HARVARD UNIVERSITY

Operator Spreading and the Emergence of Dissipation in Unitary Dynamics with Conservation Laws

Vedika Khemani

VK Vishwanath Huse 2017
VK Huse Nahum 2018

Unitary Quantum Dynamics

$$\rho(t+\delta t) = U\rho(t)U^\dagger$$

Dynamics of isolated, MB systems undergoing unitary time evolution:

spins/cold atom molecules/ black holes/…

strongly interacting, excited (no quasiparticles)

Time-independent Hamiltonian:

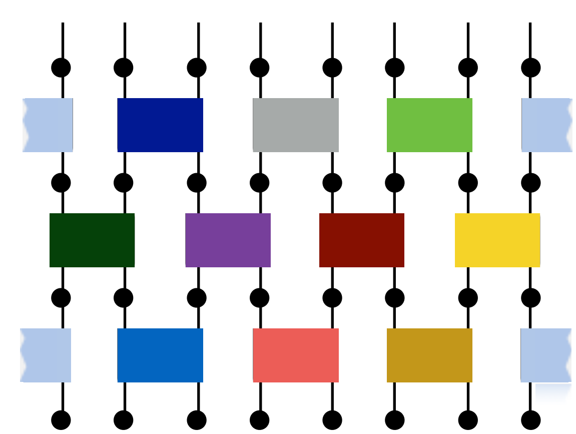
$$U(t) = e^{-iHt}$$

Floquet:

$$U(nT) = [U(T)]^n$$

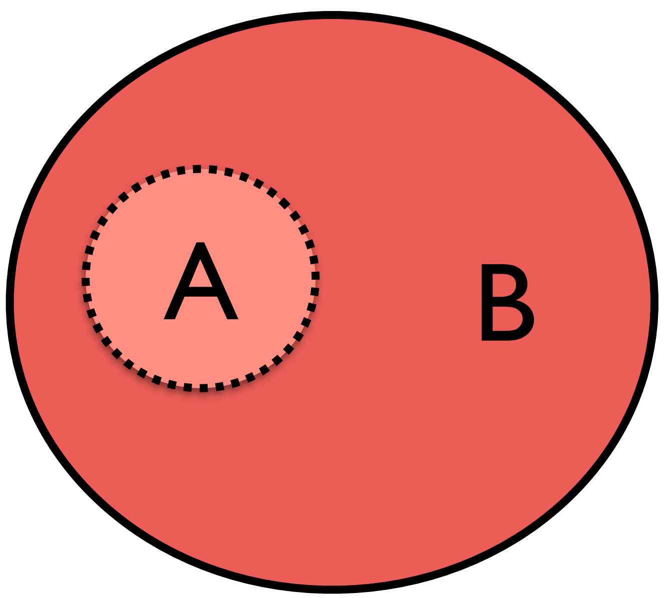
Random unitary circuit:

$$U(t) =$$



Thermalization in Isolation

Q: Can an isolated MB system act as it's own "bath" and bring its subsystems to thermal equilibrium (maximum entanglement)?



Full system remembers all details

$$|\psi(0)\rangle \longrightarrow |\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

Subsystems can thermalize

$$\rho_A(t) = \mathrm{Tr}_B |\psi(t)\rangle\langle\psi(t)|$$

Maximum entropy subject to a few constraints

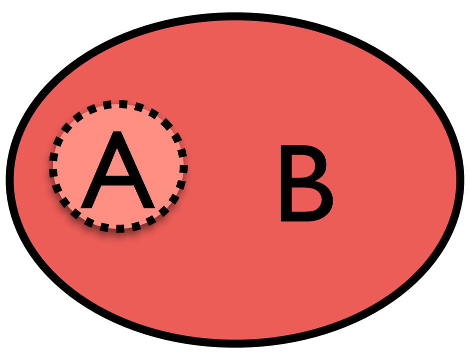
$$\lim_{t\to\infty} \rho_A(t) = \mathrm{Tr}_B \rho_{\mathrm{eq}}(T, \mu, \cdots)$$

Can *reversible* unitary time evolution bring a system to thermal equilibrium at late times?

$$|\psi(t)\rangle = U(t)|\psi_0\rangle$$

Yes: **Thermalizing**

Act as their "own bath" and bring subsystems to thermal equilibrium (maximum entropy)



$$\rho_A(t) = \mathrm{Tr}_B|\psi(t)\rangle\langle\psi(t)|$$

$$\lim_{t\to\infty} \rho_A(t) = \mathrm{Tr}_B \rho_{\mathrm{eq}}(T, \mu, \cdots)$$

No: **Many-Body Localized**

Cannot act as their own bath.

Retain *local* memory forever.

Thermalization + Conservation Law

Chaotic many-body system (ballistic information spreading)

+

locally conserved diffusive densities (energy/charge/..)

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Unitarity vs. Dissipation

Chaotic many-body system (ballistic information spreading)

+

locally conserved diffusive densities (energy/charge/..)

Q: How does unitary quantum dynamics, which is reversible, give rise to diffusive hydrodynamics, which is dissipative (increases entropy)?

Unitary Dynamics: Reversible

Diffusion: Irreversible/Dissipation

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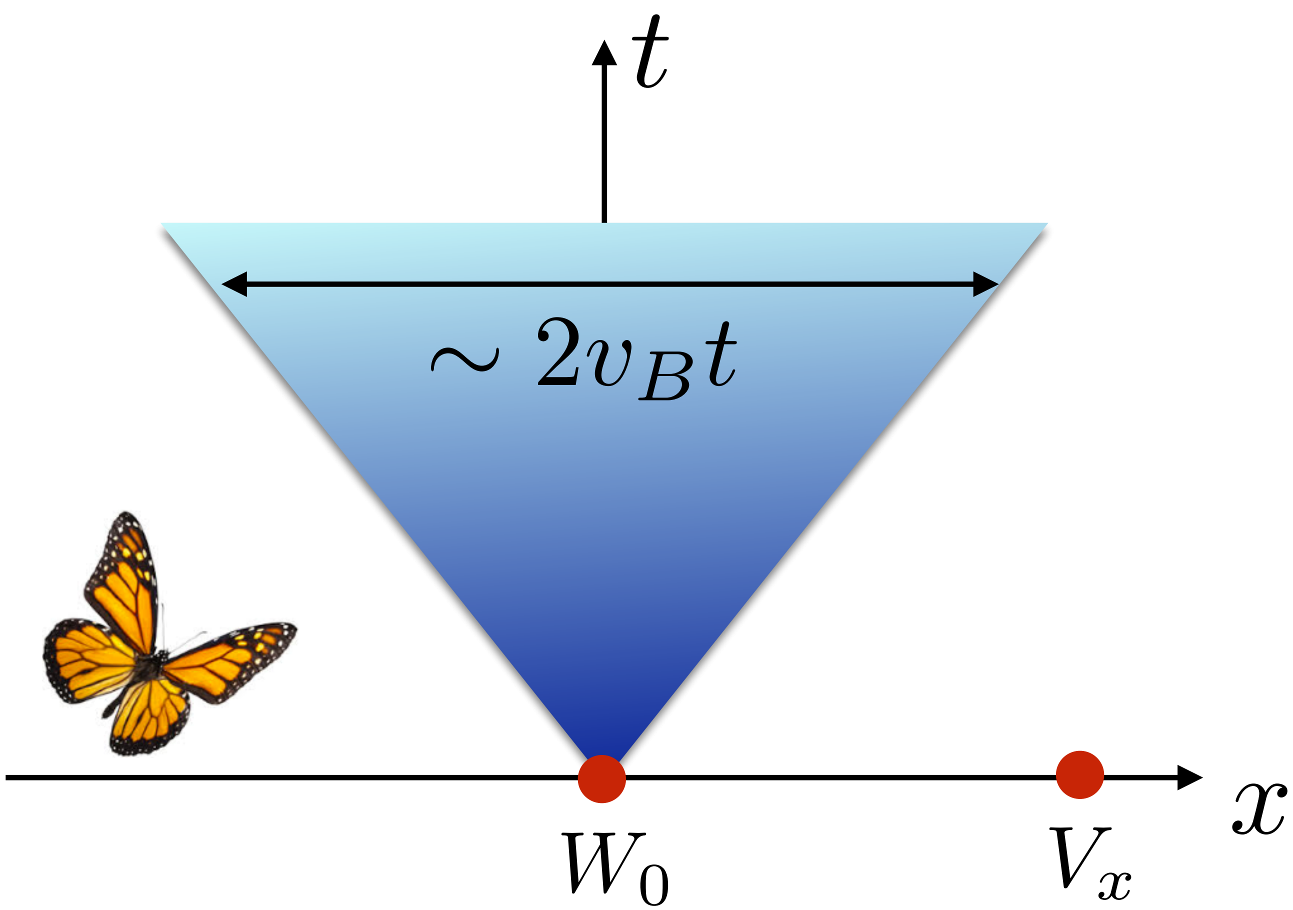
Many-Body "Quantum Chaos"

What is a precise formulation for many-body quantum chaos?

Is there a useful definition for chaos that is distinct from thermalization?

Are there distinct (universal) signatures of chaos at early/intermediate/late times? What are the most appropriate observables for probing these regimes?

Operator Spreading & OTOC



$$W(t) = U^\dagger(t) W_0 U(t)$$

$$\mathcal{C}(x,t) = \frac{1}{2}\langle |[W(t), V_x]|^2 \rangle$$

"Out-of-time-ordered-commutator"

semi-classical analog:

$$|i\hbar\{q(t), p\}|^2 = \hbar^2 \left(\frac{\partial q(t)}{\partial q(0)} \right)^2$$

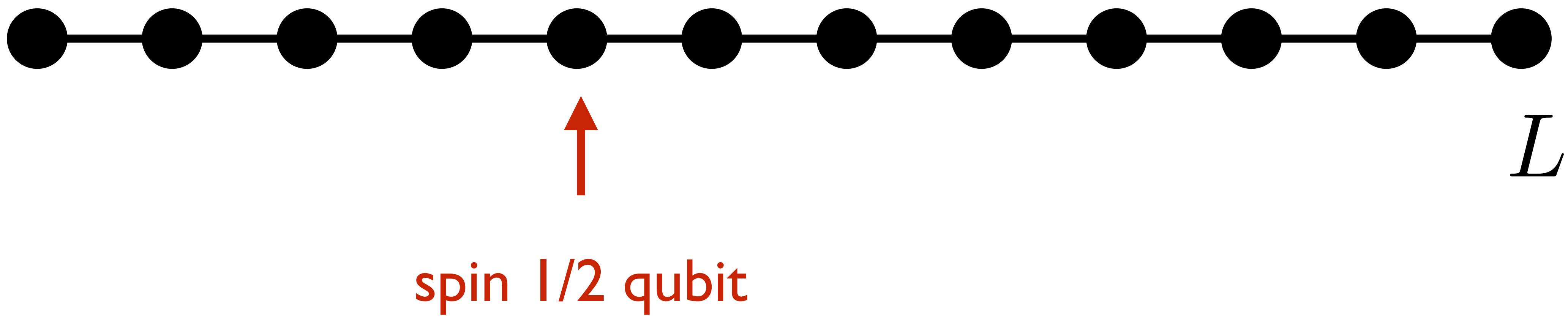
$$\sim \hbar^2 e^{\lambda t}$$

for classically chaotic systems with exponential sensitivity to initial conditions

Three aspects of dynamics

- Butterfly effect: ballistic operator growth with butterfly velocity v_B
- Diffusive hydrodynamics of conserved charges
- Lyapunov regime: exponential early-time sensitivity to perturbations

Setup



Local Hilbert space dimension: 2 (can also consider qudits with q)

4 operators per site: σ_i^μ $\quad \mu \in \{0,1,2,3\}$

Orthonormal basis of operators:

