Positivity bounds on Thermalization and Transport

Luca Delacrétaz

Stanford Institute for Theoretical Physics

September 6, 2018

based on 1805.04194 and work in progress
Tom Hartman, **Sean Hartnoll** and **Aitor Lewkowycz**

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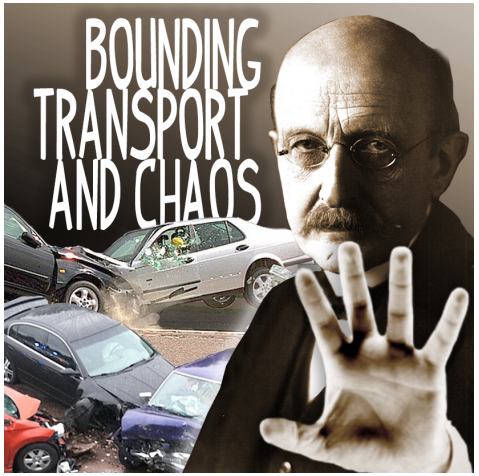

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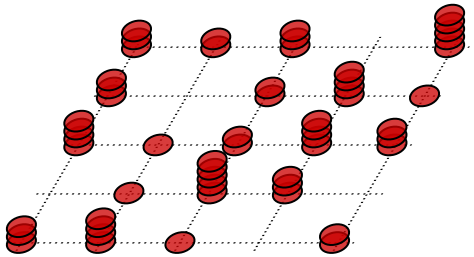
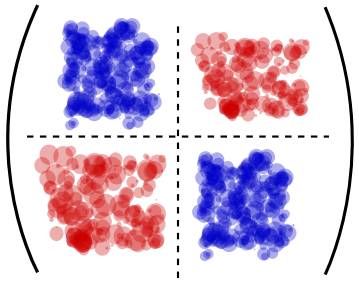
Strategy in a nutshell

Quantum system with a positive-definite, local operator $\mathcal{O}(x) > 0$



This bounds off-diagonal elements $\langle E|\mathcal{O}|E'\rangle \sim$ fluctuations in terms of diagonal elements $\langle E|\mathcal{O}|E\rangle \sim$ thermal equilibrium



This is made quantitatively precise using the ETH Ansatz

[Deutsch '91, Srednicki '94, Rigol Dunjko Olshanii '08]

For $E - E'$ small, the fluctuations are universally constrained by finite temperature hydrodynamics

The resulting bound constrains finite temperature properties of QFTs

Contents

Thermalization time and length

The expectation at finite temperature is that correlation functions decay

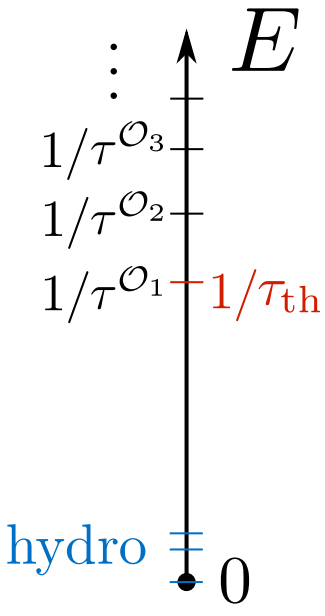
$$\langle \mathcal{O}(t)\mathcal{O}\rangle_\beta \sim e^{-t/\tau_{\rm th}^{\mathcal{O}}}$$

except those of conserved densities: ε, n, $\pi_i, \ldots$

Integrating out these 'thermally gapped' modes leads to a local effective theory for ε, n, $\pi_i, \ldots$

Hydrodynamics is valid at late times and long distances

$$t \gtrsim \tau_{\rm th} \equiv \max_{\mathcal{O}} \tau_{\rm th}^{\mathcal{O}} \qquad\qquad |x| \gtrsim \ell_{\rm th} \equiv \max_{\mathcal{O}} \ell_{\rm th}^{\mathcal{O}}$$



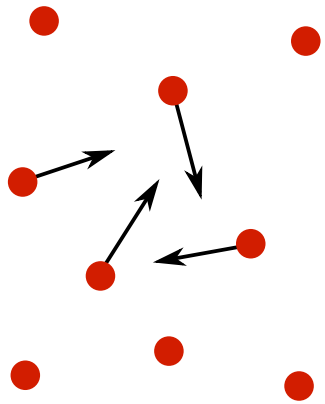
Hydro very successful (QGP, fluid-gravity, cond mat, ...)

Note: difficulty in establishing sharp definition of $\tau_{\rm th}$, $\ell_{\rm th}$
Here they are defined as the cutoff of hydrodynamics

In systems with quasiparticles, the quasiparticle life-time τ_{qp} plays an important role in transport

[Drude 1900]

$$\sigma_{\text{dc}} = \frac{ne^2}{m}\tau_{\text{qp}}$$



At weak coupling (Fermi-Liquid theory) $\tau_{\text{th}} = \tau_{\text{qp}} \sim 1/g^2$

Without quasiparticles, τ_{th} generalizes τ_{qp}

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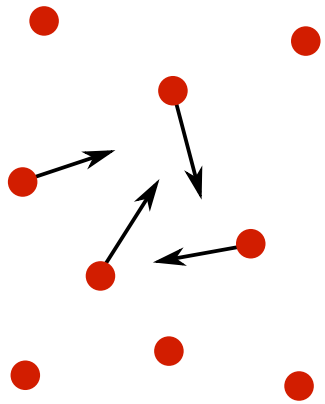

