Instabilities of 2d quantum critical metals in the $N_f \to 0$ limit

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NORDITA

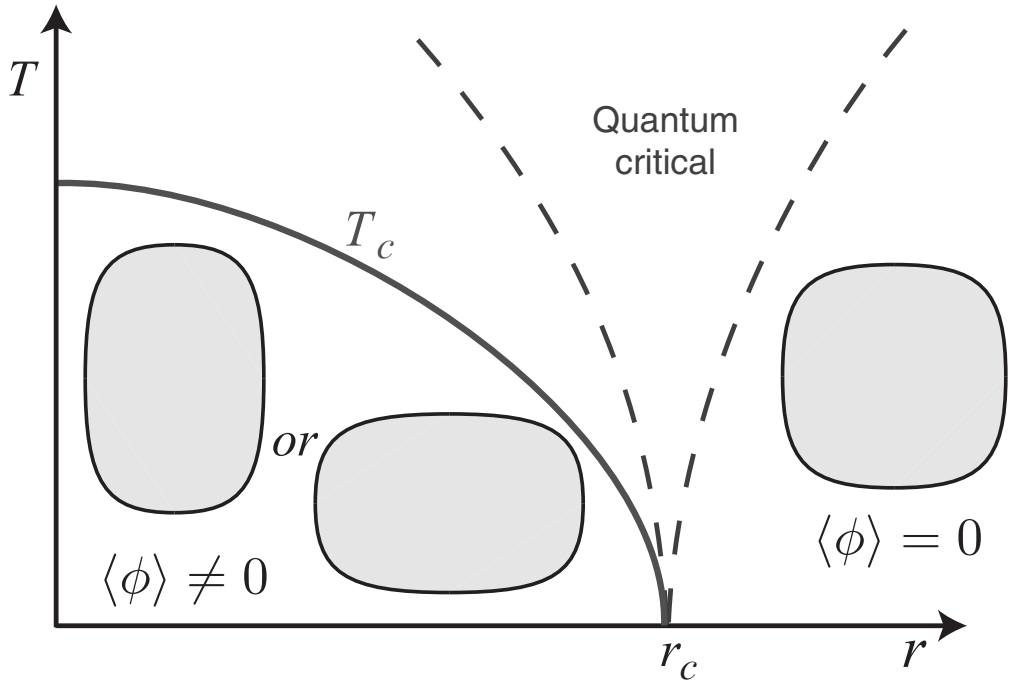
Based on:
[SciPost Phys. 4, 015 (2018)] PS
[arXiv: 18XX.XXX] PS

Quantum critical metals, $Q = 0$

Fermions at finite density coupled to critical bosons:

$$S = \int \mathrm{d}x^{d+1} \left[\psi^\dagger (\partial_\tau - \frac{\nabla^2}{2m} - \mu)\psi + \frac{(\partial_\tau \phi)^2}{2} + \frac{(\nabla \phi)^2}{2} + (r - r_c)\phi^2 + \lambda \phi \psi^\dagger \psi \right]$$

ϕ is an order parameter field of e.g. Ising-nematic transition
Putative model of some high-T_c materials.



Quantum critical metals

$$S = \int \mathrm{d}x^{d+1} \left[\psi^\dagger (\partial_\tau - \frac{\nabla^2}{2m} - \mu)\psi + \frac{(\partial_\tau \phi)^2}{2} + \frac{(\nabla \phi)^2}{2} + (r - r_c)\phi^2 + \lambda \phi \psi^\dagger \psi \right]$$

λ scaling dimension: $(d-3)/2$

$d = 2 \implies$ interaction is relevant
Fermion sign-problem generally prohibits conventional Monte-Carlo methods. However, there are some numeric results in sign-problem free cases.[1]
Full description in $d = 2$ still an open problem.

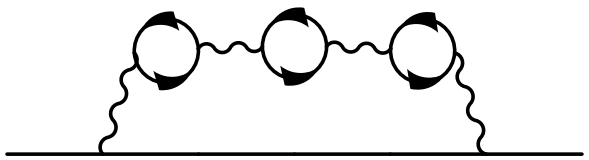
[1] Y. Schattner, S. Lederer, S. Kivelson, E. Berg, PRX **6** (2016) 031028

Previous studies: Vector large N_f, $U(N_f)$

N_f fermion flavors and interaction:

$$S_{\text{int}} = \frac{\lambda}{\sqrt{N_f}} \int \mathrm{d}x^{d+1} \phi \psi_i^\dagger \psi_i$$

Large N_f gives strong Landau-damping $\implies$ RPA two-point functions



$$D_{\text{RPA}} = \frac{1}{k^2 + \gamma \frac{|\omega|}{|k|}} \qquad \Sigma_{\text{RPA}} = \frac{-i\lambda^{4/3}\text{sgn}(\omega)|\omega|^{2/3}}{(2\pi)^{2/3}\sqrt{3}(N_f k_F)^{1/3}}$$