$(4)^L$ "Pauli strings"

$$S = \prod_i \otimes \sigma_i^{\mu_i}$$

$$xIyz, IzII, xxxx \cdots$$

$$\mathrm{Tr}[S^\dagger S']/(2^L) = \delta_{SS'}$$

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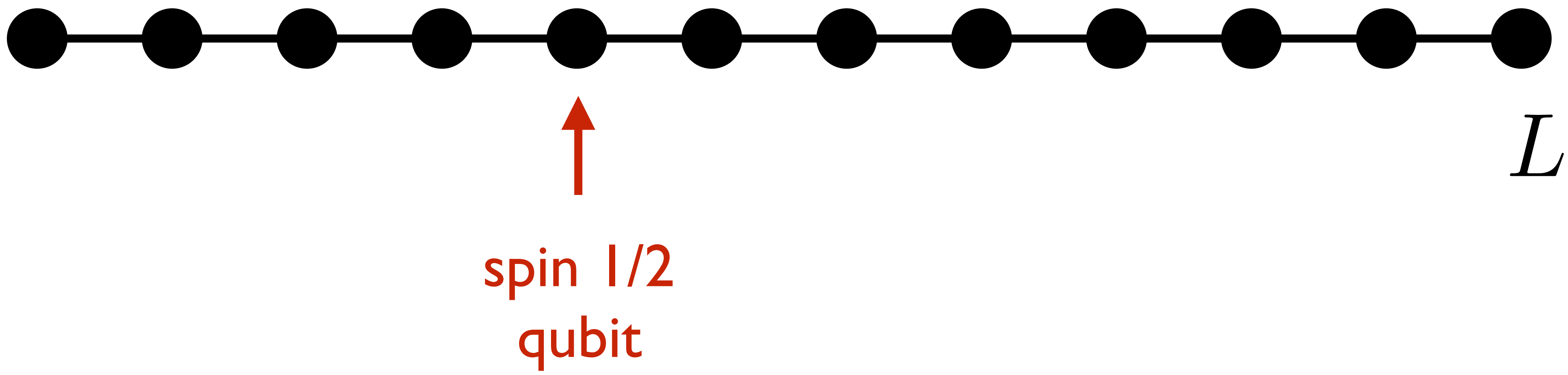
Setup



z component of spin 1/2 qubits conserved

$$S_z^{\mathrm{tot}} = \sum_i^L z_i$$

$$[U(t), S_z^{\mathrm{tot}}] = 0$$

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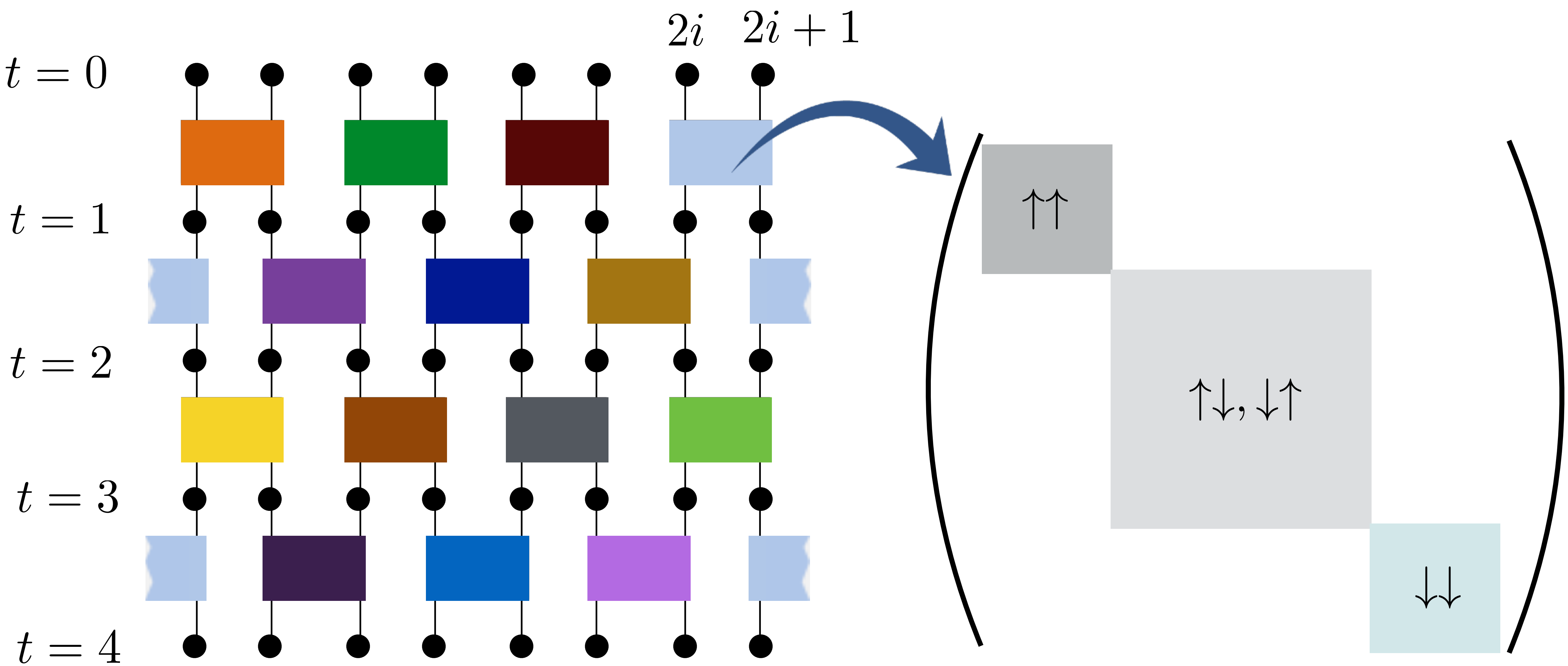
Setup: Random Conserving Circuit Model



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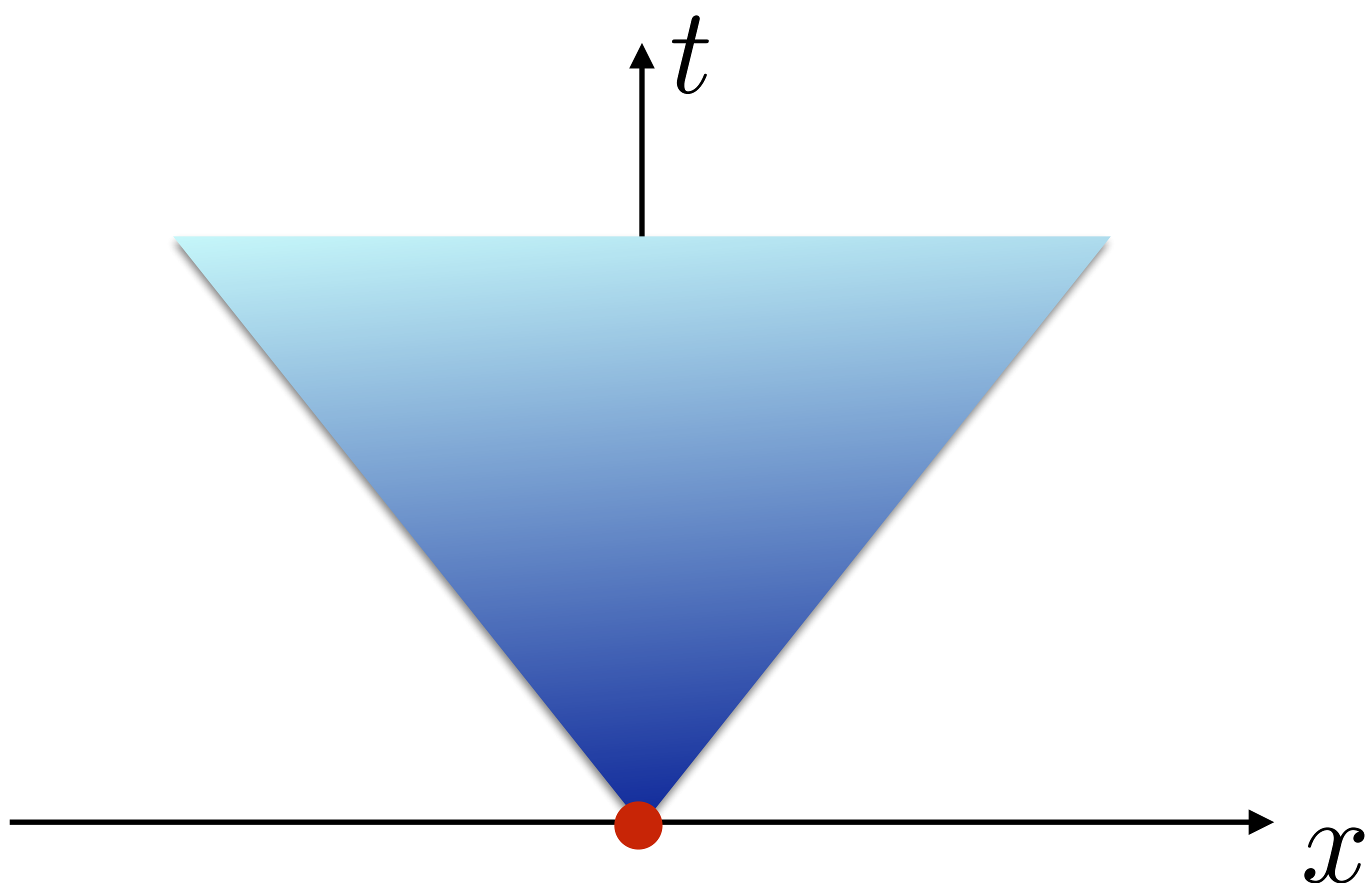
Builds on: Nahum et. al., (2016, 2017), von Keyserlingk et. al (2017).

Operator Spreading



Spreading constrained by:

- Unitarity
- Conservation Law(s)

$$O(t) = U^\dagger(t) O_0 U(t)$$

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\mathcal{S}$$

sum over $(4)^L$ strings

Operator Spreading: unitarity

Unitarity preserves operator norm

$$\mathrm{Tr}[O_0^\dagger(t)O_0(t)] = \mathrm{Tr}[O_0^\dagger O_0] = 2^L$$

$$\Downarrow$$

$$\boxed{\sum_{\mathcal{S}} |a_{\mathcal{S}}(t)|^2 = 1}$$

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\mathcal{S}$$

$$\mathrm{Tr}[S^\dagger S']/(2^L) = \delta_{SS'}$$

Operator Spreading: conservation law

Separate operator into conserved and non-conserved pieces

$$O_0(t) = O_0^c(t) + O_0^{\mathrm{nc}}(t)$$

$$O_0^c(t) = \sum_i a_i^c(t) z_i$$

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\mathcal{S}$$

$$\mathrm{Tr}[S^\dagger S']/(2^L) = \delta_{SS'}$$

Operator Spreading: conservation law

Separate operator into conserved and non-conserved pieces

$$O_0(t) = O_0^c(t) + O_0^{\rm nc}(t)$$

exp(L) mostly non-local strings, thus "hidden"

$$O_0^c(t) = \sum_i a_i^c(t) z_i$$

L local operator "strings", conserved densities

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\mathcal{S}$$

$$\mathrm{Tr}[S^\dagger S']/(2^L) = \delta_{SS'}$$

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$$O_0^c(t) = \sum_i a_i^c(t) z_i$$

L local operator "strings", conserved densities

$$\text{Tr}[O_0(t) S_z^{\text{tot}}] = \text{constant} \implies$$

$$\sum_{i=1}^{L} a_i^c(t) = \text{constant}$$

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\mathcal{S}$$

$$\text{Tr}[S^\dagger S']/(2^L) = \delta_{SS'}$$

Operator Spreading

Operator dynamics governed by the interplay between:

Unitarity: $$\sum_{\mathcal{S}} |a_{\mathcal{S}}(t)|^2 = 1$$

Conservation law: $$\sum_{i=1}^{L} a_i^c(t) = \text{constant}$$

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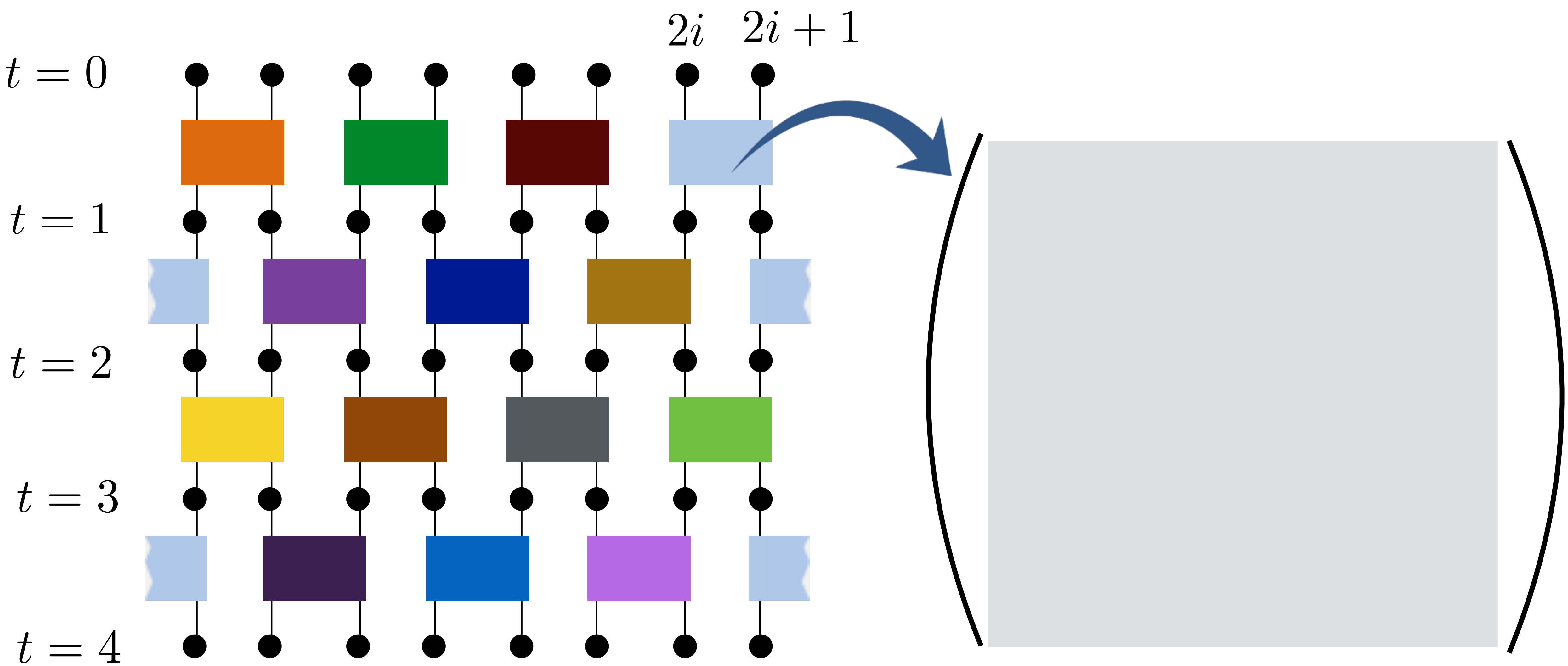
First: unconstrained circuit



$$\sum_{\mathcal{S}} |a_{\mathcal{S}}(t)|^2 = 1$$

Nahum et. al., (2016, 2017),
von Keyserlingk et. al (2017).