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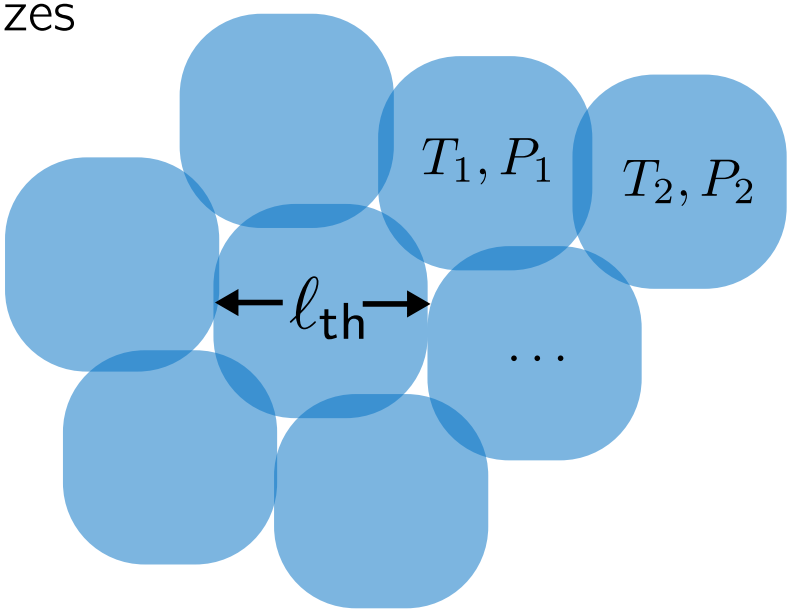
Without quasiparticles, τ_{th} generalizes τ_{qp}

Similarly, thermalization length ℓ_{th} generalizes the notion of mean free path

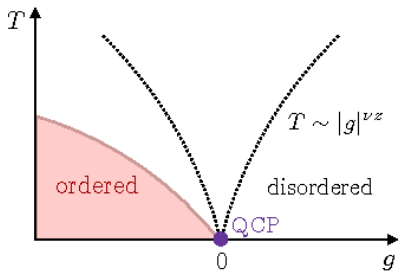
Smallest blob that can reach local thermal equilibrium

Hydrodynamics describes $T(x), P(x)$ on length scales $x \gg \ell_{\text{th}}$



$$\left.\begin{array}{rl} \text{Fermi-Liquid:} & \tau_{\text{th}} \sim \dfrac{1}{\lambda^2}\dfrac{E_F}{k_B T}\dfrac{\hbar}{k_B T} \\ \text{Large } N \text{ vector:} & \tau_{\text{th}} \sim N\dfrac{\hbar}{k_B T} \\ \epsilon\text{-expansion:} & \tau_{\text{th}} \sim \dfrac{1}{\epsilon^2}\dfrac{\hbar}{k_B T} \end{array}\right\} \gg \frac{\hbar}{k_B T}$$

Large N matrix models (SYK, holography) have $\tau_{\text{th}} \sim \dfrac{\hbar}{k_B T}$



Natural to expect a lower bound

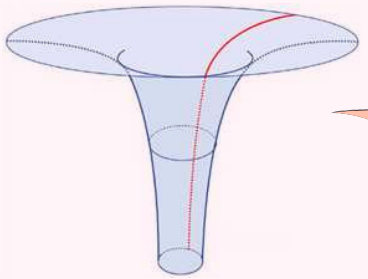
$$\tau_{\text{th}} \geq \# \frac{\hbar}{k_B T}$$

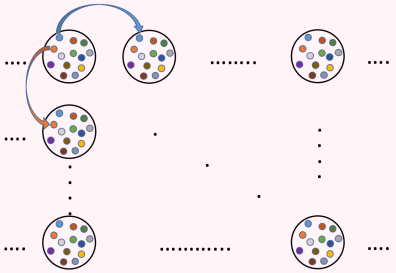
Analogous to chaos bound $\tau_L \geq \dfrac{1}{2\pi}\dfrac{\hbar}{k_B T}$ [Maldacena Shenker Stanford '15]

where $\langle [A(t), B]^2 \rangle \sim e^{t/\tau_L}$, but τ_{th} is relevant for transport!

Large N

ϵ expansion



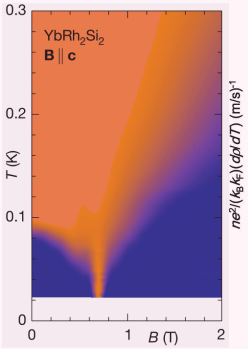


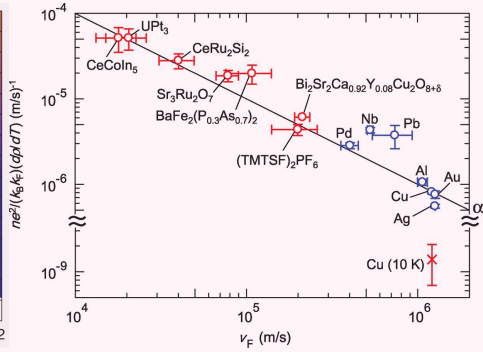
theory

Thermalization

Strange metals
[Chubukov Sachdev '93 ...