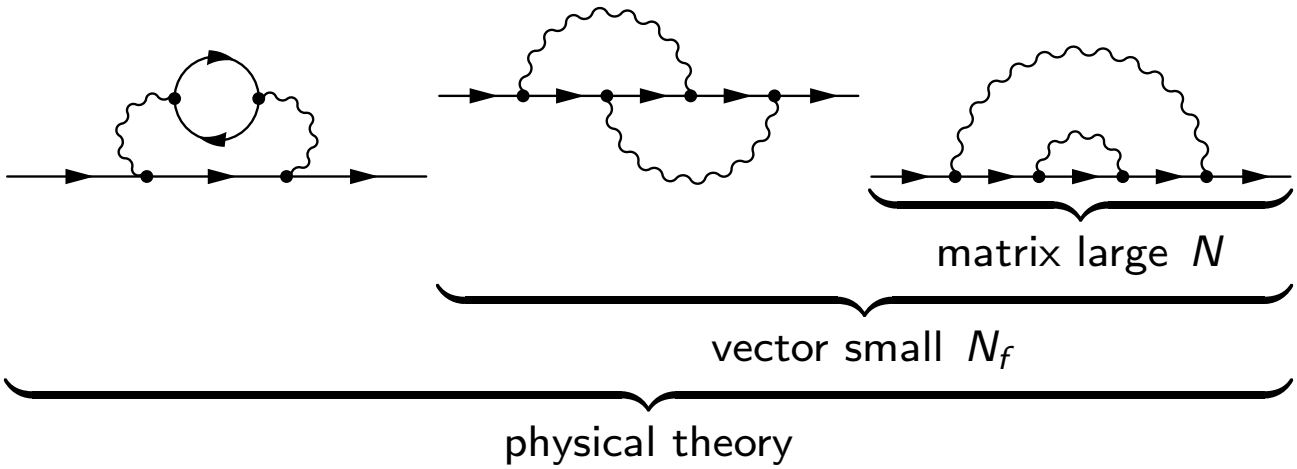
This doesn't capture the correct low energy physics[2] [3] [4]

[2] Sung-Sik Lee, PRB **80**, 165102 (2009)
[3] M. A. Metlitski and S. Sachdev, PRB **82** (2010) 075127
[4] T. Holder, W. Metzner, PRB **92** (2015), 041112

Previous studies: Boson-dominated limits

- Matrix large N [5]: $S_{\rm int} = \frac{\lambda}{\sqrt{N}} \int \mathrm{d}x^{d+1} \psi_i^\dagger \phi_{ij} \psi_j$
  Only planar diagrams, no fermion loops
- Vector small N_f: $S_{\rm int} = \lambda \int \mathrm{d}x^{d+1} \phi \psi_i^\dagger \psi_i$
  No fermion loops

Fermion self-energy contributions at order λ^4:



Boson receives no corrections, similar to the probe fermions in holography where fermions are coupled to a quantum critical system[6].

[5] A. L. Fitzpatrick, S. Kachru, J. Kaplan, S. Raghu, PRB **89** (2014), 165114
[6] H. Liu, J. McGreevy, and D. Vegh PRD **83** (2009) 065029

End of introduction

Setup

We consider the following action[7]:

$$S = \int \mathrm{d}\tau \mathrm{d}^2 x \left[\psi^\dagger_{i,\sigma} \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi_{i,\sigma} + \frac{1}{2} (\partial_\tau \phi)^2 + \frac{1}{2} (\nabla \phi)^2 - \phi(x) O(x) \right]$$

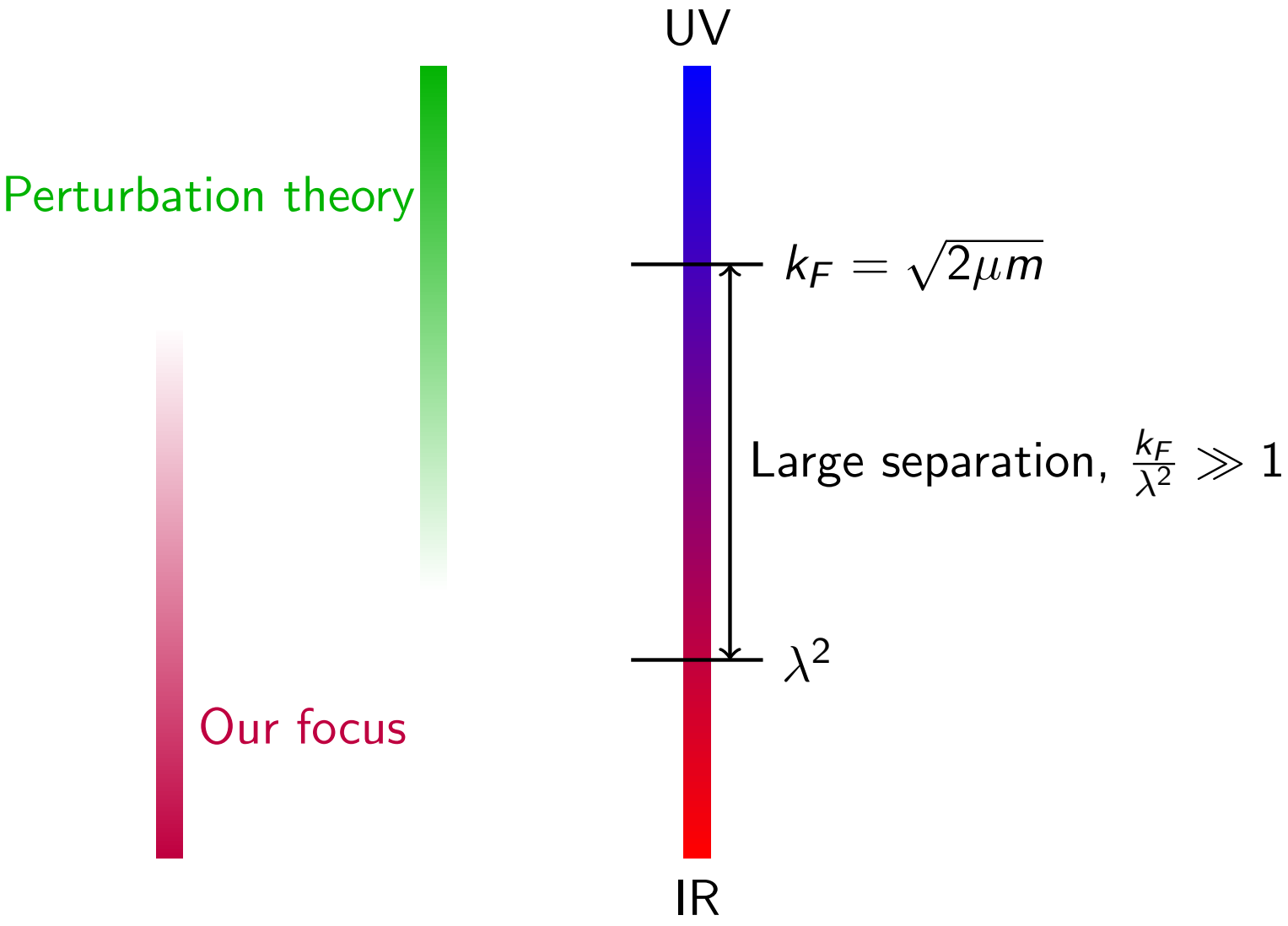
The flavor index i takes values $1 \ldots N_f$ and the spin index takes values $\sigma = \uparrow, \downarrow$. The field ϕ couples to the operator:

