

Pole-skipping

Based on:

- ① Grozdanov, Schalm, Scopelliti; PRL 120, 231601 (2018)
[1710.00921]
- ② Blake, Davison, Grozdanov, Liu; [180]

What is pole-skipping?

- A new signature of microscopic physics (quantum many-body chaos) on macroscopic, collective physics (energy dynamics, hydrodynamics)
- An analytic property of a retarded energy density two-point function $G_{T00T00}^R(\omega, k)$ in some classes of theories, and a result of a new, universal property of Einstein gravity.

Outline of the talk

1. Hydrodynamics
2. Quantum many-body chaos in holography
3. Motivation for pole-skipping and examples
4. Pole-skipping: statement and proof of its existence
5. Implications and future directions

1. Hydrodynamics

- An EFT of late-time, long-range excitations of certain states (fluids, gases, electrons in solids, quark-gluon plasma, ...)
- It encodes the conservation of global conserved operators
- Consider neutral, relativistic hydro in 4D
energy-momentum conservation : $\nabla_\mu T^{\mu\nu} = 0$
 $\Rightarrow 4 \text{ EoM's require } 4 \text{ dof's, } u^\mu(x) \text{ with } u^2 = -1$
 $T(x)$

- Gradient expansion : $\partial u^\mu \sim \varepsilon, \partial T \sim \varepsilon$

$$T^{\mu\nu} = \underbrace{\varepsilon u^\mu u^\nu + p \Delta^{\mu\nu}}_{0^{\text{th}} \text{ order}} - \underbrace{y \sigma^{\mu\nu} - \xi \nabla^\mu u^\nu}_{1^{\text{st}} \text{ order}} + \sum_{n=2}^{\infty} \left[\sum_i \lambda_i^{(n)} O_i^{(n)\mu\nu} \right]$$

- CFT : $T^{\mu\nu} = 0$ and $g_{\mu\nu} \rightarrow e^{-2\omega(x)} g_{\mu\nu} \Rightarrow T^{\mu\nu} \rightarrow e^{6\omega(x)} T^{\mu\nu}$

Order	# $\lambda_i^{(n)}$	# $\lambda_i^{(n)}$ for a CFT	
1	2	1	Navier-Stokes
2	15	5	BSSS
3	68	20	Grozdanov, Kaplis (2016)

- Hydro is a convergent expansion in momentum space

- Linearised, QNM solutions : $u^\mu = u_0^\mu + \delta u^\mu e^{-i\omega t + i\vec{k}\cdot\vec{x}}$
 $T = T_0 + \delta T e^{-i\omega t + i\vec{k}\cdot\vec{x}}$

gives dispersion relations $\boxed{\omega(k) = \sum_{n=1}^{\infty} d_n k^n}$ which are

poles of hydrodynamic Green's functions

e.g. $G_{T^{\mu\nu} T^{\rho\rho}}^R(\omega, \vec{k})$

- Example: holography in AdS_5 -Schwarzschild and $N=4$ SYM at $\lambda \rightarrow \infty$ and $N_c \rightarrow \infty$

$$\text{shear } (G_{T^0 T^0}^R) : \omega = -\frac{i}{4\pi T} k^2 - \frac{i(1-\ln 2)}{32\pi^3 T^3} k^4 + O(k^6)$$

$$\text{sound } (G_{T^0 T^0}^e) : \omega_e = \pm \frac{1}{\sqrt{3}} k - \frac{i}{6\pi T} k^2 \pm \frac{3-2\ln 2}{24\sqrt{3}\pi^2 T^2} k^3$$

$$- \frac{i(\pi^2 - 24 + 24\ln 2 - 12\ln^2 2)}{864\pi^3 T^3} k^4 + O(k^5)$$

We know all 5 transport coefficients at 2nd order to $O(\lambda^{-3/2})$

$$\lambda_i^{(2)} = \# + \# \frac{1}{\lambda^{3/2}} + \dots \quad [\text{Grozdanov, Starinets (2015)}]$$

We know 5/20 transport coefficients at 3rd order at $\lambda = \infty$

$$\lambda_i^{(3)} = \# + O(\lambda^{1/2}) \quad [\text{Grozdanov, Kaplis (2016)}]$$

- Another example: Einstein-Axion at $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ invariant point
[Davison, Gonterman (2019)]

Solutions are hypergeometric functions

${}_2F_1 \Leftrightarrow$ monodromy group

Also in Gauss-Bonnet at $\lambda_{GR} = N/4$ [Grozdanov, Starinets]