Operator shape: Right weight

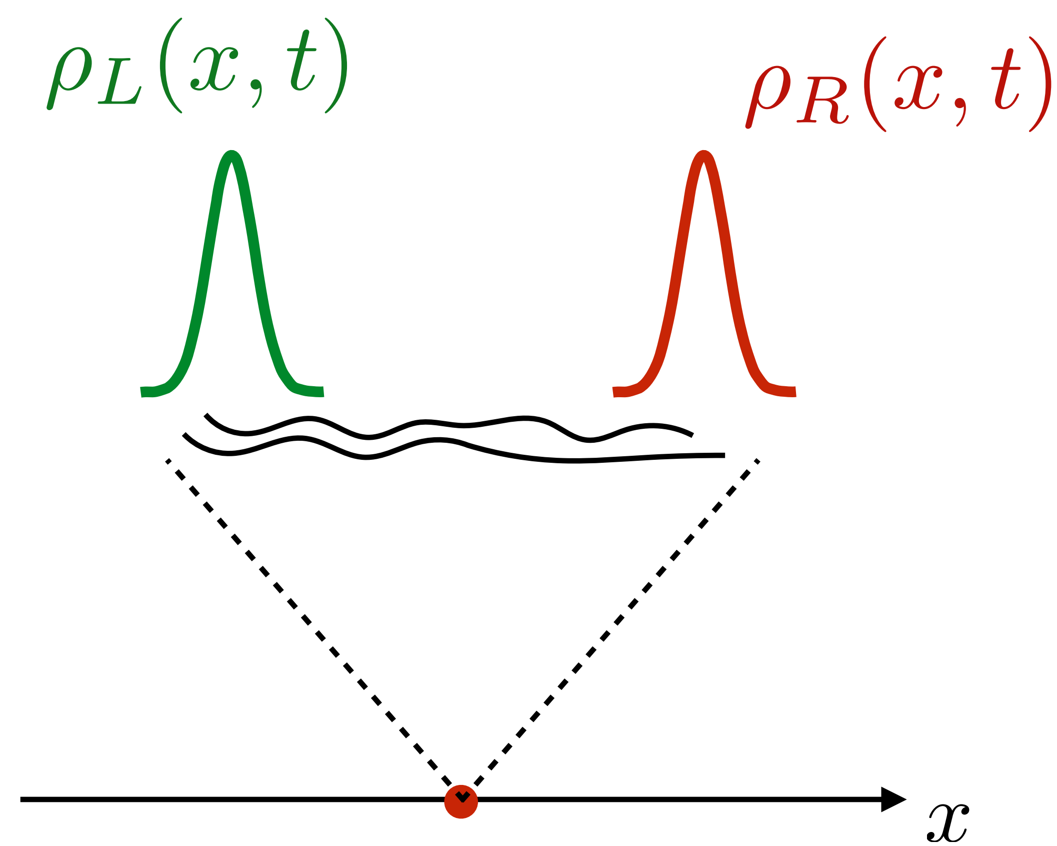
Right-Weight: "emergent" density following from unitarity

$$\rho_R(i,t) = \sum_{\substack{\text{strings } \mathcal{S} \text{ with} \\ \text{rightmost non-} \\ \text{identity on site } i}} |a_{\mathcal{S}}|^2, \qquad \sum_i \rho_R(i,t) = 1.$$

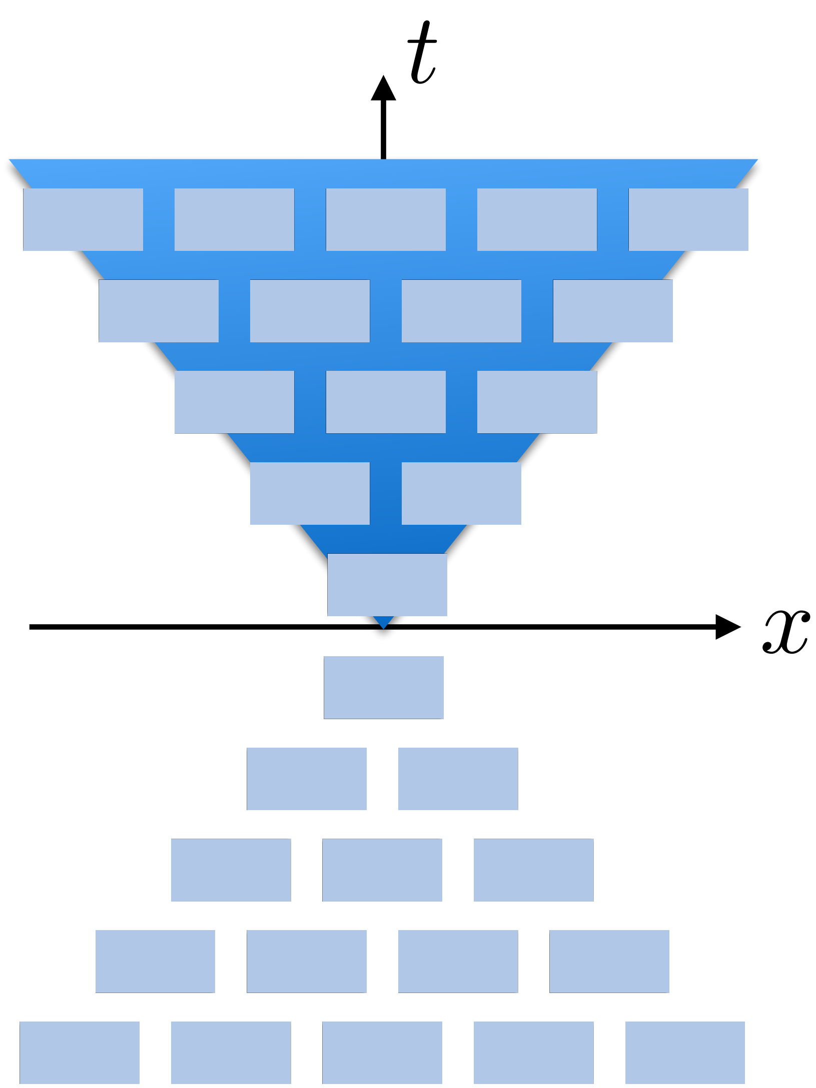
Each string has right/left edges beyond which it is purely identity.

ρ looks at the density distribution of the "right front" of the operator.

As operator spreads, weight moves to longer Pauli strings.



Operator shape: unconstrained circuit



Front dynamics: biased diffusion

$$U^\dagger(\delta t) \mathcal{S} U(\delta t)$$

has amplitudes for

making S shorter

leaving it same length

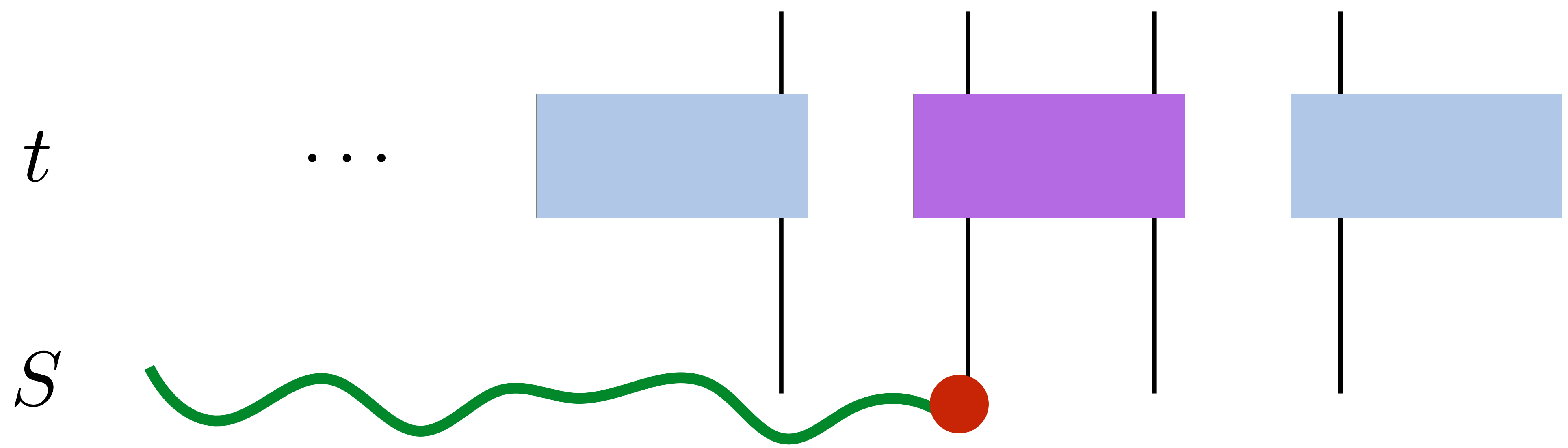
making S longer

But, biased towards making S longer.

Example, only 3/15 non-identity two-site spin 1/2 operators have identity on the right site.

Nahum et. al., (2017)
von Keyserlingk et. al (2017)

