holography itself and quantum gravity problems.

• Based on earlier Sachdev-Ye model [Sachdev & Ye '93], Sachdev-Ye-Kitaev (SYK) model was proposed as a simpler version of AdS/CFT, which is a quantum mechanical many-body system based on $N(\gg 1)$ fermionic sites. [Kitaev '15]

• From maximally chaotic behavior of the model, dual gravity theory was conjectured to be $\mathrm{AdS}_2$ black hole theory. [Kitaev '15]

• As a first step of understanding of this SYK duality, in this talk I will explore some aspects of the SYK model, in the large $N$ limit.

# Outline

# 1 SYK MODEL

- SYK model [Kitaev '15] consists of Majorana fermions on $N$ sites ($N \gg 1$):

$$H = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl} \, \chi_i \, \chi_j \, \chi_k \, \chi_l \, ,$$

with

$$\{\chi_i \, , \, \chi_j\} = \delta_{ij} \, .$$

- $J_{ijkl}$ are all-to-all & random; distributions are Gaussian:

$$P(J_{ijkl}) \sim \exp\left(-\frac{N^3 J_{ijkl}^2}{12 J^2}\right) \, .$$

- Generalization to $q$-point interaction model is straightforward.

## $O(N)$ Gauge Symmetry

- At least in the leading order of $1/N$, the random coupling of this model can be treated by Replica Trick. [Sachdev & Ye '93], [Anninos, Anous & Denef '16].

- After evaluating the disorder average [Sachdev '15]:

$$\langle Z^n \rangle_J = \int \mathcal{D}\chi_i \, \mathrm{T} \exp \left[ \frac{1}{2} \int dt \sum_{i=1}^{N} \sum_{a=1}^{n} \chi_i^a \partial_t \chi_i^a + \frac{J^2}{2qN^{q-1}} \int dt_1 dt_2 \sum_{a,b=1}^{n} \left( \sum_{i=1}^{N} \chi_i^a(t_1) \chi_i^b(t_2) \right)^q \right].$$

- Now $O(N)$ symmetry $\chi_i \to O_{ij}\chi_j$ dynamically appeared. Therefore,

SYK Model $\approx$ $O(N)$ Vector Model !

## Collective Theory

- The Large $N$ theory is represented through a (replica diagonal) bi-local collective field:

$$\Psi(t_1, t_2) \equiv \frac{1}{N} \sum_{i=1}^{N} \chi_i(t_1)\chi_i(t_2) \,.$$

The corresponding path-integral is

$$Z = \int \prod_{t_1, t_2} \mathcal{D}\Psi(t_1, t_2) \, e^{-S_{\text{col}}[\Psi]} \,,$$

where $S_{\text{col}}$ is the (Euclidean time) collective action [Jevicki, K.S. & Yoon '16]:

$$S_{\text{col}}[\Psi] = \frac{N}{2} \int dt \left[ \partial_t \Psi(t, t') \right]_{t'=t} + \frac{N}{2} \operatorname{Tr} \log \Psi - \frac{J^2 N}{2q} \int dt_1 dt_2 \, \Psi^q(t_1, t_2) \,.$$

## Saddle-Point

- In strongly coupling limit $J|t| \gg 1$ (critical IR fixed point), one can drop the Kinetic term:

$$S_c = \frac{N}{2}\mathrm{Tr}(\log\Psi) - \frac{J^2 N}{2q}\int dt_1 dt_2 \left[\Psi(t_1,t_2)\right]^q .$$

- Saddle-Point Equation:

$$0 = \frac{\delta S_c}{\delta\Psi(t_1,t_2)} = \frac{N}{2}\left[\Psi^{-1}(t_2,t_1) - J^2\,\Psi^{q-1}(t_1,t_2)\right].$$

- Critical Saddle-Point Solution [Kitaev '15]:

$$\Psi_0(t_1,t_2) = \frac{b}{J^{\frac{2}{q}}}\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}}},$$

where $b$ is a dimensionless number depending on $q$ and $t_{12} \equiv t_1 - t_2$.

# 2 QUADRATIC FLUCTUATION

- Expansion around the critical IR background $\Psi_0$ as

$$\Psi(t_1, t_2) = \Psi_0(t_1, t_2) + \sqrt{\frac{2}{N}}\, \eta(t_1, t_2)\,,$$

the collective action $S_{\rm col}$ leads to the systematic $1/N$ expansion:

$$S_{\rm col} = N\, S_{(0)} + S_{(2)} + \frac{1}{\sqrt{N}} S_{(3)} + \cdots .$$

- Bi-local Quadratic Action:

$$S_{(2)} = -\frac{1}{2} \int \prod_{i=1}^{4} dt_i \;\eta(t_1, t_2)\, \mathcal{K}(t_1, t_2; t_3, t_4)\, \eta(t_3, t_4)\,.$$