BTZ

:

$$\text{sound channel has diffusion } (G_{T^0 T^0}^e) : \omega(k) = \frac{i r_0}{2} \left(\sqrt{1 - \frac{4k^2}{r_0^2}} - 1 \right)$$

$$\approx \frac{i r_0}{2} \sum_{n=1}^{\infty} (-1)^n \binom{1/2}{n} \left(\frac{4k^2}{r_0^2} \right)^n$$

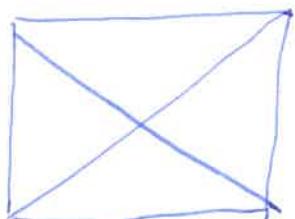
converges for $|k| < r_0/2$

2. Quantum many-body chaos in holography

- A microscopic phenomenon $l_{\text{chaos}} \ll l_{\text{hydro}}$
 \Rightarrow it is clear that quantum chaos \Rightarrow hydro (somehow)
Note that the role of quantum chaos in thermalisation is not clear.
- It is far from obvious that chaos can be inferred from hydro in certain cases! (content of ①)
- At large N , chaos can be characterised by OTOC
$$\langle V(t, \vec{x}) W(0) V(t, \vec{x}) W(0) \rangle_{\beta_0} = 1 - e^{-\lambda(t - |\vec{x}|/\sqrt{k_B})^2} + \dots$$
$$= 1 - e^{-i\omega t + ik|\vec{x}|} \quad (\text{by definition})$$

Def: $\boxed{\Phi_c : \omega = i\lambda, k = ik_0 \text{ such that } k_0 = \lambda/\sqrt{k_B}}$

- $\lambda \leq \lambda_{\max} = 2\pi T \cdot \frac{k_B}{\hbar}$ [Maldacena, Shenker, Stanford (2015)]
- In holography, OTOC is computed from considering a shock wave in the maximally extended geometry



$$ds^2 = A(uv) du dv + B(uv) d\vec{x}^2 - A(uv) h(\vec{x}) \delta(u) du^2$$

u, v are Kruskal - Szekeres coordinates

3. Motivation for pole-skipping and examples [from ①]

- Consider $N=4$ SYM dual to $S = \int d^5x \sqrt{-g} (R + 12)$
- The shock wave is a solution to linear and non-linear Einstein's equations. $\delta(ds^2) \sim dU^2$ is the sound channel
- Study sound channel at the linear level : $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu} e^{i\omega t + ik_x x}$
 $\{h_{tt}, h_{tx}, h_{tr}, h_{rx}, h_{rr}, h_{yy} + h_{zz}, h_{xx}\}$