Operator shape: unconstrained circuit



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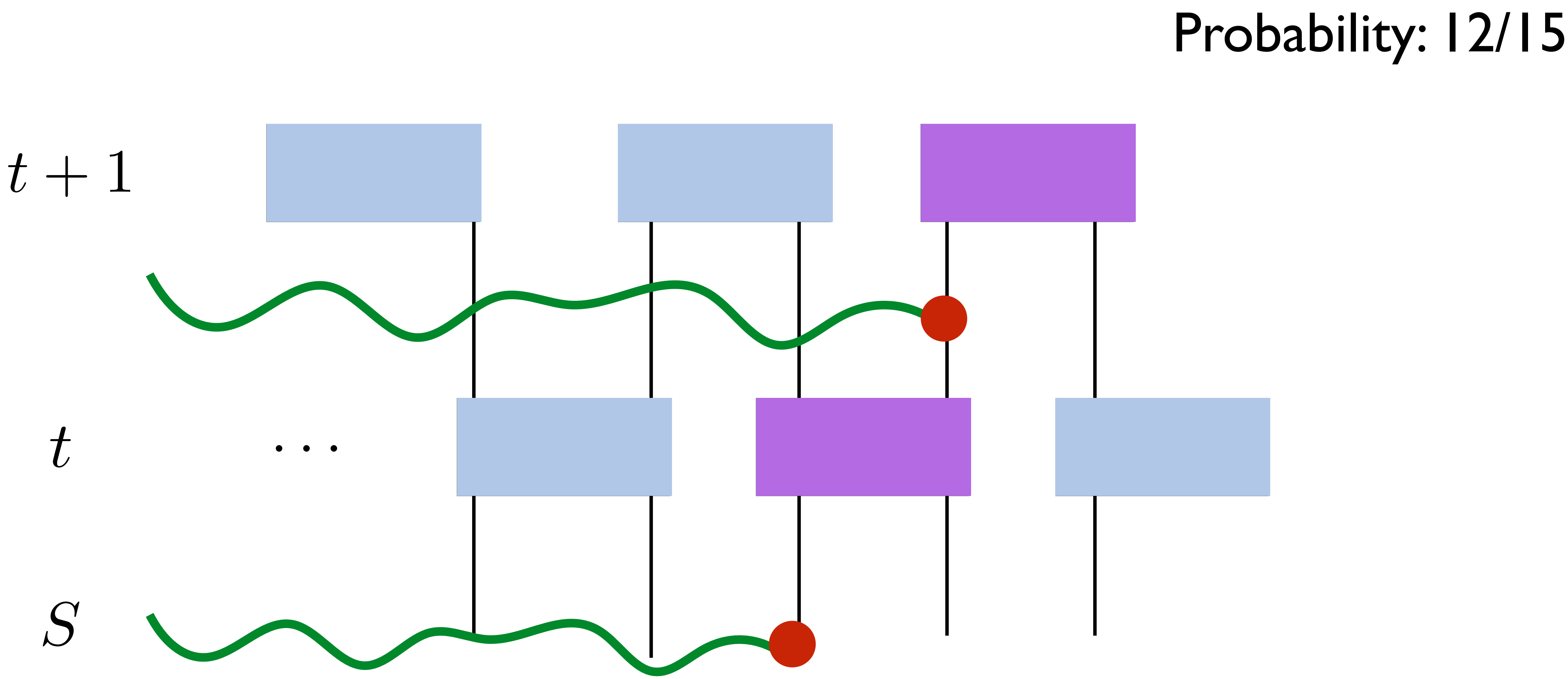
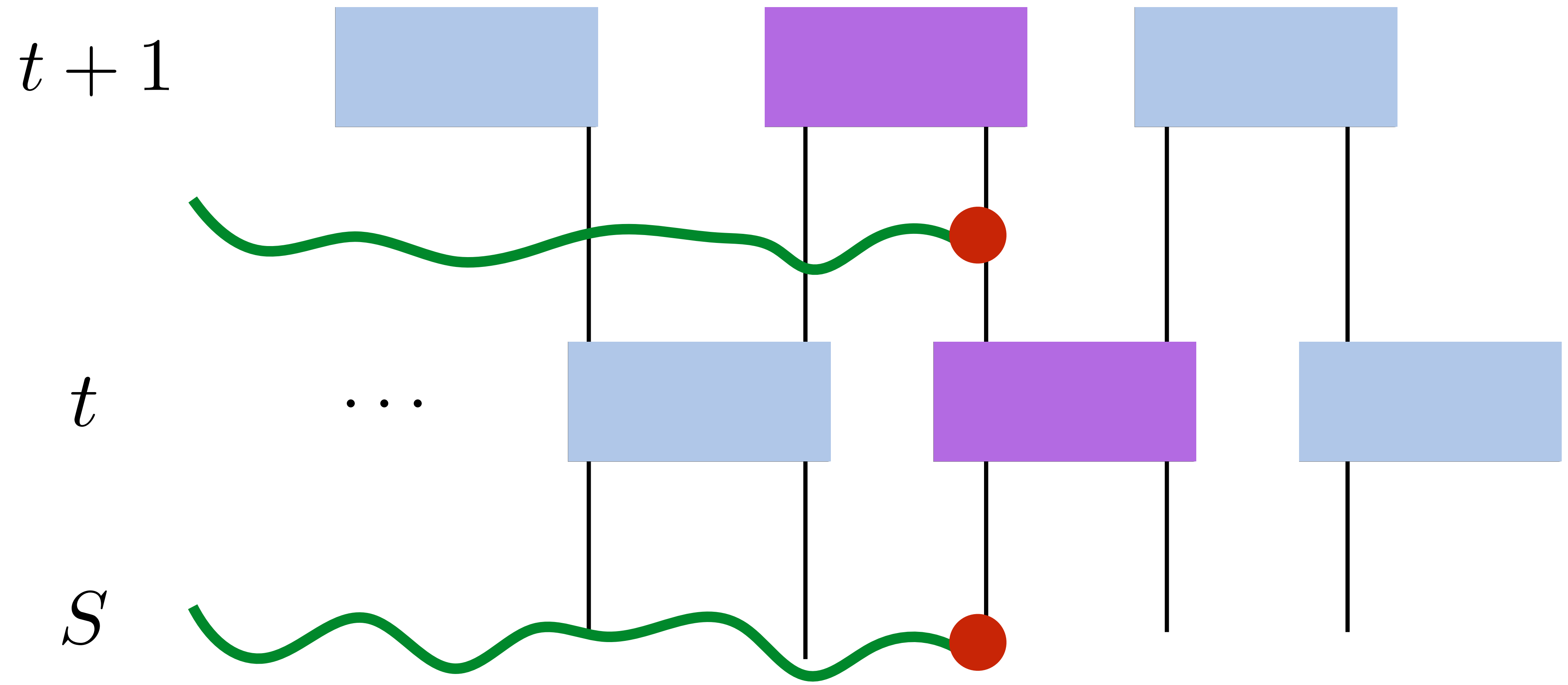


Nahum et. al., (2017)
von Keyserlingk et. al (2017)

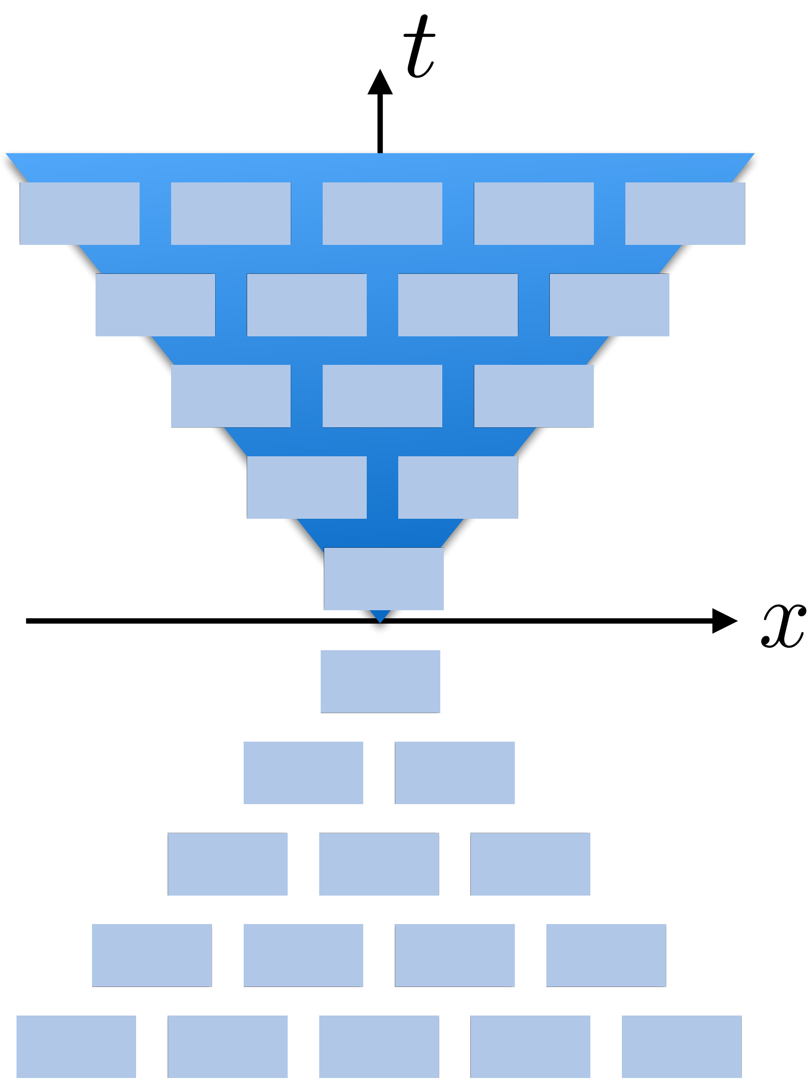
Operator shape: unconstrained circuit

Probability: 3/15



Nahum et. al., (2017)
von Keyserlingk et. al (2017)

Operator shape: unconstrained circuit



Front dynamics: biased random-walk

Emergent hydrodynamics:

$$\partial_t \rho_R(x,t) = v_B \partial_x \rho_R(x,t) + D_\rho \partial_x^2 \rho_R(x,t)$$

$$\rho_R(x,t) \approx \frac{1}{\sqrt{4\pi D_\rho t}} e^{-\frac{(x - v_B t)^2}{4 D_\rho t}}$$

$$v_B \sim 1 - \frac{2}{q^2};\; D_\rho \sim \frac{2}{q^2}$$

Nahum et. al., (2017)
von Keyserlingk et. al (2017)

Operator shape: unconstrained circuit

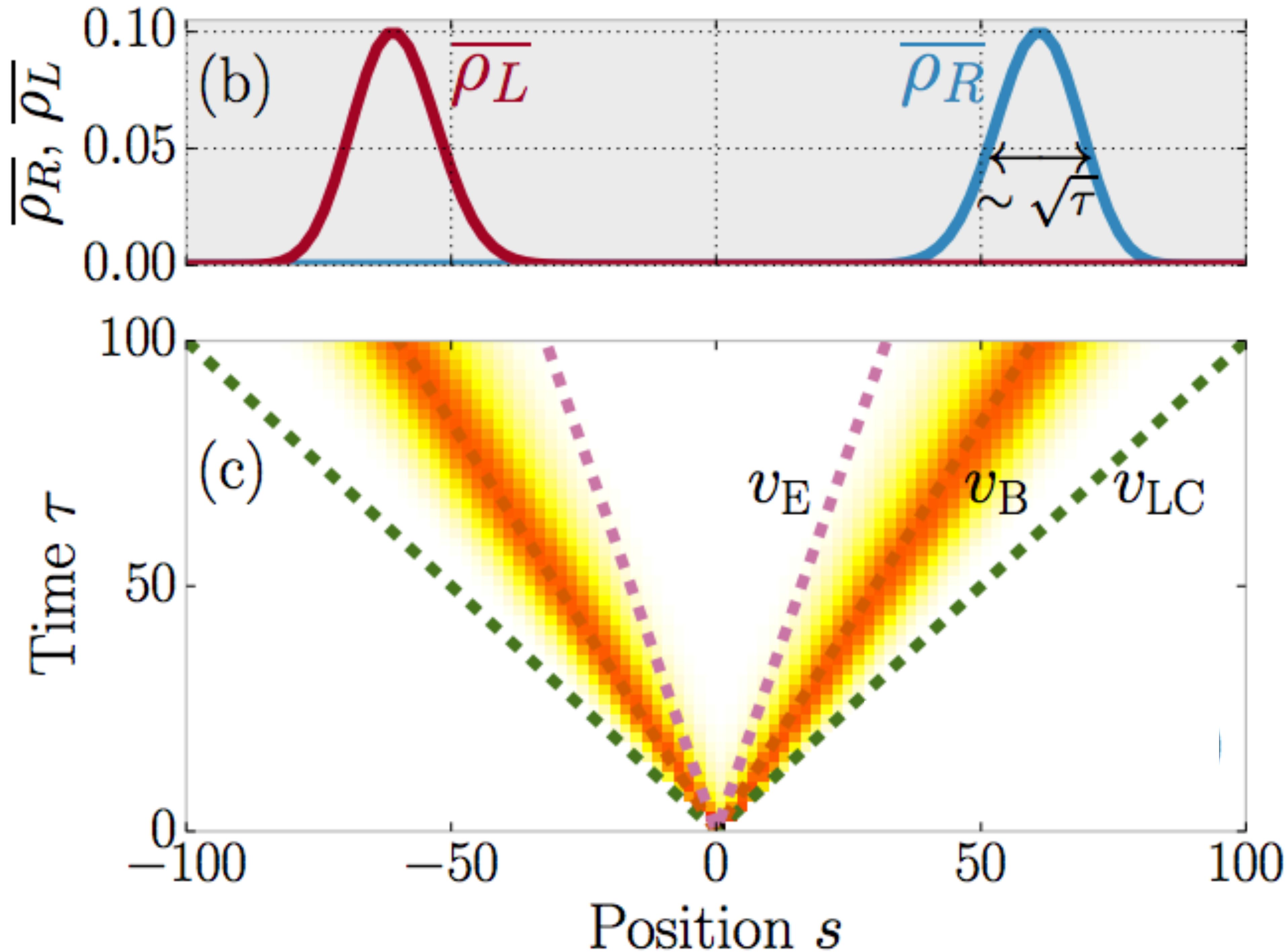
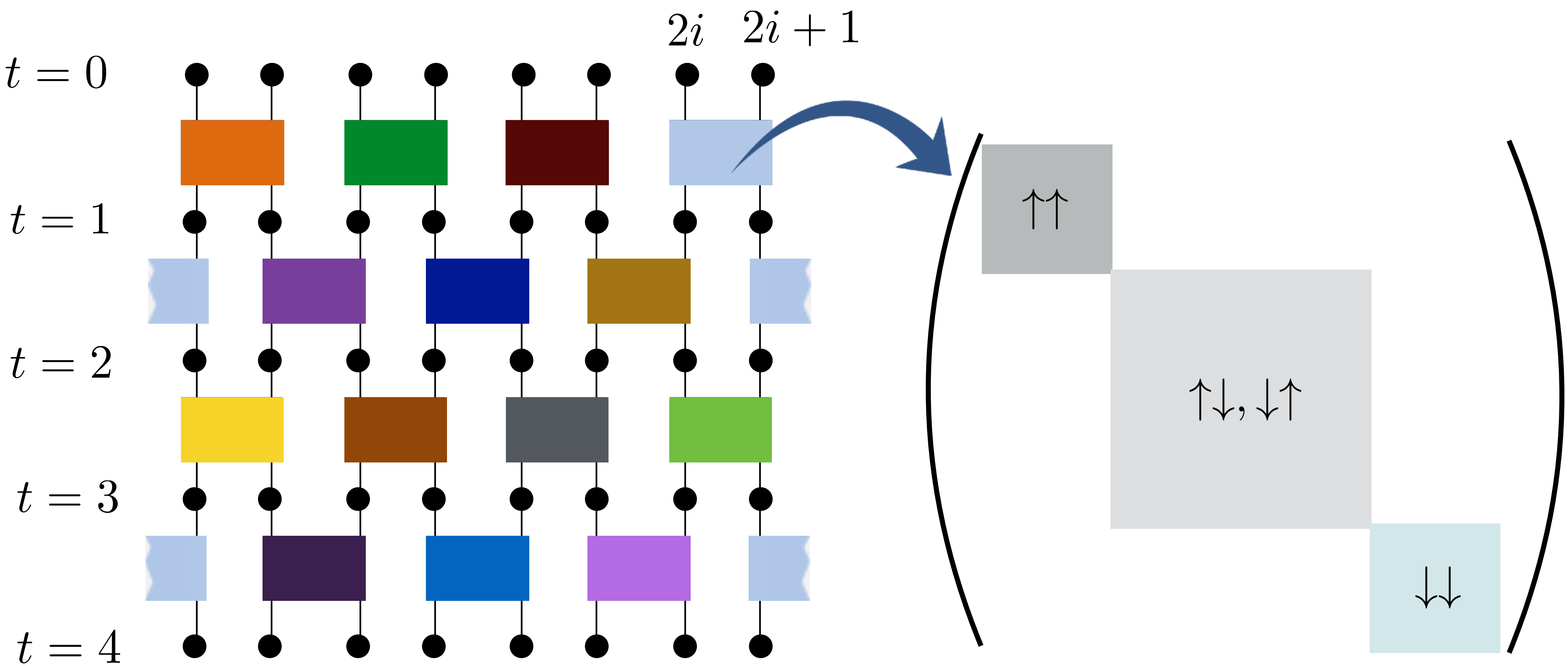