Custers et al '03, Mackenzie et al '13]





Heavy fermions, cuprates

...

numerics

ED – small systems

QMC – hard to access real time dynamics ...

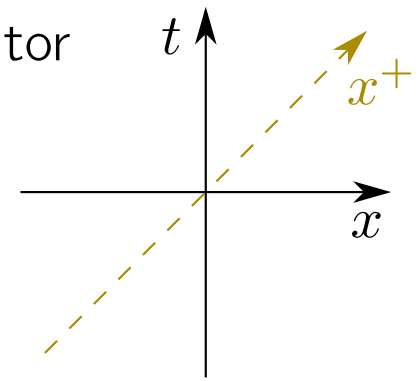
experiment

Contents

Posivity in relativistic QFT

Lorentz invariant QFTs contain a positive definite operator

$$\mathcal{O} \equiv \int dx^+ T_{++} \geq 0$$



where $x^+ = x + t$ and $T_{++} = T_{tt} + 2T_{tx} + T_{xx}$

energy density momentum density 'pressure'

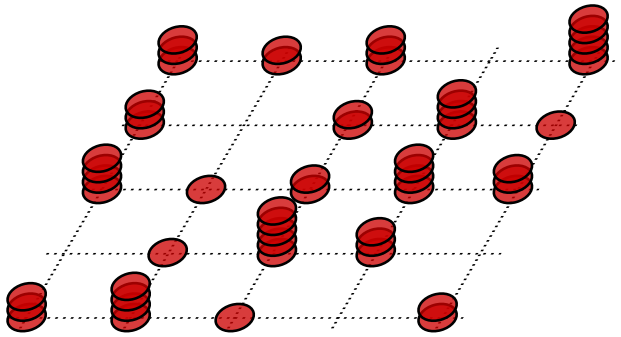
[Faulkner Leigh Parrikar Wang '16]

[Hartman Kundu Tajdini '16]

Positivity in lattice systems

Occupation number, energy density, etc. are bounded below in lattice systems

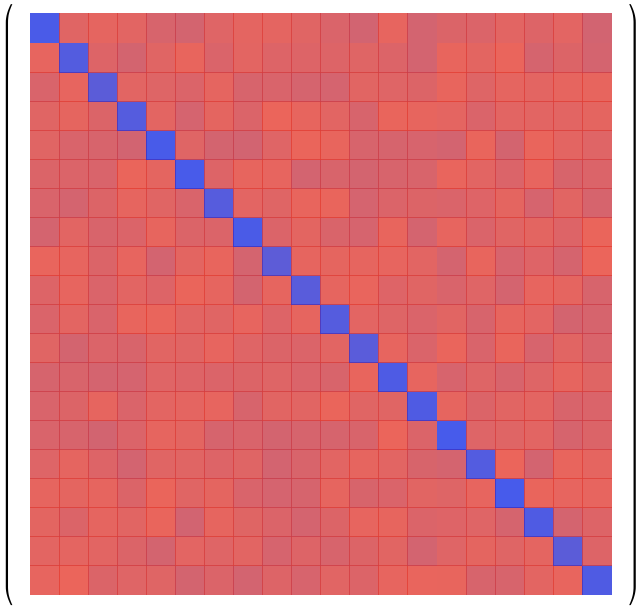
$$\mathcal{O} \equiv \hat{n} - n_{\min} \geq 0$$



$\mathcal{O}^{ab} \equiv \langle a|\mathcal{O}|b\rangle$ is a positive matrix in any sub-Hilbert space

diagonal $\geq$ off-diagonal

$\mathcal{O}^{ab} =$



Equilibrium thermodynamics

Fluctuations (thermalization, transport)

To make this picture quantitative, we will use the ETH Ansatz:

[Deutsch '91]

[Srednicki '94]

$$\langle a|\mathcal{O}|b\rangle = \delta_{ab}\langle\mathcal{O}\rangle_T + e^{-S/2}\sqrt{G_{\mathcal{O}\mathcal{O}}}R_{ab}$$

with states in a microcanonical window $E_a,\, E_b \in [E - \frac{\Delta E}{2}, E + \frac{\Delta E}{2}]$

'generic' operator thermal exp. value thermal Green's function

ETH Ansatz: $$\langle a|\mathcal{O}|b\rangle = \delta_{ab}\langle\mathcal{O}\rangle_T + e^{-S/2}\sqrt{G_{\mathcal{O}\mathcal{O}}(\omega)}R_{ab}$$ [Deutsch '91] [Srednicki '94]

energy eigenstates density of states random matrix

The operator need not be local [Garrison Grover '15] (but it helps)

Proof: Let us write $\langle a|\mathcal{O}|b\rangle = \delta_{ab}\langle\mathcal{O}\rangle_T + f(E_a, E_b)R_{ab}$ and find f

'generic' operator thermal exp. value thermal Green's function

ETH Ansatz: $$\langle a|\mathcal{O}|b\rangle = \delta_{ab}\langle\mathcal{O}\rangle_T + e^{-S/2}\sqrt{G_{\mathcal{O}\mathcal{O}}(\omega)}R_{ab}$$ [Deutsch '91] [Srednicki '94]