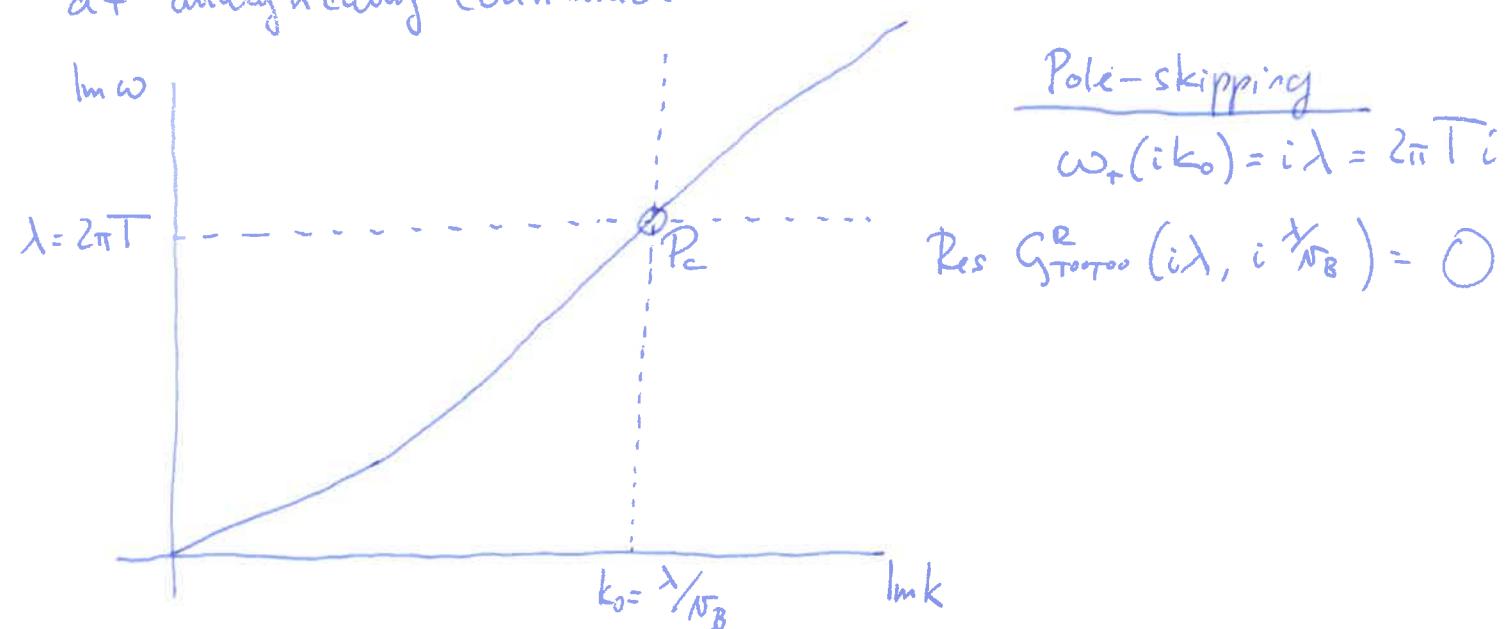
Gauge: $h_{\mu x} = 0$

- There exists a radially-null solution at the horizon, which is regular, iff $(\omega, k) \in P_c$

In KS coordinates : $ds^2 = A(uv)dUdV + B(uv)dx^2 - A(uv)e^{ikx} C_+ \frac{du^2}{u}$

$\delta(\omega)$ is summed to $1/\lambda$

- QNM (sound mode ω_+) in the spectrum of $G_{T00T00}^R(\omega, k)$ at analytically continued ω and k to $\text{Im } \omega$ values:



- SYK chain [Gu, Qi, Stanford (2016)]

$$G_{\text{TOPO}}^e(\omega, k) = C \frac{i\omega \left(\frac{\omega^2}{\lambda^2} + 1 \right)}{-i\omega + Dk^2}$$
$$\lambda = \frac{2\pi T}{N_B \sqrt{AD}}$$
$$\omega(k) = -iDk^2$$

- EFT of chaos : a re-summed theory of hydro dof's in
1+1 D [Blake, Lee, Liu (2018)]

- large- N CFT [Haehl, Rozali (2018)]

4. Pole-skipping: statement and proof of its existence [from ②]

The energy-density correlator $G_{T^0 T^0}^e(\omega, k) = \frac{b(\omega, k)}{a(\omega, k)}$

$a(\omega, k)$ contains the hydro pole

$$\omega = \omega_h(k) = \begin{cases} \omega_0 + \dots \\ -iDk^2 + \dots \end{cases}$$

1. $\omega_h(k)$ passes through P_c

2. $b(\omega, k)$ has a line of zeros which passes through P_c

At P_c , $G_{T^0 T^0}^e = \frac{0}{0}$; it is ill-defined

Perturb: $\omega = i\lambda + \delta\omega$, $k = ik_0 + \delta k$. Then
 as $\delta\omega, \delta k \rightarrow 0$: $G_{T^0 T^0}^e(\omega, k) = \frac{\partial_\omega b(P_c) \frac{\delta\omega}{\delta k} + \partial_k b(P_c)}{\partial_\omega a(P_c) \frac{\delta\omega}{\delta k} + \partial_k a(P_c)}$

The function is infinitely multiple-valued at P_c ,
 depending on the slope $\frac{\delta\omega}{\delta k}$ it is approached.

Proof and explanation

Consider the following class of theories:

$$\text{Einstein-matter } S = \int d^{d+2}x \sqrt{g} (R - 2\Lambda + L_m)$$

Metric in infalling EF coordinates: $ds^2 = -r^2 f(r) dv^2 + 2dvdr + h(r)d\vec{x}^2$,
 $r = t + r_*$, $\frac{dr_*}{dr} = \frac{1}{r^2 f}$

Chaos parameters: $\lambda = 2\pi T$, $k_0^2 = \frac{(2\pi T)^2}{\alpha_B^2} = d\pi T h'(r_*)$

- Longitudinal (sound) channel perturbations ($\sim e^{-i\omega r + ikx}$)

δg_{vv} , δg_{vr} , δg_{vx} , δg_{vv} , $\delta g_{xi}^{(0)}$, $\delta \varphi$ (matter)

sources energy density

- Horizon regularity

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}^{(1)}(r - r_0) + \dots$$

$$\delta \varphi = \delta \varphi^{(0)} + \delta \varphi^{(1)}(r - r_0) + \dots$$