Figure from:
von Keyserlingk et. al (2017)

Setup: Random Conserving Circuit Model



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Operator Spreading

Operator dynamics governed by the interplay between:

Unitarity:

$$\sum_{\mathcal{S}} |a_{\mathcal{S}}(t)|^2 = 1$$

Conservation law:

$$\sum_{i=1}^{L} a_i^c(t) = \text{constant}$$

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Spreading of conserved charges

First, consider spreading of conserved density

$$O_0 = z_0$$

$$a_i^c(t=0) = \delta_{i0}$$

$$\sum_i a_i^c(t) = 1$$

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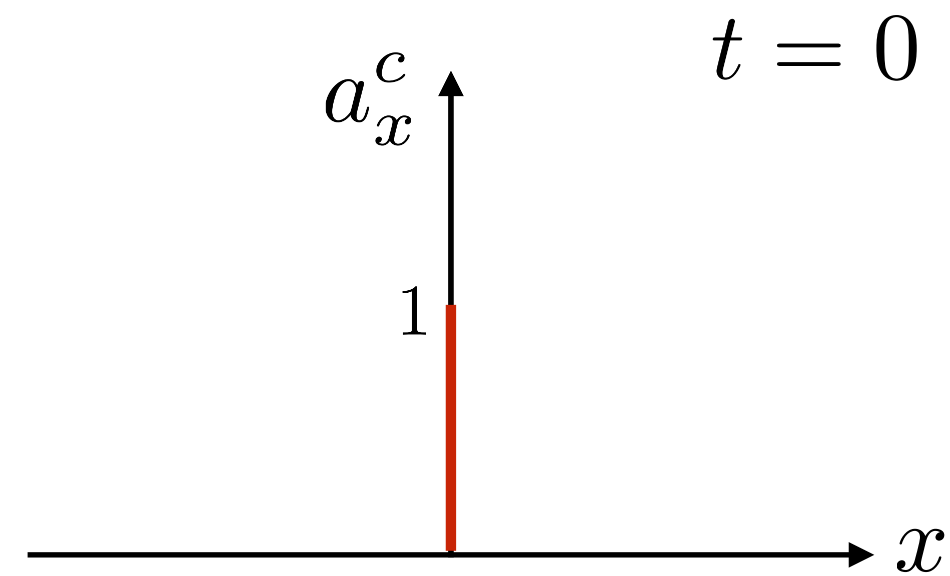
Diffusion & conserved amplitudes: intuition

Initial state: Infinite temperature equilibrium
+ local charge perturbation

$$\rho_0 = \frac{1}{2^L}\left[\mathbb{I} + \epsilon O_0\right]$$

$$O_0 = z_0$$



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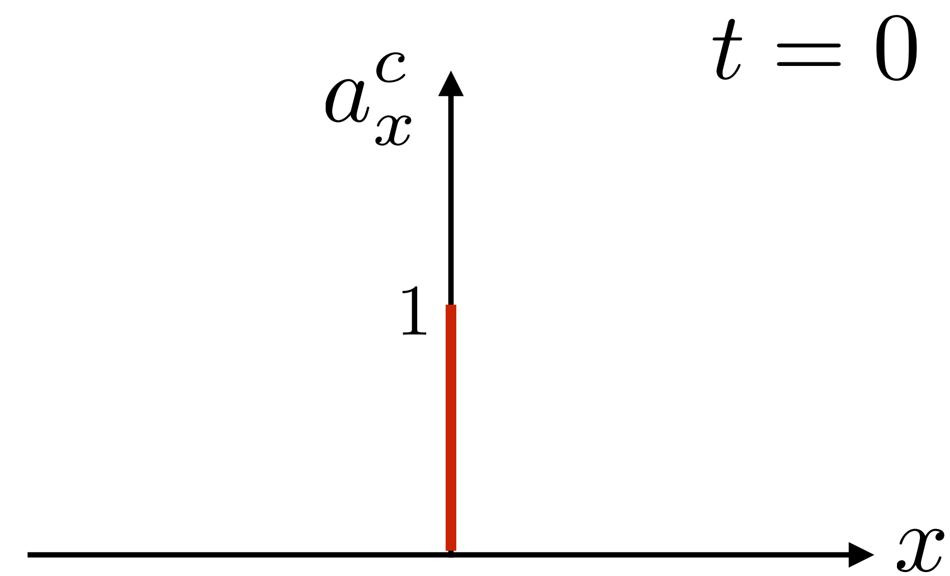
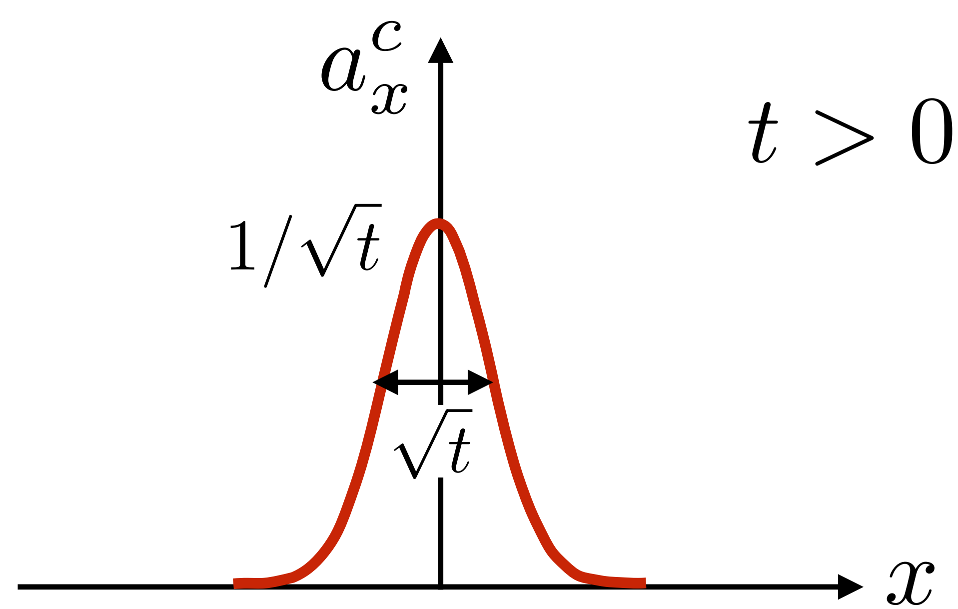


Diffusive charge spreading (coarse grained):

$$\begin{aligned}\langle z\rangle(x,t) &= \mathrm{Tr}[\rho(t) z_x] \\ &= \frac{\epsilon}{2^L}\mathrm{Tr}[\rho(t) z_x] \\ &= \epsilon\, a_x^c(t) \sim \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4D_c t}}\end{aligned}$$



Diffusion & conserved amplitudes

$$O_0 = z_0$$

Random conserving circuit model

$$\overline{a_i^c(t)} = \frac{1}{2^t}\binom{t-1}{\lfloor \frac{i+t-1}{2} \rfloor}$$

$$\sum_i a_i^c(t) = 1$$

$$\overline{a^c(x,t)} \approx \sqrt{\frac{1}{2\pi t}} e^{-\frac{x^2}{2t}}$$

coarse grain+
scaling limit

$$D_c = \frac{1}{2}$$

independent of q

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Diffusive Lump

$$\sum_i a_i^c(t) = 1$$

Total operator **weight** in the diffusive lump of conserved charges decreases as a power-law in time.

$$\rho^c_{\text{tot}} \equiv \sum_i |a_i^c(t)|^2$$

$$\overline{\rho^c_{\text{tot}}(t)} \approx \int dx |\overline{a_x^c(t)}|^2 = \int dx \, \frac{1}{2\pi t} e^{-\frac{x^2}{t}} = \frac{1}{2\sqrt{\pi t}}$$

Significant weight in a "diffusive cone" near the origin, even at late times.

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Slow emission of non-conserved operators

- No net loss in operator weight (unitarity).
- Conserved parts emit a steady flux of "non-conserved" operators.
- The local production of non-conserved operators is proportional to the square of the diffusion current, as in Ohm's law:

$$\delta\rho_i^{\mathrm{nc}}(t) \sim (a_i^c(t) - a_{i+1}^c(t))^2 \quad \sim (\partial_x a^c(x,t))^2$$

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Emergence of dissipation

The dissipative process is the **conversion** of operator weight from locally observable conserved parts to non-conserved, non-local (non-observable) parts at a *slow* hydrodynamic rate.

Observable entropy increases, while total von Neumann entropy of the full system is conserved.

Increase in *observable* entropy

$$\rho(t) = \frac{1}{2^L}[\mathbb{I} + \epsilon O_0(t)] \qquad S_{\mathrm{vn}}(t) = \mathrm{const}$$

$$O_0^c(t) = \sum_i a_i^c(t) z_i$$

$$S_{\mathrm{vn}}^c(t) = -\mathrm{Tr}[\rho^c(t)\log\rho^c(t)]$$
$$= L\log(2) - \frac{1}{2}\sum_i |a_i^c(t)|^2 + \cdots$$

$$\frac{d}{dt}S_{\mathrm{vn}}^c(t) \sim \frac{1}{2D_c}\int dx |j_c(x)|^2$$

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Putting it all together

- **Diffusion of conserved densities:** Local conserved densities spread diffusively. The *weight* of O(t) on the conserved parts (which live in a diffusive cone near the origin) slowly decreases as a power-law in time. Thus significant weight near the origin even at late times.

Putting it all together

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- **Ballistic spreading of non-conserved operators:** Once emitted, the non-conserved parts spread ballistically, quickly becoming non-local and hence non-observable.