## $SL(2,\mathcal{R})$ Casimir

• Quadratic kernel $\mathcal{K}$ is a function of the bi-local $SL(2,\mathcal{R})$ Casimir

$$\begin{aligned} C_{1+2} &= \big(\hat{D}_1+\hat{D}_2\big)^2 - \frac{1}{2}\big(\hat{P}_1+\hat{P}_2\big)\big(\hat{K}_1+\hat{K}_2\big) - \frac{1}{2}\big(\hat{K}_1+\hat{K}_2\big)\big(\hat{P}_1+\hat{P}_2\big) \\ &= -\,(t_1-t_2)^2\,\partial_1\partial_2 \,. \end{aligned}$$

• The eigenfunctions of the bi-local $SL(2,\mathcal{R})$ Casimir $C_{1+2}$ are, because of the properties of conformal blocks, given by the three-point function of the form

$$\Big\langle \mathcal{O}_h(t_0)\,\mathcal{O}_\Delta(t_1)\,\mathcal{O}_\Delta(t_2)\Big\rangle = \frac{\mathrm{sgn}(t_{12})}{|t_{10}|^h|t_{20}|^h|t_{12}|^{2\Delta-h}} \,.$$

## Diagonalizing the Kernel

- It is more useful to Fourier transform from $t_0$ to $\omega$ by

$$\left\langle \widetilde{\mathcal{O}_h}(\omega)\, \mathcal{O}_\Delta(t_1)\, \mathcal{O}_\Delta(t_2) \right\rangle \equiv \int dt_0\, e^{i\omega t_0} \left\langle \mathcal{O}_h(t_0)\, \mathcal{O}_\Delta(t_1)\, \mathcal{O}_\Delta(t_2) \right\rangle$$
$$\propto \frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{2\Delta-\frac{1}{2}}}\, e^{i\omega(\frac{t_1+t_2}{2})}\, Z_\nu(|\tfrac{\omega t_{12}}{2}|)\,,$$

where $h = \nu + 1/2$ and

$$Z_\nu(x) \equiv J_\nu(x) + \xi_\nu\, J_{-\nu}(x)\,, \qquad \xi_\nu \equiv \frac{\tan(\pi\nu/2)+1}{\tan(\pi\nu/2)-1}\,.$$

- The complete set of $\nu$ can be fixed from the representation theory of the conformal group [Kitaev '17]. The discrete modes $\nu = 2n + 3/2$ ($n = 0, 1, 2, \cdots$) and the continuous modes $\nu = ir$ ($0 < r < \infty$).

## Bi-local Propagator

- Therefore, the bi-local propagator ($\mathcal{D} = \mathcal{K}^{-1}$) is

$$\mathcal{D}(t_1, t_2; t_1', t_2') \propto J^{-1} \Bigg[ \int_0^\infty dr \int_{-\infty}^\infty dw \, \frac{r}{\sinh(\pi r)} \frac{u^*_{ir,w}(t_1, t_2)\, u_{ir,w}(t_1', t_2')}{\widetilde{g}(ir) - 1} + \sum_{n=0}^\infty \int_{-\infty}^\infty dw \, \frac{\nu\, u^*_{\nu w}(t_1, t_2)\, u_{\nu w}(t_1', t_2')}{\widetilde{g}(\nu) - 1} \Bigg|_{\nu = \frac{3}{2} + 2n} \Bigg],$$

where for $q = 4$

$$\widetilde{g}(\nu) \equiv -\frac{2\nu}{3} \cot\left(\frac{\pi\nu}{2}\right).$$

- The zero mode $\nu = \frac{3}{2}$, $(n = 0)$ gives a divergence since

$$\widetilde{g}\left(\tfrac{3}{2}\right) = 1, \qquad (\text{for any } q)$$

# 3 RADON TRANSFORM and LEG FACTORS

- After change of coordinates by

$$t \equiv \frac{t_1 + t_2}{2}, \qquad \eta \equiv \frac{t_1 - t_2}{2},$$

the bi-local $SL(2, \mathcal{R})$ Casimir becomes

$$C_{1+2} = -\eta^2(-\partial_\eta^2 + \partial_t^2),$$

which is the Laplacian of Lorentzian $\mathrm{dS}_2$ space (or $\mathrm{AdS}_2$)

$$ds^2 = \frac{-d\eta^2 + dt^2}{\eta^2}.$$

- In fact, the eigenfunctions

$$u_{\nu,\omega}(\eta, t) = \mathrm{sgn}(\eta)\, |\eta|^{\frac{1}{2}}\, e^{i\omega t}\, Z_\nu(|\omega\eta|),$$

are particular choice of $\alpha$-vacuum wave functions in Lorentzian $\mathrm{dS}_2$ space.