- Split Einstein's equations $E_{\mu\nu} = \{E_{vv}, X\} = 0$

- $E_v = 0$:

$$(-i\frac{d}{2}\omega h'(r_0) + k^2)\delta g_{vv}^{(0)} - i(2\pi T + i\omega)[\omega \delta g_{xi}^{(0)} + 2k \delta g_{vx}^{(0)}] = -2h(r_0) \underbrace{\left[T_{vv}(r_0) \delta g_{vv}^{(0)} - \delta T_{vv}^{(0)}(r_0) \right]}_{(2)} = 0$$

$$f_n = -\frac{3(d)}{4}F - \frac{1}{2}(\partial \phi)^2 + V(\phi) - \frac{V(\phi)}{2} \sum_{i=1}^d (\partial \phi_i)^2$$

At $\omega = 2\pi T$: $(d\pi T h'(r_0) + k^2)\delta g_{vv}^{(0)} = 0$, hence at P_c this EoM is identically 0. $E_{\mu\nu} = 0$ thus provide 1 fewer constraints on the solution!

\Rightarrow There exists an extra linearly independent solution to Einstein's equations at P_c .

- To make sense of the holographic dictionary, must perturb from P_c :

$$\omega = i\lambda + \varepsilon \delta \omega, \quad k = i k_0 + \varepsilon \delta k$$

$$(-\frac{d}{2}\delta \omega h'(r_0) + 2k_0 \delta k)\delta g_{vv}^{(0)} + \delta \omega [2\pi T \delta g_{xi}^{(0)} + 2k_0 \delta g_{vx}^{(0)}] = 0$$

This equation is non-trivial. It depends on the slope $\frac{\delta \omega}{\delta k}$!

$$\frac{\delta \omega}{\delta k} = \frac{2k_0 \delta g_{vv}^{(0)}}{\frac{d}{2}h'(r_0)\delta g_{vv}^{(0)} - 2\pi T \delta g_{xi}^{(0)} - 2k_0 \delta g_{vx}^{(0)}}$$

Boundary conditions at the AdS boundary are still free.

Eg. $\phi = A(\omega, k) r^{-\alpha_1} + B(\omega, k) r^{-\alpha_2}$, $\alpha_1 < \alpha_2$, as $r \rightarrow \infty$

\uparrow non-normalisable \downarrow normalisable

Choose either normalisable or non-normalisable grow at P_c .

$\Rightarrow \delta g_{\mu\nu}^{(0)}$, which computes the slope of QNM.

Or, compute the slope of zeros in $b(\omega, k)$.

Hence $a(\omega, k)$ and $b(\omega, k)$ have a zero at P_c with a computable slope $\frac{\delta \omega}{\delta k}$.

Example : Einstein-Axion $S = \int d^4x \sqrt{g} (R + 6 + \frac{1}{2} \sum_{i=1}^2 (\partial \varphi_i)^2)$
 $\varphi_i = m x_i$, $f(r) = 1 - \frac{m^2}{2r^2} - (1 - \frac{m^2}{2r_0^2}) \frac{r_0^3}{r^3}$

2 parameters : m, r_0

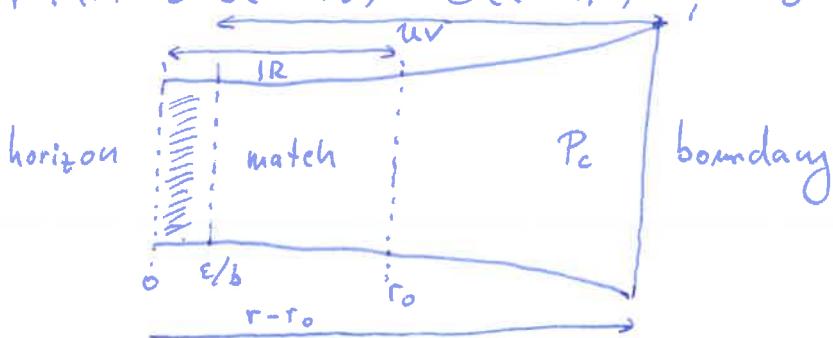
- Single decoupled gauge-invariant mode (sound) $\Psi(r)$