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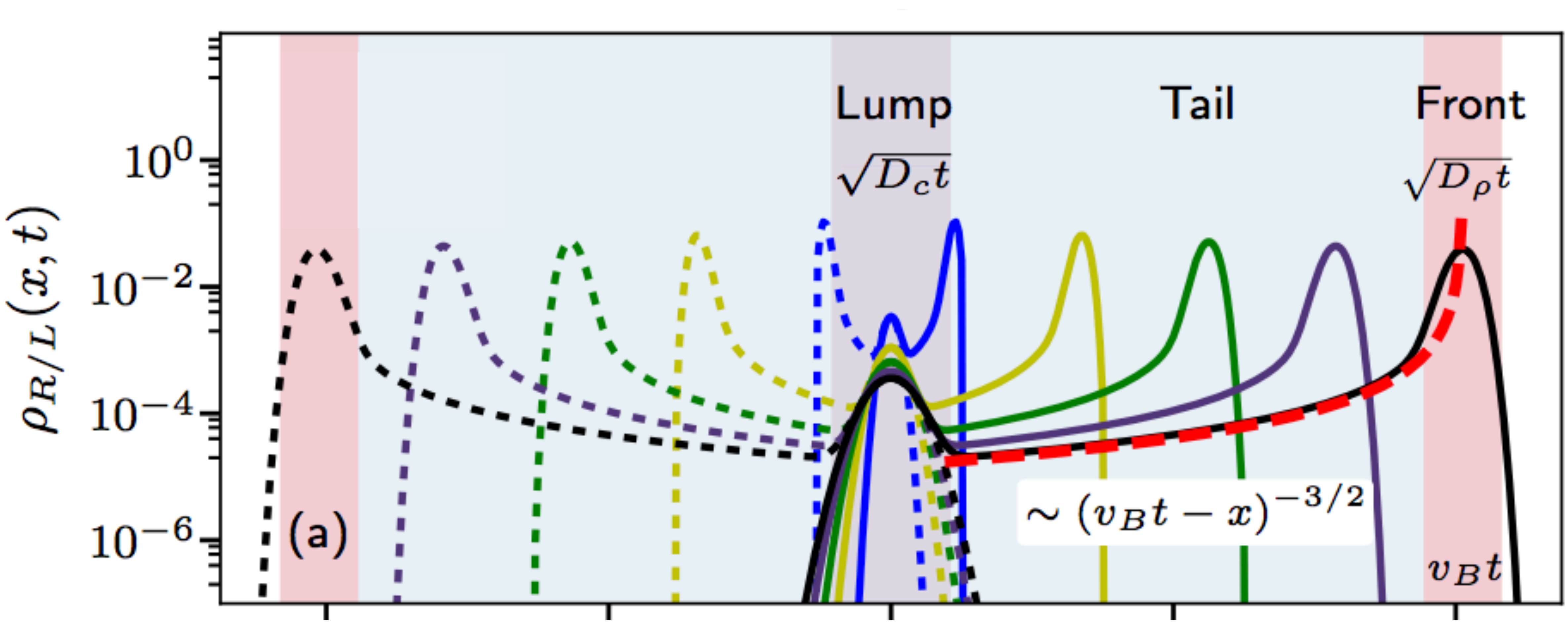
- **Ballistic spreading of non-conserved operators:** Once emitted, the non-conserved parts spread ballistically, quickly becoming non-local and hence non-observable.

- **Diffusive tails behind ballistic front:** Slow diffusive modes lead to power-law "tails" behind the leading ballistic front, coming from ``lagging" fronts emitted at later times. Show up in the OTOC.

Operator shape: conserving circuit



Coupled hydrodynamic description

Diffusion of conserved charges

$$\partial_t a^c(\mathbf{x},t) = D_c \nabla^2 a^c(\mathbf{x},t)$$

Biased diffusion of non-conserved fronts emitted from local gradients in the conserved charges

$$\partial_t \rho_R^{nc}(x,t) = v_B \partial_x \rho_R^{\mathrm{nc}}(x,t) + D_\rho \partial_x^2 \rho_R^{\mathrm{nc}}(x,t) + 2D_c |\partial_x a^c(x,t)|^2$$

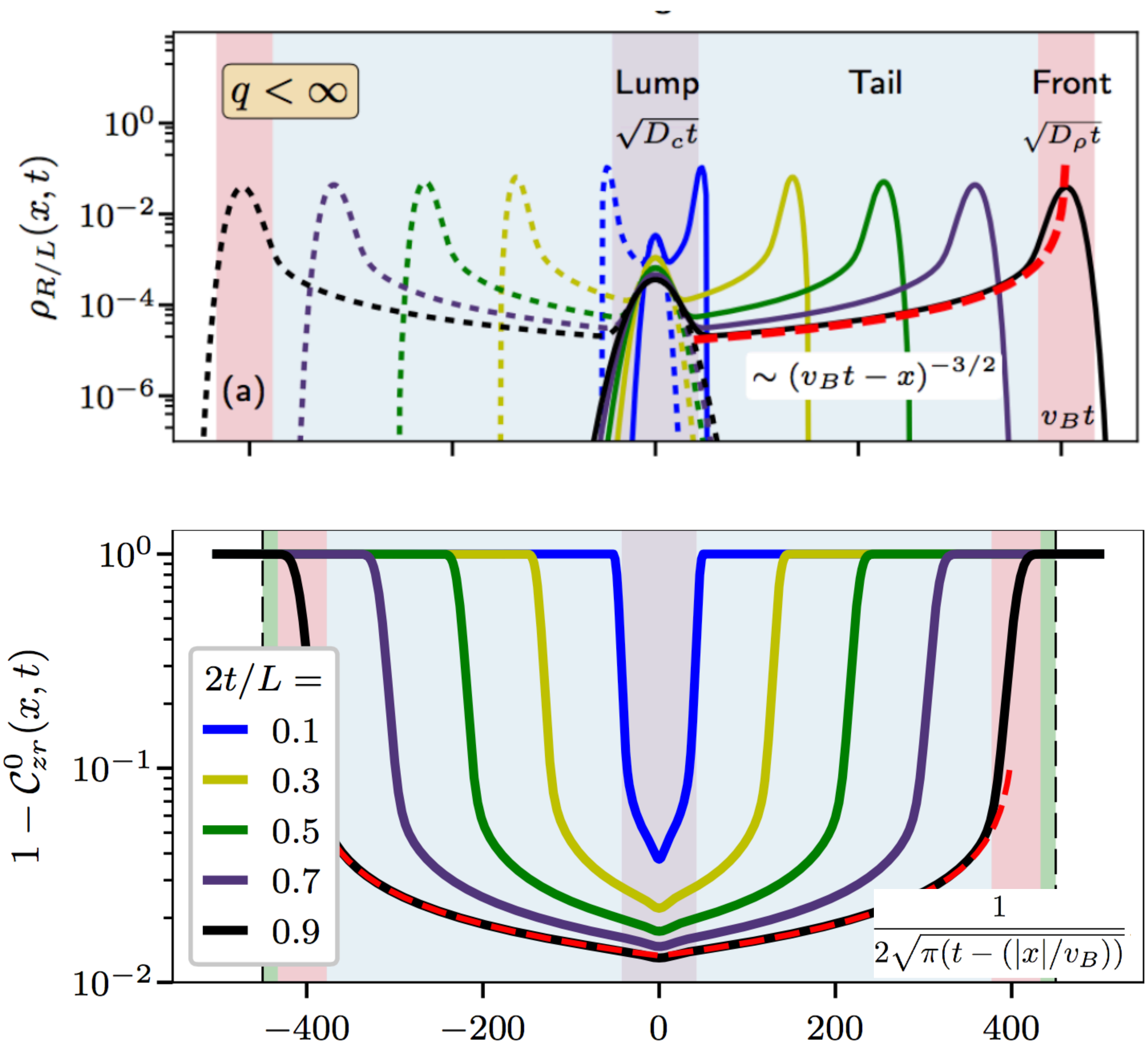
OTOC

OTOC's sensitive to both shape and internal structure

$$\mathcal{C}^{\mu}_{\alpha\beta}(x,t) = \frac{1}{2}\mathrm{Tr}\left\{\rho^{\mathrm{eq}}_{\mu}|[\sigma^{\alpha}_{0}(t), \sigma^{\beta}_{x}]|^{2}\right\}$$

Consider zero "chemical potential" / "infinite temperature"

OTOC: z(t), r



$q < \infty$
Lump
Tail
Front
$\sqrt{D_c t}$
$\sqrt{D_\rho t}$
$\rho_{R/L}(x,t)$
$\sim (v_B t - x)^{-3/2}$
(a)
$v_B t$
$1 - \mathcal{C}^0_{zr}(x,t)$
$2t/L =$
0.1
0.3
0.5
0.7
0.9
$\frac{1}{2\sqrt{\pi(t-(|x|/v_B))}}$

Existence of a Lyapunov Regime

- No simple exponential growth in OTOC

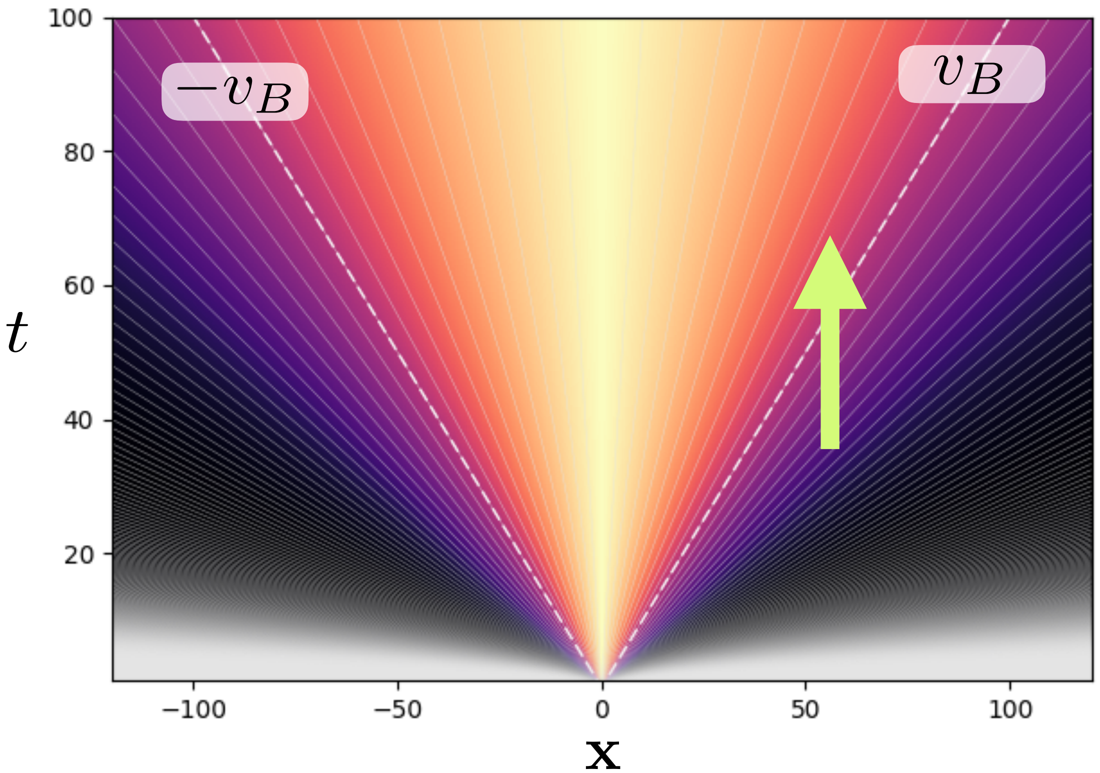
$$C(x_0, t) \sim \exp\left[-\frac{(x_0 - v_B t)^2}{2Dt}\right]$$

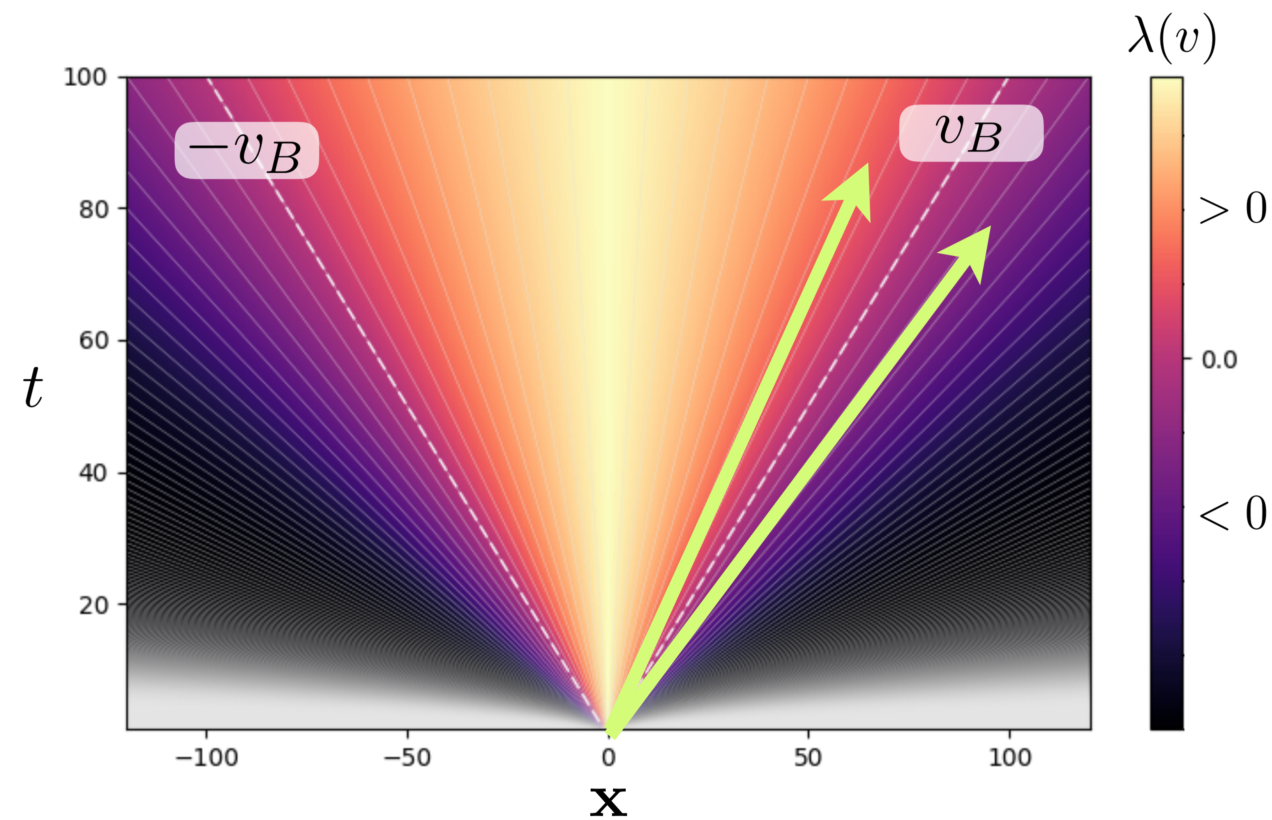
- OTOC does show early-time exponential growth in large N/holographic/semiclassical models