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Proof: Let us write $\langle a|\mathcal{O}|b\rangle = \delta_{ab}\langle\mathcal{O}\rangle_T + f(E_a, E_b)R_{ab}$ and find f

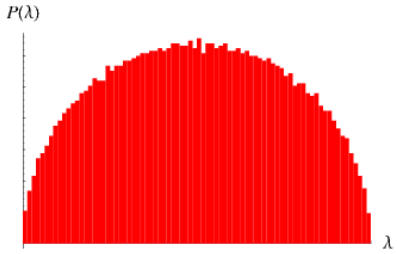
$$\begin{aligned} G_{\mathcal{O}\mathcal{O}}(\omega) &= \int dt\, e^{i\omega t}\langle\mathcal{O}(t)\mathcal{O}\rangle_T \\ &= \sum_b \int dt\, e^{i\omega t}\langle a|\mathcal{O}(t)|b\rangle\langle b|\mathcal{O}|a\rangle \\ &= \sum_b \delta(\omega - E_b + E_a)|f_{ab}|^2 \underbrace{|R_{ab}|^2}_{\to 1} \end{aligned}$$

Then taking $\sum_b \to \int dE_b\, \Omega(E_b)$ gives $|f_{ab}|^2 = \frac{1}{\Omega(E_b)}G_{\mathcal{O}\mathcal{O}}(E_b - E_a)$

ETH Ansatz: $\langle a|\mathcal{O}|b\rangle = \delta_{ab}\langle\mathcal{O}\rangle_T + e^{-S/2}\sqrt{G_{\mathcal{O}\mathcal{O}}(\omega,k)}R_{ab}$

applied to positive $\mathcal{O} = \int dx^+ T_{++}$ or $\mathcal{O} = \hat{n} - n_{\min}$.

Eigenvalue repulsion of R_{ab} can overcome $e^{-S/2}$ suppression



Specifically, a real symmetric matrix A_{ab} satisfies

$$\lambda_{\max}^2 \geq \frac{1}{N}\sum_{ab}|A_{ab}|^2$$

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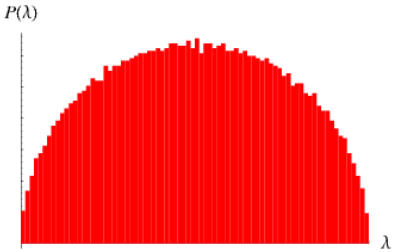

Specifically, a real symmetric matrix A_{ab} satisfies

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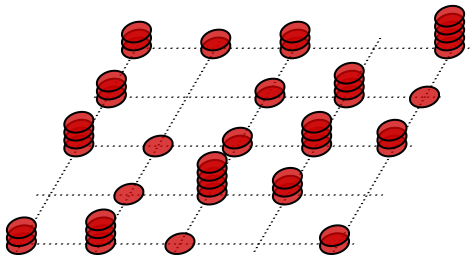
which gives schematically

$$|\langle\mathcal{O}\rangle|^2 \geq \frac{1}{N}\sum_{ab} e^{-S} G_{\mathcal{O}\mathcal{O}}(\omega,k)$$

$(n - n_{\min})^2$

choose narrow window so that ω, k small, and $G_{\mathcal{O}\mathcal{O}}(\omega,k)$ controlled by hydrodynamics

$$|\langle \mathcal{O} \rangle|^2 \geq \frac{1}{N} \sum_{ab} e^{-S} G_{\mathcal{O}\mathcal{O}}(\omega, k) \quad \text{with} \quad \mathcal{O} = \hat{n} - n_{\min}$$

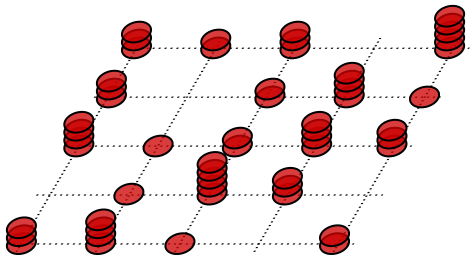


The RHS (fluctuations) will be universally fixed by hydrodynamics if

$$\omega \lesssim \frac{2\pi}{\tau_{\text{th}}} \qquad\qquad k \lesssim \frac{2\pi}{\ell_{\text{th}}}$$

Many things can happen in the IR (e.g. symmetry breaking) and $G_{\mathcal{O}\mathcal{O}}$ will reflect that. Generically, bound gives UV constraints on IR physics.

$$|\langle \mathcal{O} \rangle|^2 \geq \frac{1}{N} \sum_{ab} e^{-S} G_{\mathcal{O}\mathcal{O}}(\omega, k) \quad \text{with} \quad \mathcal{O} = \hat{n} - n_{\min}$$



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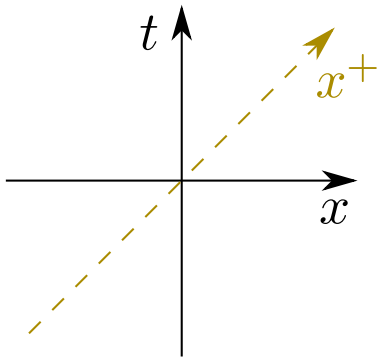
Diffusive dynamics at late times

$$\langle j_i \rangle = -D\nabla_i n + O(\ell_{\text{th}}^2 \nabla^2) \qquad \Rightarrow \qquad G^R_{\mathcal{O}\mathcal{O}}(\omega, k) = \frac{\chi D k^2}{-i\omega + Dk^2}$$