$$O(x) = \sum_{i=1, \sigma=\uparrow,\downarrow}^{N_f} \int \frac{\mathrm{d}^3 k \mathrm{d}^3 q}{(2\pi)^6} \lambda_\sigma(\mathbf{k}) \psi^\dagger_{\sigma,i}\big(k - \frac{q}{2}\big) \psi_{i,\sigma}\big(k + \frac{q}{2}\big) \mathrm{e}^{i\mathbf{q}\mathbf{x}}$$

The coupling function $\lambda_\sigma(\mathbf{k})$ characterizes how the fermions couple to the order parameter fluctuations. Only the dependence on the direction of $\mathbf{k}$ is relevant for low energy excitations close to the Fermi surface.

[7] M. A. Metlitski, S. Sachdev, PRB **82** (2010) 075127

Energy scales



Order of limits is important! 1: $N_f \to 0$, 2: $\omega/k_F \to 0$

Calculating $N_f \to 0$ fermion correlation functions

We calculate the fermion two-point function. Integrating out the fermion we have

$$Z[J, J^\dagger] = \int \mathcal{D}\phi \, \mathrm{e}^{\left(-S_b[\phi] - S_{\mathrm{det}}[\phi] - \int d^3z d^3z' \, J_i^\dagger(z_1) G^i{}_j[\phi](z_1, z_2) J^j(z_2)\right)}$$

with

$$\Big(-\partial_{\tau_1} + \frac{\nabla_1^2}{2m} + \mu + \lambda\phi(z_1) \Big) G[\phi](z_1, z_2) = \delta^3(z_1 - z_2)$$

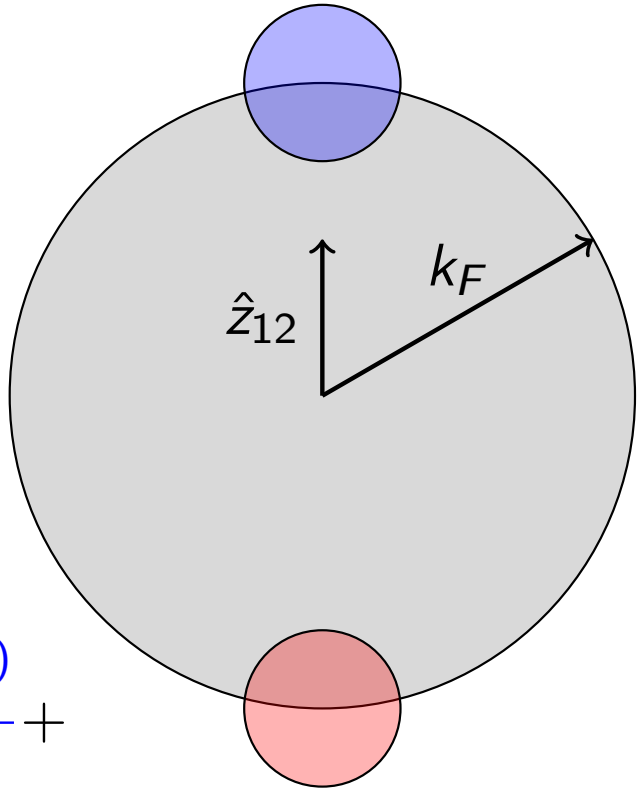
$$S_{\mathrm{det}}[\phi] = \int \mathrm{d}x \mathrm{d}y \mathrm{d}\tau \left[-N_f \, \mathrm{Tr} \ln G^{-1}[\phi] \right] = \mathcal{O}(N_f)$$

Taking functional derivatives with respect to the sources, the full fermion Green's function is then given by a path integral over only the bosonic field:

$$\langle \psi_j^\dagger(z) \psi_i(0) \rangle = \delta_j^i G(z, z') = \delta_j^i \frac{\int \mathcal{D}\phi \, G[\phi](z, z') e^{-S_b[\phi]}}{\int \mathcal{D}\phi \, e^{-S_b[\phi]}}$$

Large separation limit

Only two patches contribute to scattering for $|z_2 - z_1| \gg k_F^{-1}$ assuming all modes of ϕ have $k \ll k_F$.



$$G_{IR}(z_1, z_2)[\phi] = \frac{\sqrt{k_F}}{(2\pi)^{3/2}\sqrt{|z_{12}|}} \times$$
$$\times \left[\frac{e^{-ik_F|z_{12}| + i\pi/4 + \lambda I_{\hat{z}_{12}}[\phi](\tau_1, z_1; \tau_2, z_2)}}{-i|z_{12}| + \tau_2 - \tau_1} + \right.$$
$$\left. + \frac{e^{ik_F|z_{12}| - i\pi/4 + \lambda I_{-\hat{z}_{12}}[\phi](\tau_1, z_1; \tau_2, z_2)}}{i|z_{12}| + \tau_2 - \tau_1} \right] + \text{subleading}$$

n-point functions

$h_{\hat{n}_1,\hat{n}_2}(z)$ depends on the boson propagator $D(k)$:

$$h_{\hat{n}_1,\hat{n}_2}(z) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1-\cos(k\cdot z)}{(i\omega - v_F \hat{n}_1 \cdot k)(i\omega - v_F \hat{n}_2 \cdot k)} D(k)$$