- Requires a small parameter ε in the quantum setting (furnished by 1/N or a weak scattering rate)

$$C \sim \epsilon \, e^{\lambda t} \qquad\qquad t_* = \frac{1}{\lambda} \log \frac{1}{\epsilon}$$

- Spatially local systems potentially have a small parameter because it takes a large time $t_* \sim |x|/v_B$ for a large commutator to build up. Simple exponential regime may *still* not exist due to front broadening





$$C(\mathbf{x_0}, t) = \langle |[V(0,t), W(\mathbf{x_0})]|^2 \rangle$$

$$C(\mathbf{x}, t) \sim e^{\lambda(\mathbf{v})t} \quad \text{for} \quad \mathbf{x} = \mathbf{v}t$$

OTOC at fixed x_0 → OTOC at fixed *v*

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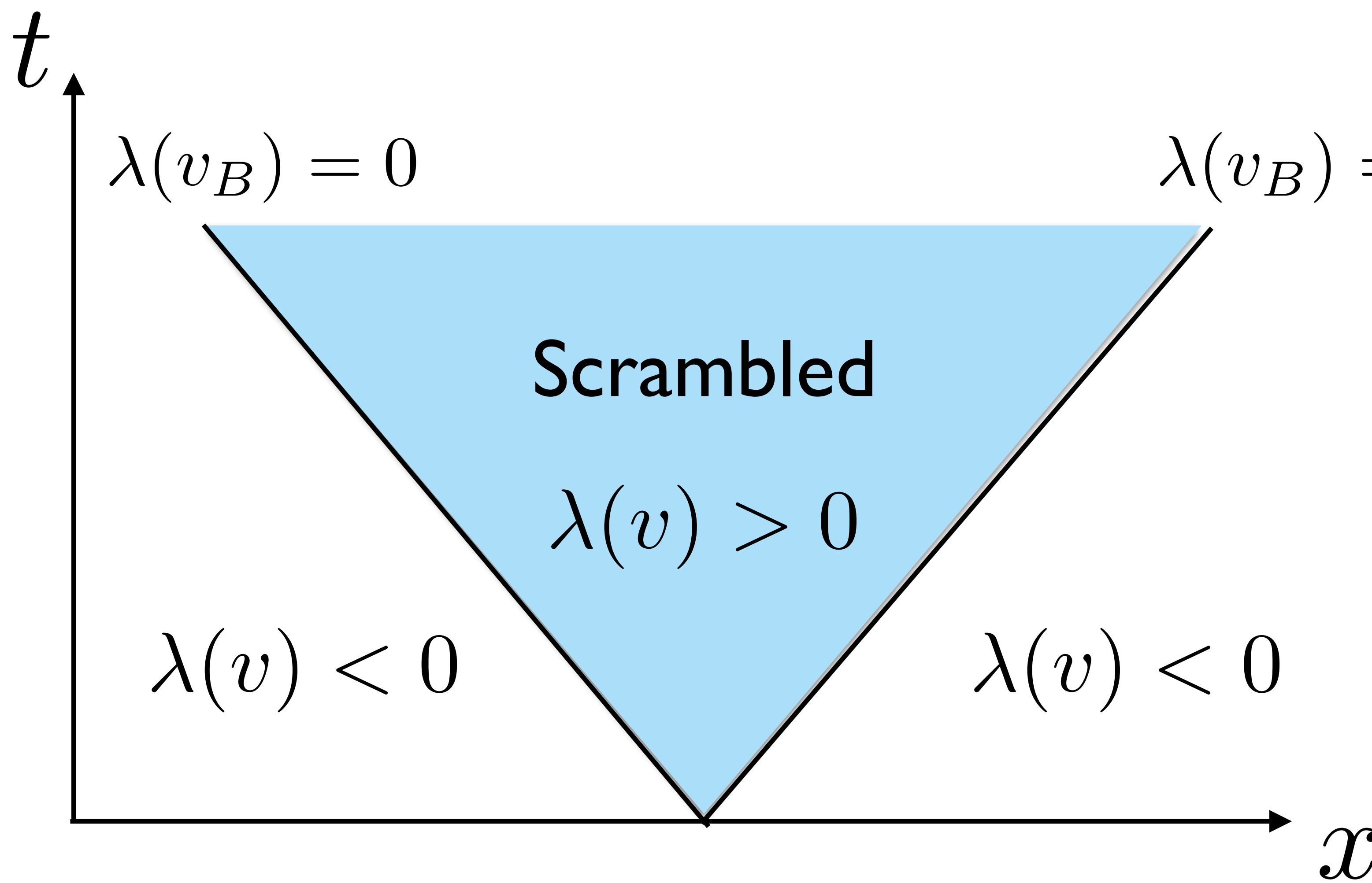
Classical chaos



Classically, C(x,t) grows or decays in time along rays with a velocity dependent Lyapunov exponent

$$C(x = vt, t) \sim e^{\lambda(v)t}$$

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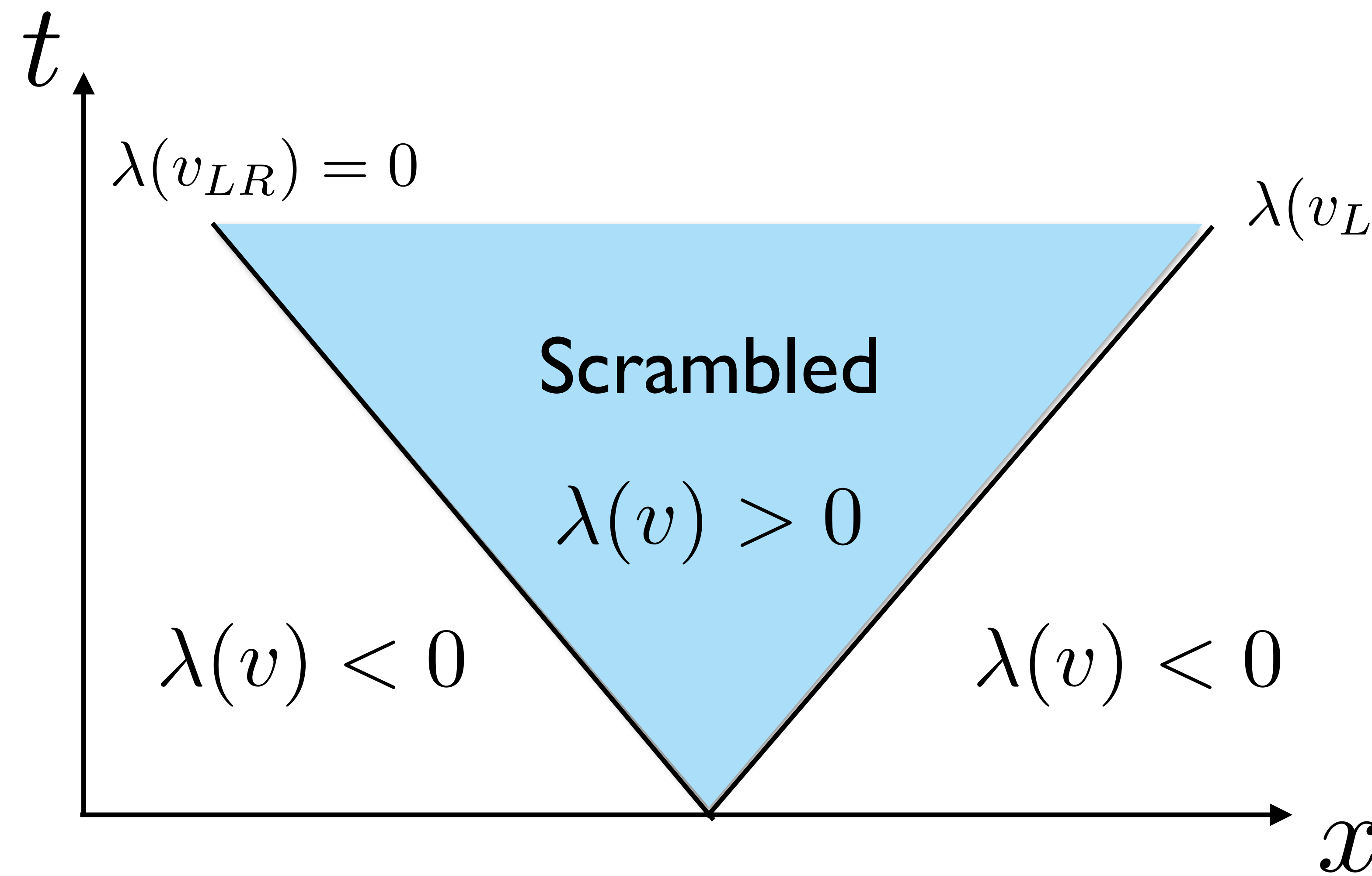
Lieb-Robinson 1972, Deissler, Kaneko 1986

Quantum chaos: large N/ semiclassical



Large N/ semiclassical quantum models show exponential regime:

$$C(x,t) \sim \frac{1}{N^2} e^{\lambda_L (t - |x|/v_B)}$$

e.g. SYK chain (Gu, Qi, Stanford 2016), weakly interacting diffusive metals (Patel et. al. 2017, Aleiner et. al 2016)

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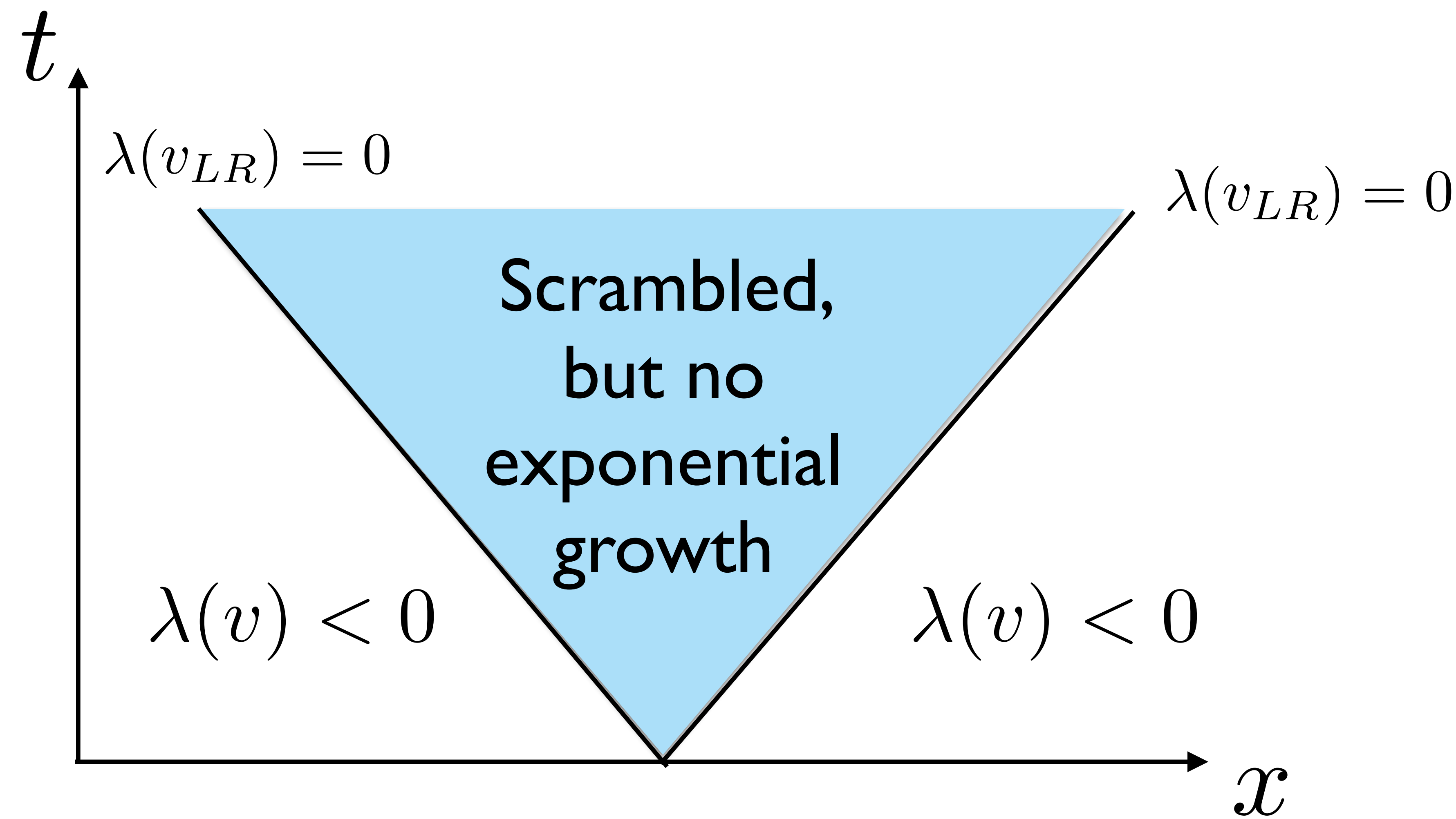
"Strongly quantum chaos"