Corrections due to hydrodynamic fluctuations can (and should) be taken into account – they can give important corrections even for $\omega < 2\pi/\tau_{\text{th}}$

[Mukerjee Oganesyan Huse '05, Kovtun '03]

$$|\langle \mathcal{O} \rangle|^2 \geq \frac{1}{N} \sum_{ab} e^{-S} G_{\mathcal{O}\mathcal{O}}(\omega, k) \quad \text{with} \quad \mathcal{O} = \int dx^+ \, T_{++}$$



IR divergent expectation value $|\langle \mathcal{O} \rangle| \sim \varepsilon L$!
need to regulate with finite volume states

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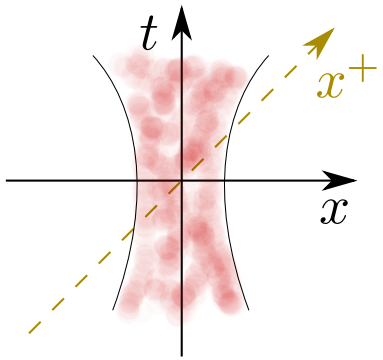

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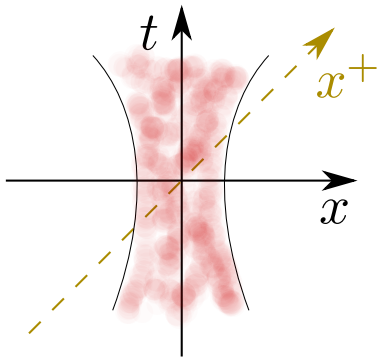

IR divergent expectation value $|\langle\mathcal{O}\rangle| \sim \varepsilon L$!
need to regulate with finite volume states

Hydrodynamics: **Lorentz** and **translation** invariant IR

$$\langle T^{\mu\nu}\rangle = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} - \eta\Delta^{\mu\alpha}\nabla_{\langle\alpha}u_{\beta\rangle}\Delta^{\beta\nu} - \zeta\Delta^{\mu\nu}\nabla_\alpha u^\alpha + \mathcal{O}(\ell_{\text{th}}^2\nabla^2)$$

with $\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$.

Solve hydro eom for u^μ and ϵ to get $\langle T^{\mu\nu}(g)\rangle \quad \rightsquigarrow \quad G_{TT} \sim \frac{\delta\langle T\rangle}{\delta g}$.

sound: $\omega = \pm c_s k - \frac{i}{2}\Gamma_s k^2 + \ldots \qquad \Gamma_s = \frac{\zeta + \frac{4}{3}\eta}{sT}$

diffusion: $\omega = -iD_\perp k^2 + \ldots \qquad D_\perp = \frac{\eta}{sT}$

Contents

The bound:

$$(\varepsilon+P)^2 \geq \frac{64\pi^3}{L^5 \sinh \frac{\Delta\omega}{2T}} \int_{-\Delta\omega}^{\Delta\omega} d\omega \frac{\sin^2 \frac{L\omega}{2}}{\omega^2} \frac{\sinh \frac{1}{2T}(\Delta\omega - |\omega|)}{\sinh \frac{\omega}{2T}} \operatorname{Im} G^R_{T_{++}T_{++}}(\omega, k_{\mathsf{min}})$$

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$|\langle \mathcal{O} \rangle|^2$ size of window from dx^+ integral fluctuation-dissipation fixed by hydro