At P_c : $\Psi(r) = C_1 + C_2 \int_{r_0}^r dr \frac{\exp\left(\frac{m^2 - 3r_0^2}{r_0/3r_0^2 - 2m^2} \tan^{-1}\left(\frac{2r+r_0}{\sqrt{3r_0^2 - 2m^2}}\right)\right)}{r \sqrt{2(r^2 + rr_0 + r_0^2) - m^2}}$

Near P_c : $k^2 = -k_0^2 + \varepsilon$, $\omega = i\lambda - 2i\lambda \varepsilon q$

$$\Rightarrow \frac{\delta \omega}{\delta k} = \frac{4\lambda^2}{\pi s_B} q + O(\varepsilon)$$

$k^2 + r^3 f'(r) = \varepsilon + b(r - r_0) + O((r - r_0)^2)$, $b = 3r_0^2 f'(r_0) + r_0^3 f''(r_0)$



Explicitly derive :

$$\frac{1}{N_B} \frac{\delta \omega}{\delta k} = \frac{8(3r_0^2 - m^2)}{3(2r_0^2 - m^2)} + \frac{4(6r_0^2 - m^2)}{3(m^2 - 2r_0^2)} - \frac{F(r_0)r_0}{\tilde{N}(m, r_0)}$$

$$F(r_0) = \frac{\exp\left(\frac{m^2 - 3r_0^2}{r_0 \sqrt{3r_0^2 - 2m^2}} \tan^{-1}\left(\frac{3r_0}{\sqrt{3r_0^2 - 2m^2}}\right)\right)}{r_0 \sqrt{6r_0^2 - m^2}}$$

$$\tilde{N}(m, r_0) = N(m, r_0) + \frac{1}{\sqrt{2} 2\pi T} \exp\left(\text{sgn}(3r_0^2 - 2m^2) \frac{\pi (m^2 - 3r_0^2)}{2r_0 \sqrt{3r_0^2 - 2m^2}}\right)$$

$$N(m, r_0) = \int_{r_0}^{\infty} dr \frac{\exp\left(\frac{m^2 - 3r_0^2}{r_0 \sqrt{3r_0^2 - 2m^2}} \tan^{-1}\left(\frac{2r+r_0}{\sqrt{3r_0^2 - 2m^2}}\right)\right)}{r \sqrt{2(r^2 + rr_0 + r_0^2) - m^2}}$$

The curve $\frac{\delta \omega}{\delta k}$ precisely matches our ~~numerical~~ ^{numerical} check.

5. Implications and future directions

- Microscopic chaos leaves a precise analytic imprint on collective, real-time dynamics — hydrodynamics
- A stringent constraint on the structure of all-order hydrodynamics expansion.
- What happens away from large coupling and large N ?
- Weak coupling? Some hints are provided by Grzadov, Schalm, Scopelliti 1804.09182
- Experiments?

Appendix : $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ invariant point

At $m = \sqrt{2}r_0$, the full Green's function is

$$G_{T^{00}T^{00}}^{\text{R}}(\omega, k) = -\frac{k^2(k^2 + 2r_0^2)}{2r_0} \frac{\Gamma\left(\frac{1}{4} - \frac{i\omega}{2r_0} - \frac{1}{4}\sqrt{1-4\frac{k^2}{r_0^2}}\right) \Gamma\left(\frac{1}{4} - \frac{i\omega}{2r_0} + \frac{1}{4}\sqrt{1-4\frac{k^2}{r_0^2}}\right)}{\Gamma\left(\frac{3}{4} - \frac{i\omega}{2r_0} - \frac{1}{4}\sqrt{1-4\frac{k^2}{r_0^2}}\right) \Gamma\left(\frac{3}{4} - \frac{i\omega}{2r_0} + \frac{1}{4}\sqrt{1-4\frac{k^2}{r_0^2}}\right)}$$

Poles: $\frac{1}{4} - \frac{i\omega}{2r_0} - \frac{1}{4}\sqrt{1-4\frac{k^2}{r_0^2}} = -n$

$$\frac{1}{4} - \frac{i\omega}{2r_0} + \frac{1}{4}\sqrt{1-4\frac{k^2}{r_0^2}} = -p$$

n, p are non-negative integers

$n=0$: Hydro pole $\omega = -i\Omega k^2 - \dots$

$$\omega = i\frac{r_0}{2} \left(\sqrt{1 - \frac{4k^2}{r_0^2}} - 1 \right)$$

passes through $\omega = i\lambda = ir_0$ at $k^2 = -2r_0^2 = -k_0^2$

Slope: $\frac{1}{N_B} \frac{\delta \omega}{\delta k} = \frac{4}{3}$