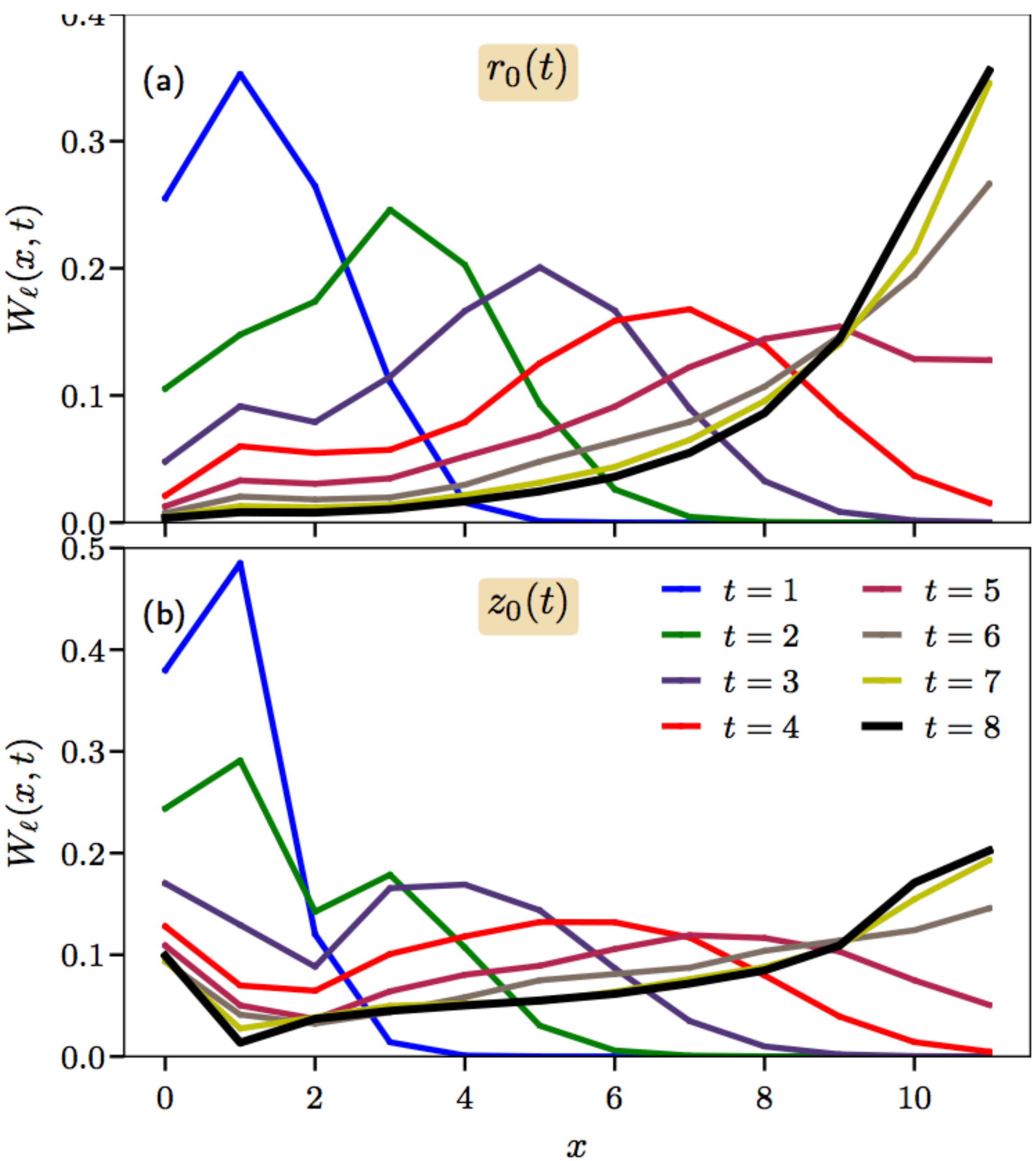
No exponentially growing regime with positive Lyapunov exponents seems to exist (yet?) for "strongly quantum" many-body chaos.

VK, Huse Nahum 2018

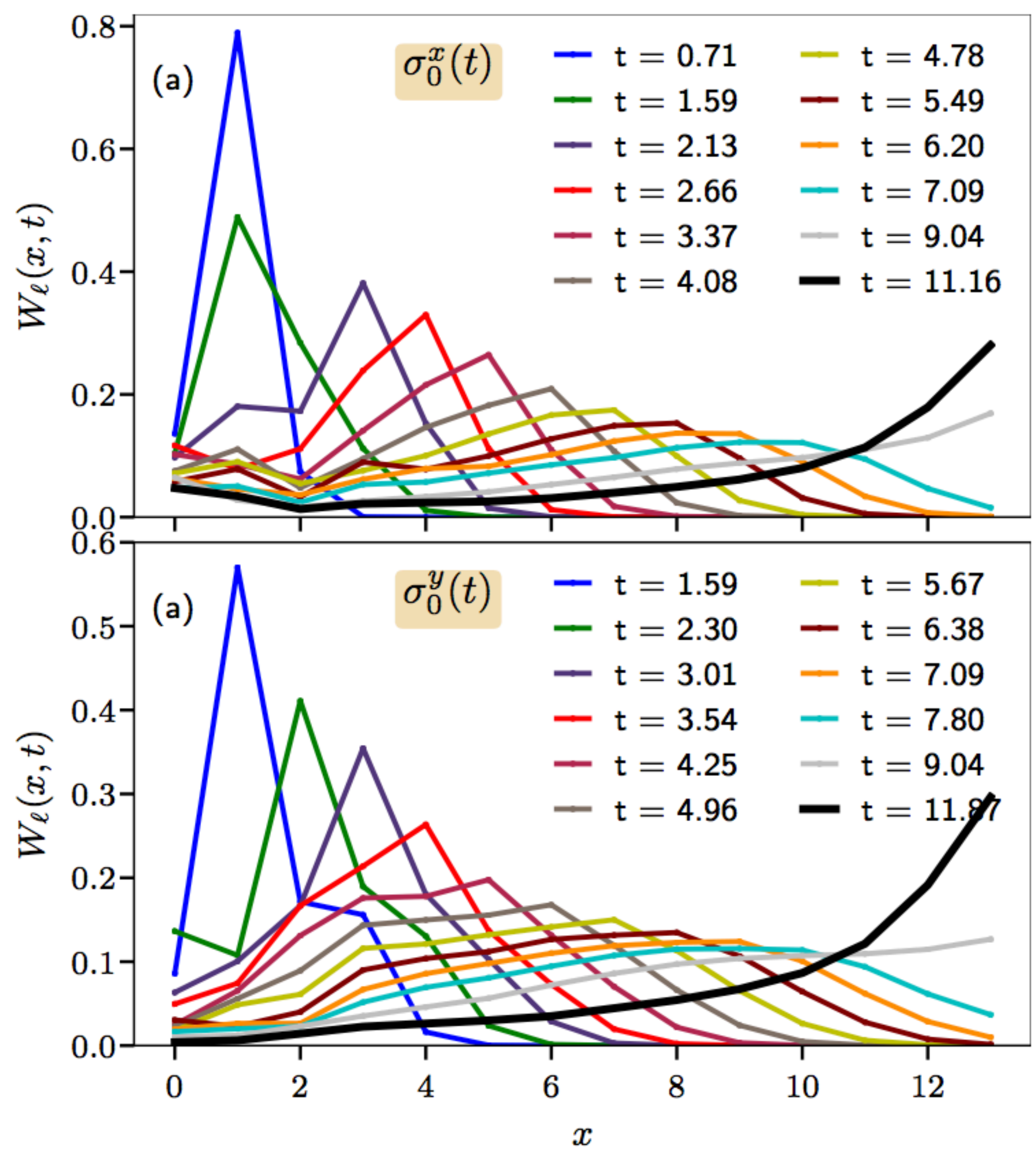
Summary & Outlook

- Operator spreading with diffusive conserved densities governed by multiple coupled diffusive hydrodynamic processes. Distributed over conserved and not conserved operators, with separation of time scales
- Concrete resolution of the fundamental tension between unitarity and dissipation.
- Some Extensions:
  - Understand operator dynamics in other classes of systems? With broken symmetries? Integrable systems?
  - Positive Lyapunov exponents for strongly quantum systems?
  - Entanglement dynamics with conservation laws
  - Connections with Hamiltonian dynamics at finite temperature?
  - Connections with bounds relating D and v_B? Implications for charged black-holes?

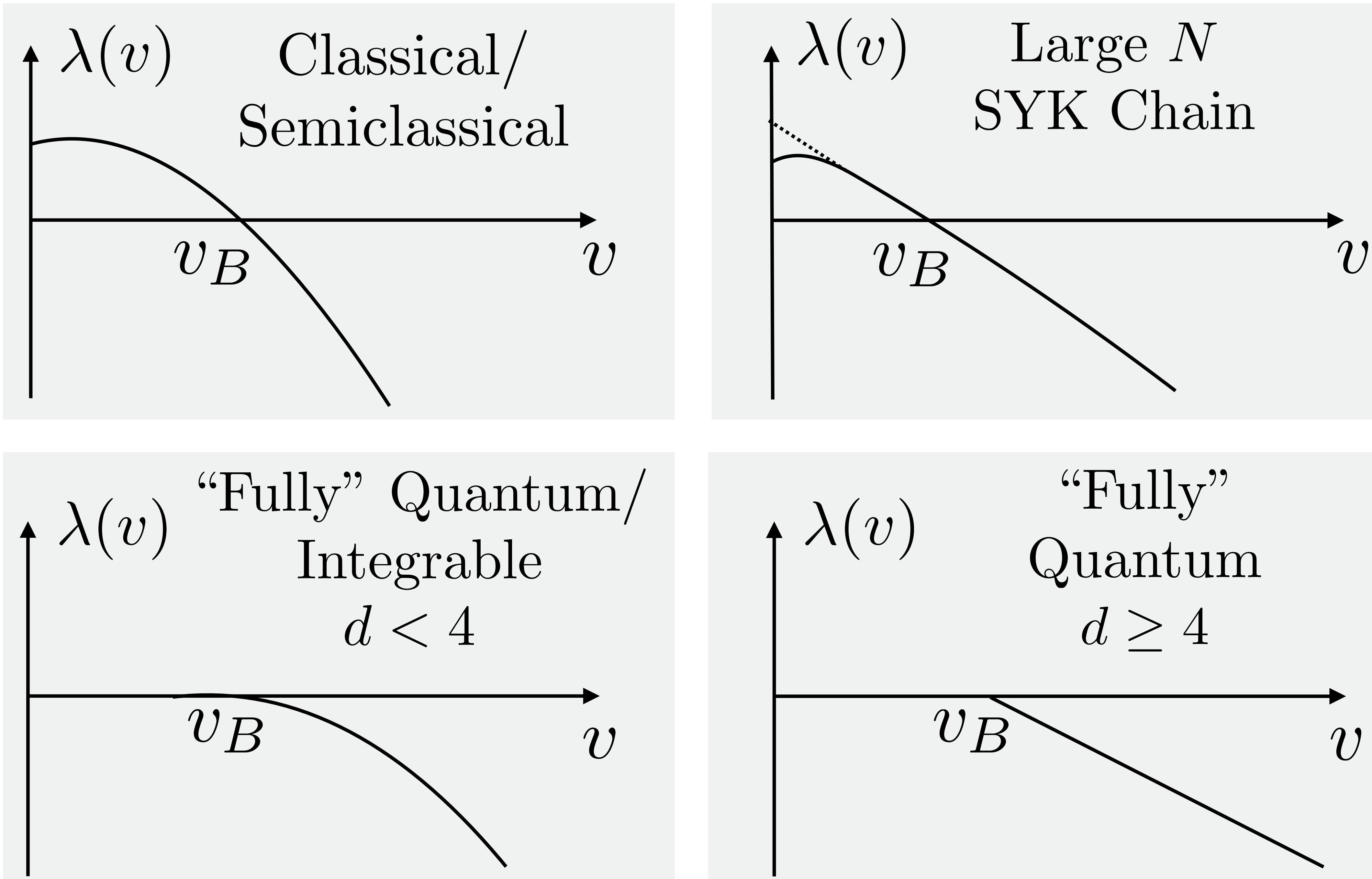
U(1) Floquet



Hamiltonian



λ(v) Forms



Also: Aleiner et. al 2016,
Patel Sachdev 2017