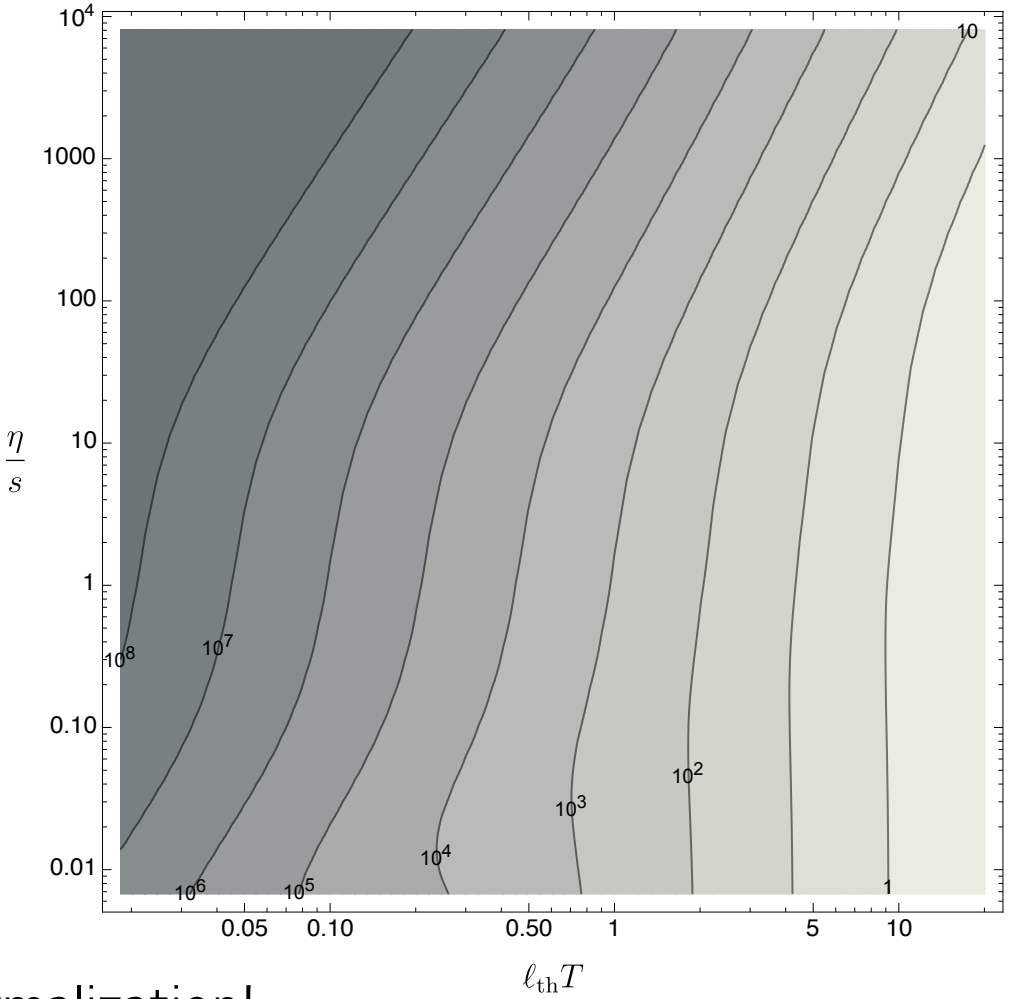
Writing $s = s_o T^3$, it has the form

$$s_o \geq \mathcal{F}\left(\frac{\eta}{s}, \frac{\zeta}{s}, c_s, \ell_{\text{th}} T, \tau_{\text{th}} T\right)$$

for $N \to \infty$, $s_o \sim N^2$ bound becomes weak.

(free scalar: $s_o = \frac{4\pi^2}{45}$. QGP at large T: $s_o \simeq 20$)

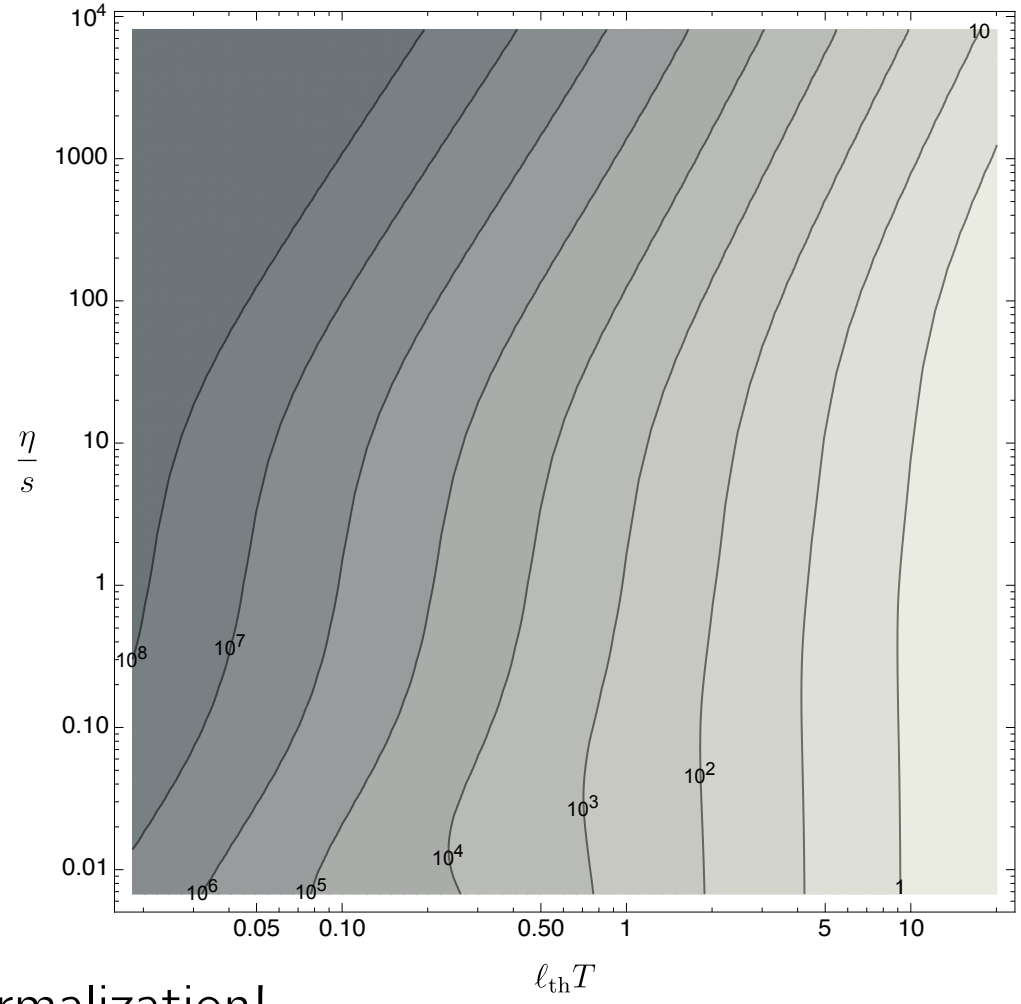
Exclusion plot $s_o \geq \mathcal{F}\left(\frac{\eta}{s}, \frac{\zeta}{s}, c_s, \ell_{\text{th}}T, \tau_{\text{th}}T\right)$