Has closed form solution when $v_F = c$

$\Longrightarrow$

All real space fermion n**-point functions in closed form for** $N_f = 0$[8]

For correlation functions where all insertions are spatially collinear we only need $h_{\hat{n}_1,\hat{n}_2}(z)$ and for $\hat{n}_1 = \pm\hat{n}_2 \parallel z$ which considerably simplifies the above integral. We define:

$$h_s(\tau, r) \equiv h_{\hat{n},\hat{n}}(\tau, r\hat{n}) = \frac{(r+i\tau)^2}{12\pi\sqrt{r^2+\tau^2}}$$

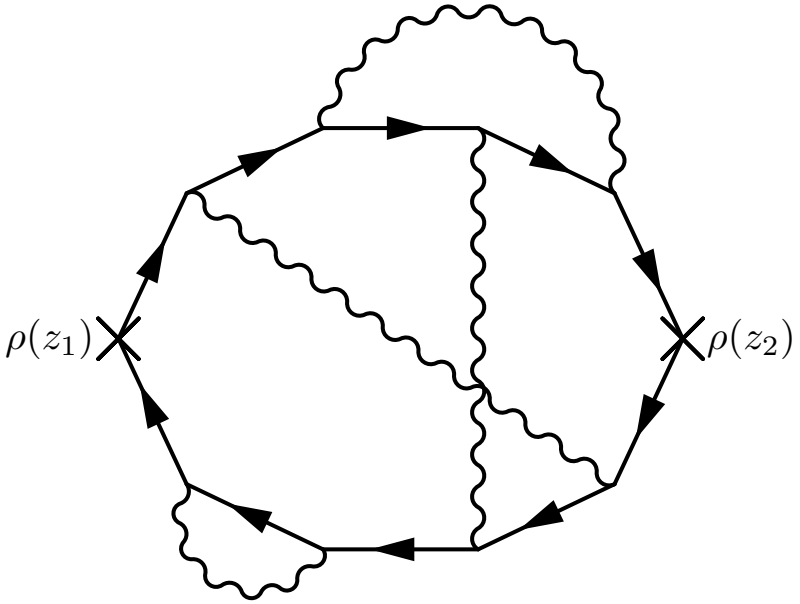
$$h_a(\tau, r) \equiv h_{\hat{n},-\hat{n}}(\tau, r\hat{n}) = -\frac{\sqrt{r^2+\tau^2}}{4\pi}$$

[8] PS, SciPost Phys. **4**, 015 (2018)

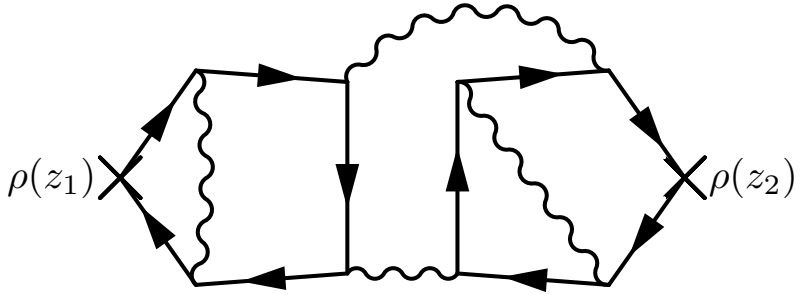
Composite operator correlators

$$\rho_\sigma(z) = \sum_i \psi^\dagger_{i,\sigma}(z)\psi_{i,\sigma}(z) \quad \text{density operator}$$

$$b(z) = \sum_i \psi_{i,\uparrow}(z)\psi_{i,\downarrow}(z) \quad \text{pair operator}$$



(a) $\propto N_f$



(b) $\propto N_f^2$

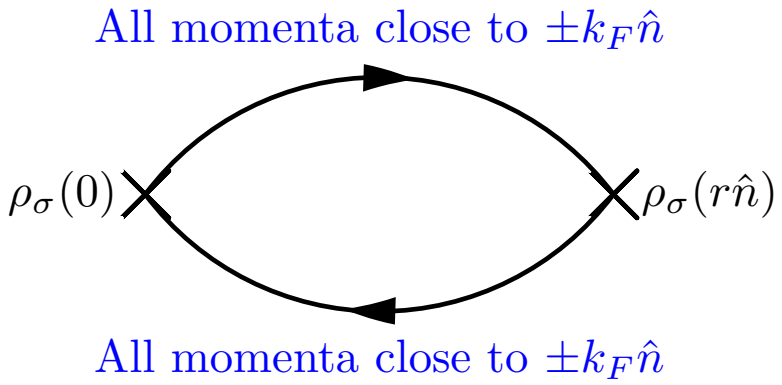
Only (a) should be included in the small N_f limit. This is good since (b) can not be obtained using the large separation limit we considered for $G[\phi](z_1, z_2)$. Generalizes to density n-point functions.