## $\alpha$-Vacuum

• There is a unique vacuum for Lorentzian de-Sitter space, which satisfies the Hadamard condition: Euclidean (Bunch-Davis) vacuum [Bunch & Davis '78]

$$\phi_\omega^{\mathrm{E}}(\eta) = \eta^{\frac{1}{2}}\, H_\nu^{(2)}(|\omega|\eta)\,.$$

• The $\alpha$-vacuum wave function is defined by Bogoliubov transformation from this Euclidean wave function [Mottola '84] [Allen '85] as

$$\begin{aligned}\phi_\omega^{\alpha}(\eta) \;&\equiv\; N_\alpha \Big[\phi_\omega^{E}(\eta) \,+\, e^{\alpha}\phi_\omega^{E*}(\eta)\Big] \\ &=\; N_\alpha\, \eta^{\frac{1}{2}} \Big[H_\nu^{(2)}(|\omega|\eta) \,+\, e^{\alpha} H_\nu^{(1)}(|\omega|\eta)\Big]\,.\end{aligned}$$

• The choice with $\alpha = i\pi(\nu + 1/2)$ recovers the SYK eigenfunctions.

## "$i$"-Problem

- However, this Lorentzian $\mathrm{dS}_2$ space cannot be the correct dual spacetime.

- If the dual gravity theory is on Lorentzian $\mathrm{dS}_2$, it must has

$$Z = \int \mathcal{D}\Phi_m \exp\left[ i \Big( S_{\mathrm{grav}}[\Phi] + S_{\mathrm{matter}}[\Phi] \Big) \right],$$

while the SYK partition function is

$$Z = \int \mathcal{D}\Psi \, e^{-S_{\mathrm{col}}[\Psi]}.$$

- The mismatch of the factor "$i$" proceeds for all $n$-point functions [Das, Ghosh, Jevicki & K.S. '17].

## Radon Transform

- There is a natural transform from Euclidean AdS to Lorentzian dS, given by Radon transform:

$$\left[\mathcal{R}f\right](\eta,t) \;=\; 2\eta\int_{t-\eta}^{t+\eta} d\tau \int_0^\infty \frac{dz}{z}\,\delta\Big(\eta^2-(\tau-t)^2-z^2\Big)\,f\,(\tau,z)\ .$$

- The inverse transformation is

$$\mathcal{R}^{-1}\,\overline{\psi}_{\omega,\nu}^{(\mathrm{dS}_2)}(\eta,t) \;=\; L^{-1}(\nu)\,\overline{\phi}_{\omega,\nu}^{(\mathrm{EAdS}_2)}(\tau,z)\,.$$

where

$$L(\nu)\;\equiv\;(\text{Leg Factor})\;=\;-2i\sqrt{\pi}\,\frac{\Gamma(\frac{1}{4}+\frac{\nu}{2})}{\Gamma(\frac{3}{4}+\frac{\nu}{2})}\,.$$

## Bi-local Propagator and Leg factors

- The bi-local propagator with $\eta = (t_1 - t_2)/2$ and $t = (t_1 + t_2)/2$:

$$G(\eta, t; \eta', t') \propto \int d\omega \Bigg[ \sum_{n=0}^{\infty} \frac{4 \sin \pi\nu_n}{\widetilde{g}(\nu_n) - 1} \, \overline{\psi}^{\,*}_{\omega,\nu_n}(\eta, t) \, \overline{\psi}_{\omega,\nu_n}(\eta', t') \Big|_{\nu_n = 2n + \frac{3}{2}} + \int_0^{\infty} dr \, \frac{\overline{\psi}^{\,*}_{\omega,\nu}(\eta, t) \, \overline{\psi}_{\omega,\nu}(\eta', t')}{\widetilde{g}(\nu) - 1} \Big|_{\nu = ir} \Bigg] .$$

- Using the inverse Radon transform to bring the dS wave functions into the EAdS wave functions:

$$G(\tau, z; \tau', z') \propto \int d\omega \Bigg[ \sum_{n=0}^{\infty} \frac{4 \sin \pi\nu_n}{\widetilde{g}(\nu_n) - 1} \, |L^{-1}(\nu_n)|^2 \, \overline{\phi}^{\,*}_{\omega,\nu_n}(\tau, z) \, \overline{\phi}_{\omega,\nu_n}(\tau', z') + \int_0^{\infty} dr \, |L^{-1}(\nu)|^2 \, \frac{\overline{\phi}^{\,*}_{\omega,\nu}(\tau, z) \, \overline{\phi}_{\omega,\nu}(\tau', z')}{\widetilde{g}(\nu) - 1} \Big|_{\nu = ir} \Bigg] .$$

# 4 IMPLICATIONS of the LEG FACTORS

• The extra Leg factors $|L^{-1}(\nu)|^2$ in the integrand leads to additional poles in the propagator.

• This is analog to the "old" $c = 1$ matrix model/$2D$ string duality, where the extra leg pole factors are characterizing the so-called "discrete modes" (gauge d.o.f.) in the $2D$ string theory.

• In fact, higher spin theory in AdS$_2$ also has similar discrete modes.
[Work in progress; Jevicki, K.S. & Yoon]

• We speculate that our leg factors in the SYK model also play a central role to identify the gauge degree of freedom in the dual gravity theory.
[Das, Ghosh, Jevicki & K.S. '17]

# Thank you!