Bound on thermalization!

$$s\ell_{\text{th}}^3 \geq 4\pi^3 \int_0^\infty \frac{y(a_1 + a_2 y^2)}{(e^y - 1)(a_3 + a_4 y^2)} dy$$

Exclusion plot $s_o \geq \mathcal{F}\left(\frac{\eta}{s}, \frac{\zeta}{s}, c_s, \ell_{\text{th}}T, \tau_{\text{th}}T\right)$



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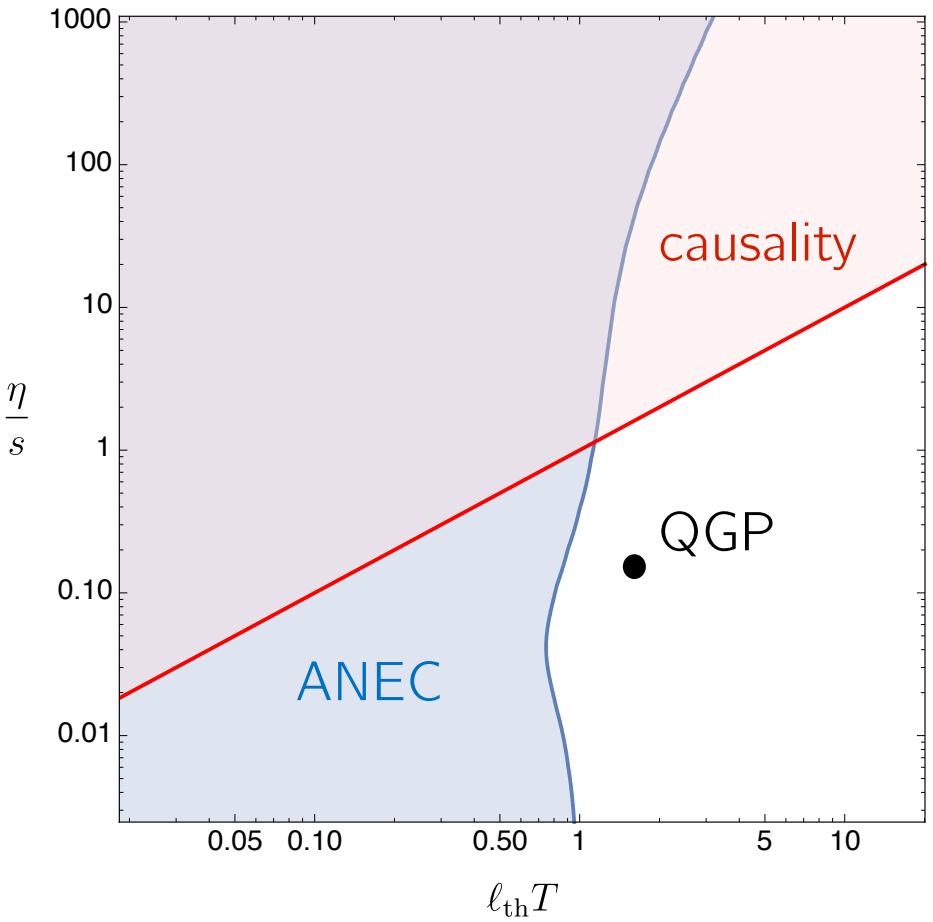
$$\sim \frac{1}{c_s^2}$$

$$\sim \frac{\eta}{s}$$

Application of relativistic bound: Quark Gluon Plasma



causality bound:
$D_\perp \leq c\,\ell_{\rm th}$
[Hartman Hartnoll Mahajan '17]