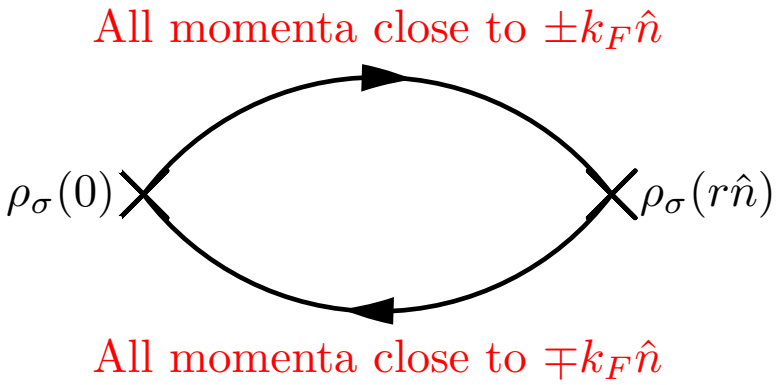
Density-density correlator

$$\langle \rho_\sigma \rho_\sigma(\tau, r\hat{n}) \rangle = N_f k_F \frac{\left(\tau^2 - r^2\right)}{4\pi^3 r \left(r^2 + \tau^2\right)^2} +$$

$$+ N_f k_F \frac{\sin(2k_F r) \exp\left(2\lambda_\sigma^- \lambda_\sigma^+ h_a(\tau, r) - \lambda_\sigma^{-2} h_s(\tau, -r) - \lambda_\sigma^{+2} h_s(\tau, r)\right)}{4\pi^3 r \left(r^2 + \tau^2\right)}$$

where $\lambda_\sigma^+ = \lambda_\sigma(k_F \hat{n})$ and $\lambda_\sigma^- = \lambda_\sigma(-k_F \hat{n})$.





For the same patch contribution, all interacting diagrams cancel out at leading order for large separations. This is expected [9],[10].

[9] J. Feldman, H. Knörrer, R. Sinclair, and E. Trubowitz, in *Singularities*

[10] A. Neumayr, W. Metzner, Phys. Rev. B **58** (1998) 15449

Large separations

We consider these susceptibilities at large separations, $r \gg \lambda^{-2}$.

$$\langle \rho_\sigma \rho_\sigma \rangle = ... + N_f k_F \frac{\sin(2k_F r) \exp(...h_a(\tau, r) + ...h_s(\tau, r) + ...h_s(\tau, -r))}{4\pi^3 r \left(r^2 + \tau^2\right)}$$

Now we consider the functions $h_s(\tau, \pm r)$ and $h_a(\tau, r)$. We calculate these for $|\tau| \ll r$:

$$h_s(\tau, r) = \frac{(r + i\tau)^2}{12\pi\sqrt{r^2 + \tau^2}} = \frac{r}{12\pi} + ... = h(r) + ...$$

$$h_a(\tau, r) = -\frac{\sqrt{r^2 + \tau^2}}{4\pi} = -\frac{r}{4\pi} + ... = -3h(r) + ...$$

We thus have

$$\langle \rho_\sigma \rho_\sigma \rangle = ... - N_f k_F \frac{\sin(2k_F r) \exp\left(-h(r)\left(\lambda_\sigma^{-2} + 6\lambda_\sigma^- \lambda_\sigma^+ + \lambda_\sigma^{+2}\right)\right)}{4\pi^3 r^3}$$

and we see that this grows exponentially for $\lambda_\sigma^{-2} + 6\lambda_\sigma^- \lambda_\sigma^+ + \lambda_\sigma^{+2} < 0$.

Pair-pair correlator

We can do the same analysis for the pair-pair correlator:

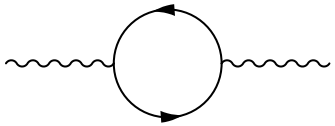
$$\begin{aligned}\langle b^\dagger(0)b(\tau, r\hat{n})\rangle =& N_f k_F \frac{\exp\left(-h(r)\left(\lambda_\downarrow^{-2} - 6\lambda_\downarrow^- \lambda_\uparrow^+ + \lambda_\uparrow^{+2}\right)\right)}{8\pi^3 r^3} \\ &+ N_f k_F \frac{\exp\left(-h(r)\left(\lambda_\uparrow^{-2} - 6\lambda_\uparrow^- \lambda_\downarrow^+ + \lambda_\downarrow^{+2}\right)\right)}{8\pi^3 r^3} \\ &+ iN_f k_F \frac{\exp\left(-h(r)(\lambda_\downarrow^- + \lambda_\uparrow^-)^2 + 2ik_F r\right)}{8\pi^3 r^3} \\ &- iN_f k_F \frac{\exp\left(-h(r)(\lambda_\downarrow^+ + \lambda_\uparrow^+)^2 - 2ik_F r\right)}{8\pi^3 r^3}\end{aligned}$$

This correlator grows exponentially if either $\lambda_\downarrow^{-2} - 6\lambda_\downarrow^- \lambda_\uparrow^+ + \lambda_\uparrow^{+2} < 0$ or $\lambda_\uparrow^{-2} - 6\lambda_\uparrow^- \lambda_\downarrow^+ + \lambda_\downarrow^{+2} < 0$.

Landau damping corrections

Integrating out the fermions only keeping 2 vertex fermion loops gives correction to boson kinetic term:

$$S_{b,\text{eff}} = \int \frac{\mathrm{d}\omega \mathrm{d}k^2}{(2\pi)^3} \frac{1}{2} \phi \Big(\omega^2 + k_x^2 + k_y^2 + \underbrace{\frac{\lambda^2 N_f k_F |\omega|}{2\pi v \sqrt{v^2(k_x^2 + k_y^2) + \omega^2}}}_{} \Big) \phi$$



Using this corrected boson kinetic term we can once again calculate h_s, h_a:

$$h_s(\tau, r) = \frac{(2r)^{1/3} \Gamma(\frac{2}{3})}{3\sqrt{3}(\pi^2 \lambda^2 N_f k_F)^{1/3}} + \ldots = h(r) + \ldots$$

$$h_a(\tau, r) = -\frac{(2r)^{1/3} \Gamma(\frac{2}{3})}{\sqrt{3}(\pi^2 \lambda^2 N_f k_F)^{1/3}} + \ldots = -3h(r) + \ldots$$

These functions still have unbounded growth in r and crucially their ratio is preserved. $\implies$ **The same instability regions with/without these N_f corrections.**

Couplings of quantum critical theories[12]

Breaking of point group symmetry.
$\lambda_\uparrow^+ = \lambda_\downarrow^+ = \lambda_\uparrow^- = \lambda_\downarrow^-$.
E.g.: Ising-nematic transition
$N_f \to 0$ analysis: "s-wave superconductor" instability
Sign-problem free Monte Carlo: s-wave superconductor groundstate[11]

Breaking of point group symmetry and time-reversal symmetry.
Varma circulating current, Spin liquid emergent gauge field
$\lambda_\uparrow^+ = \lambda_\downarrow^+ = -\lambda_\uparrow^- = -\lambda_\downarrow^-$.
$N_f \to 0$ analysis: "$Q = 2k_F$ CDW/SDW" instability

Breaking spin-inversion symmetry.
Ising ferromagnet, Ising spin-nematic
$\lambda_\uparrow^+ = \lambda_\uparrow^- = -\lambda_\downarrow^+ = -\lambda_\downarrow^-$.
Neither, so far.

[11] PRX **6** (2016), 031028, Y. Schattner, S. Lederer, S. A. Kivelson, E. Berg
[12] PRB **82**, 075127, M. A. Metlitski, S. Sachdev

Summary

- n-point functions of the $Q = 0$ quantum critical metal are solvable in the $N_f \to 0$ limit.
- Density-density and pair-pair correlation functions show instabilities for some combinations of couplings.
- Adding 2-vertex fermion loops includes some of the N_f corrections but does not change the conditions for instabilities.

n-point functions

For general fermion n-point functions we need to evaluate expressions of the form:

$$\lim_{N_f \to 0} \langle \psi^\dagger_{i_1}(z_1)\psi_{j_1}(w_1)...\psi^\dagger_{i_n}(z_n)\psi_{j_n}(w_n)\rangle =$$

$$= Z[0]^{-1} \int \mathcal{D}\phi \sum_{\sigma \in S_n} \prod_k \mathrm{sgn}(\sigma)\delta^{i_k}_{j_{\sigma_k}} G[\phi](z_k, w_{\sigma_k})\mathrm{e}^{-S_b[\phi]}$$

The ϕ integral is of the form:

$$Z[0]^{-1} \int \mathcal{D}\phi \exp\left(\lambda \sum_i I_{\hat{n}_i}[\phi](z_i, w_i) - S_B[\phi]\right)$$

$$= \exp\left(\lambda^2 \sum_{i<j} \Big[h_{\hat{n}_i,\hat{n}_j}(z_j - z_i) - h_{\hat{n}_i,\hat{n}_j}(z_j - w_i) - h_{\hat{n}_i,\hat{n}_j}(w_j - z_i) + \right.$$

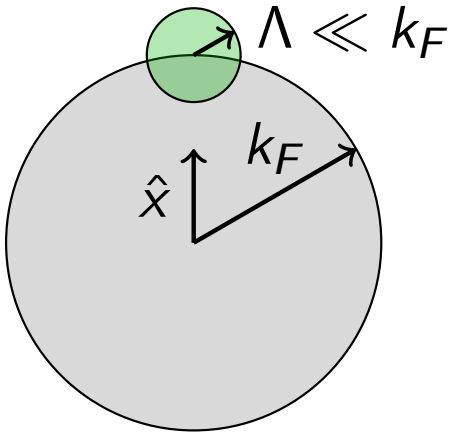
$$\left. + h_{\hat{n}_i,\hat{n}_j}(w_j - w_i)\Big] - \lambda^2 \sum_i h_{\hat{n}_i,\hat{n}_i}(z_i - w_i)\right)$$

where $h_{\hat{n}_i,\hat{n}_j}(z)$ is defined on the next slide.

Solving single patch background field Greens function PDE

For momenta in a single patch around $k_F\hat{\mathbf{x}}$ we can approximate the PDE as:

$$\Big(-\partial_{\tau_1} + iv_F\partial_x + k_F + \lambda\phi(z_1)\Big)G_{\hat{x}}[\phi](z_1, z_2) = \delta^3(z_1 - z_2)$$



The solution to this first order PDE can be written

$$G_{\hat{x}}[\phi](z_1, z_2) = G_0(z_1 - z_2)\times$$
$$\times \exp\left(ik_F(x_1 - x_2) + \lambda\int \mathrm{d}^3z\phi(z)(G_0(z - z_1) - G_0(z - z_2))\right)$$