ANEC bound:
$\ell_{\rm th}T \gtrsim 1$

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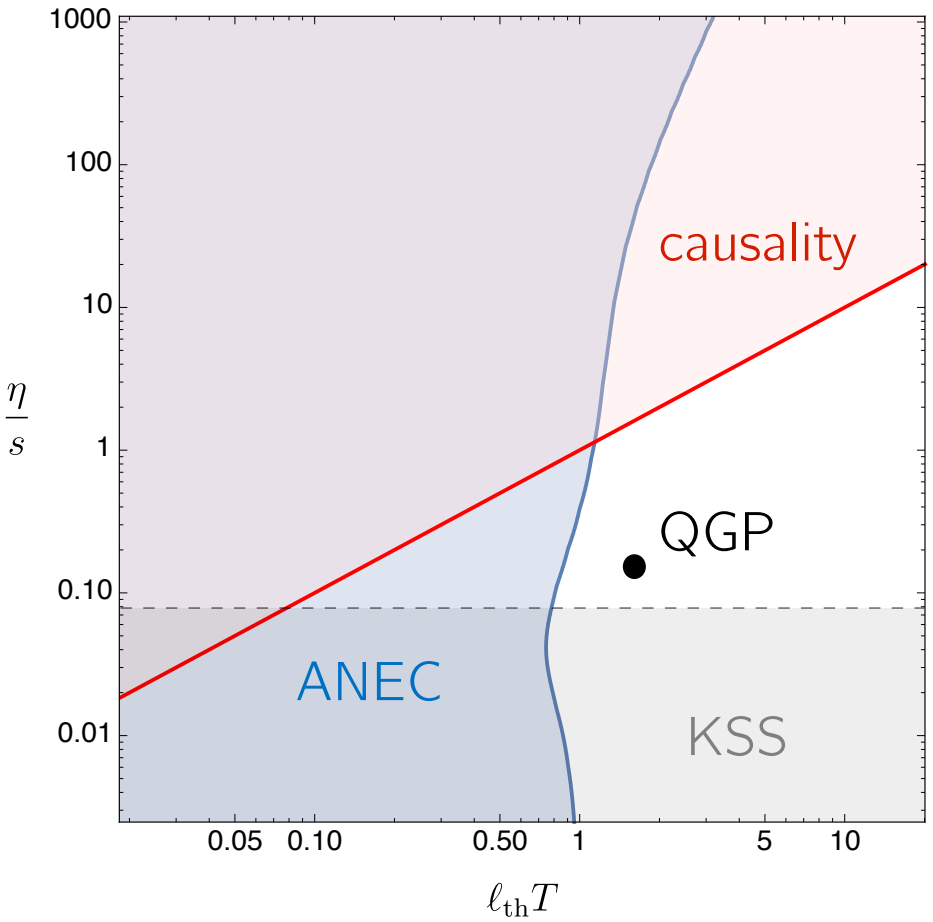

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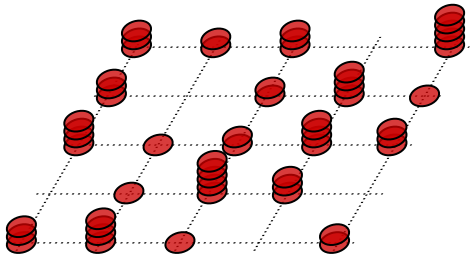
$D_\perp \leq c\,\ell_{\rm th}$
[Hartman Hartnoll Mahajan '17]

ANEC bound:
$\ell_{\rm th}T \gtrsim 1$

KSS limit:
$\eta/s \geq 1/4\pi$
[Kovtun Son Starinets '04]

Lattice bound: (with a few extra assumptions)



$$(n - n_{\min})^2 \ell_{\text{th}}^d \gtrsim \chi T$$

Application: Hubbard model

$$H = t \sum_{<ij>} c_i^\dagger c_j + U \sum_i n_i^2$$

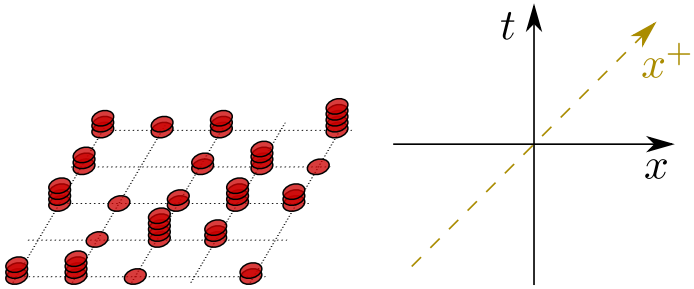
At high T, $\chi \sim n/T$ so we find

$$n \ell_{\text{th}}^d \gtrsim 1$$

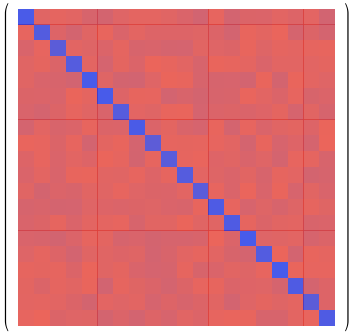
Non-quasiparticle derivation of a MIR-like bound!

Summary:

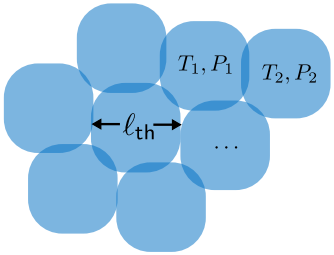
(1) relativistic QFTs and latticess have $\mathcal{O} \geq 0$



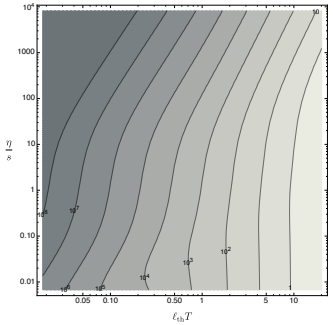
(2) ETH makes precise diagonal $\geq$ off-diagonal



(3) off-diagonal terms universally controlled by hydrodynamics



(4) certain parameter regions violate positivity!



Outlook:

- Can positivity+ETH bounds explain $\rho \sim T$?
- Look for violations of these bounds to probe for breakdown of ETH
- CFT perspective: ETH $\Rightarrow$ transport=4pt function. Conformal bootstrap?
- Sharper definition of τ_{th}, ℓ_{th} needed for sharp bounds!

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Thanks!