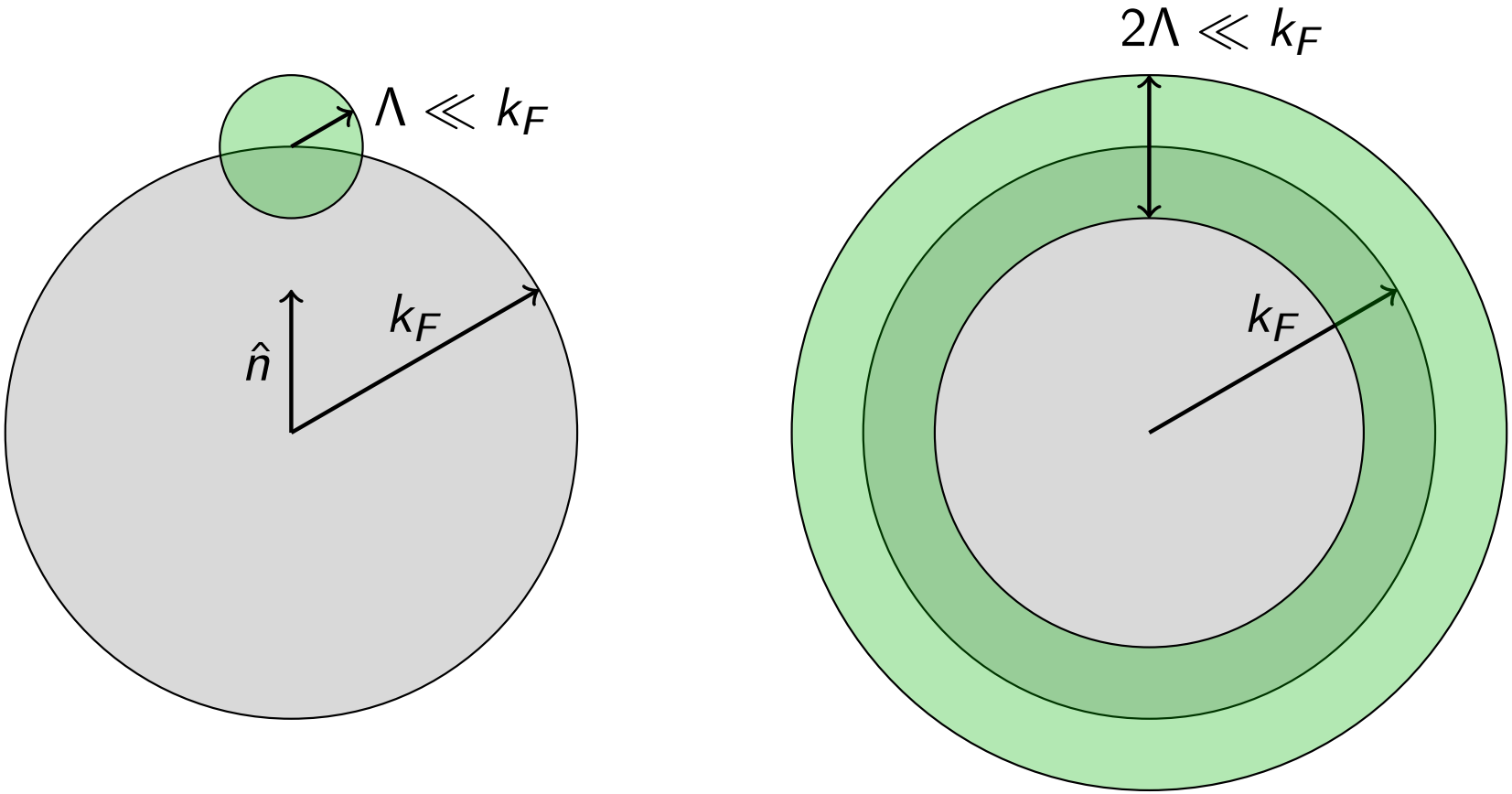
where

$$G_0(z) = \frac{\delta(y)}{2\pi(ix - v_F\tau)}$$

Extending $G_{\hat{n}}[\phi]$ to whole FS

Project onto $k\hat{n}$-momenta, apply $G_{\hat{n}}[\phi]$, integrate over all $\hat{n}$.

$$G_{\hat{n}}[\phi] \longrightarrow G_{\mathrm{IR}}[\phi]$$

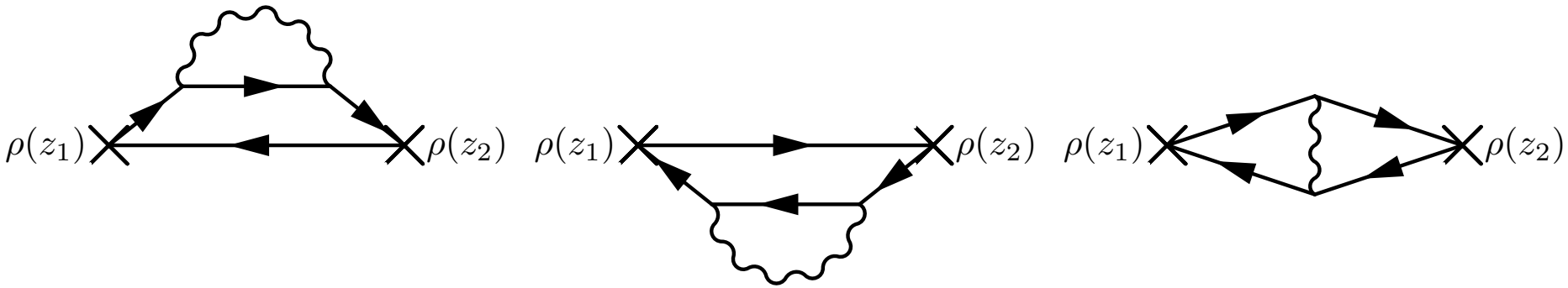


Extending $G_{\hat{n}}[\phi]$ to whole FS

$$
\begin{aligned}
G_{IR}[\phi](z_1, z_2) &= \int \mathrm{d}^3 z' \int \frac{\mathrm{d}k k \mathrm{d}\theta}{(2\pi)^2} G_{\hat{n}(\theta)}[\phi](z_1, z') \delta(\tau', \tau_2) \mathrm{e}^{ik\hat{n}(\theta)\cdot(z'-z_2)} \\
&= \int \frac{\mathrm{d}\eta \mathrm{d}\theta \mathrm{d}k k}{(2\pi)^3} \frac{1}{-i\eta + \tau_2 - \tau_1} \mathrm{e}^{i((k-k_F)\eta + k\hat{n}(\theta)\cdot(z_1-z_2))} \times \\
&\quad \times \mathrm{e}^{\lambda I_{\hat{n}(\theta)}[\phi](\tau_1, z_1; \tau_2, z_1 + \eta\hat{n}(\theta))}
\end{aligned}
$$

For $|z_2 - z_1| \gg k_F^{-1}$ and with the understanding that relevant ϕ will not contain frequencies of order k_F we can perform a saddle point approximation around $(k, \hat{n}, \eta)$ equal to $(k_F, \hat{z}_{12}, |z_{12}|)$ and $(k_F, -\hat{z}_{12}, -|z_{12}|)$

Perturbative verification up to loop order 2



The first two can be verified in real space using the already verified two-point function.
The third is more involved. Both the diagram and Fourier transform of the λ^2 contribution to the density-density correlator give

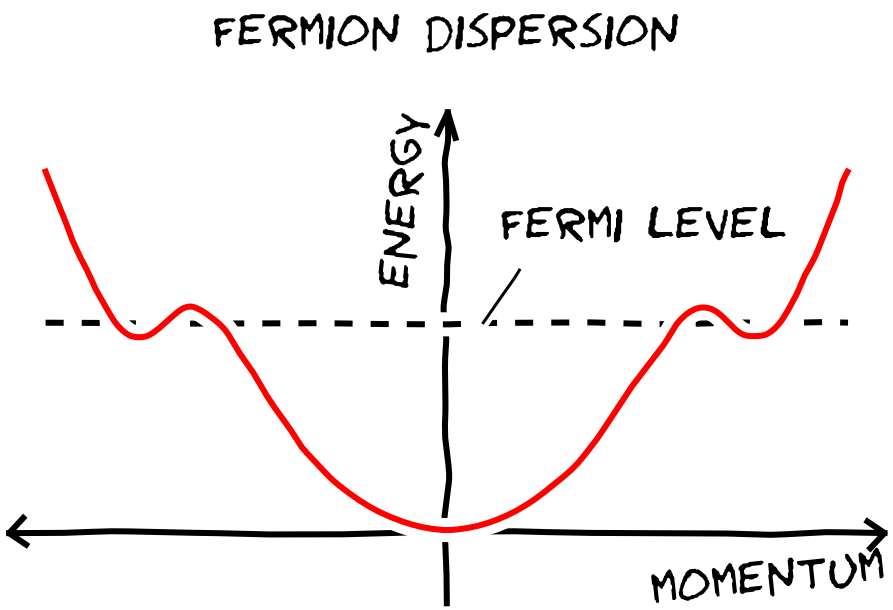
$$-\frac{N_f \lambda^2 \sqrt{k_F}}{4\pi^3 \sqrt{|\omega|}} \operatorname{Re}\left[(1+i)\, K\left(\frac{i(|k|-2k_F)+|\omega|}{2|\omega|}\right)\right] + \text{subleading.}$$

when expanded around $k = 2k_F$. K is the complete Elliptic integral of the first kind.

Search for instabilities

$G = \langle \psi^\dagger \psi \rangle$ shows non-Fermi liquid behavior.

[1602.05360, 1612.05326]
PS, B. Meszéna, A. Bagrov, and K Schalm



To search for instabilities we consider higher order correlators.
Define:

$$\rho_\sigma(z) = \sum_i \psi^\dagger_{i,\sigma}(z)\psi_{i,\sigma}(z) \quad \text{density operator}$$

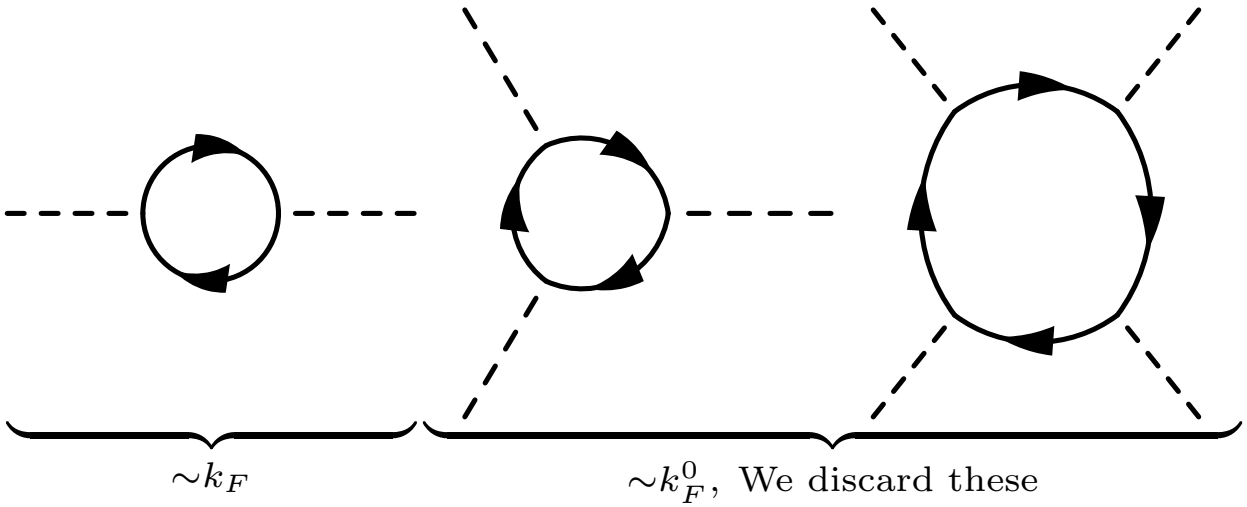
$$b(z) = \sum_i \psi_{i,\uparrow}(z)\psi_{i,\downarrow}(z) \quad \text{pair operator}$$

Multi-loop cancellation at large k_F

Fermion loops with $n = 3$ or more vertices cancel upon symmetrization in Luttinger liquids.

Large cancellations also happen in higher dimensions for $n > 2$ as $\Lambda_b \ll k_F$ [13]

$\Longrightarrow$ Only $n = 2$ loops survive a **double scaling limit** $N_f \to 0, k_F \to \infty$ with $N_f k_F$ constant.[14]



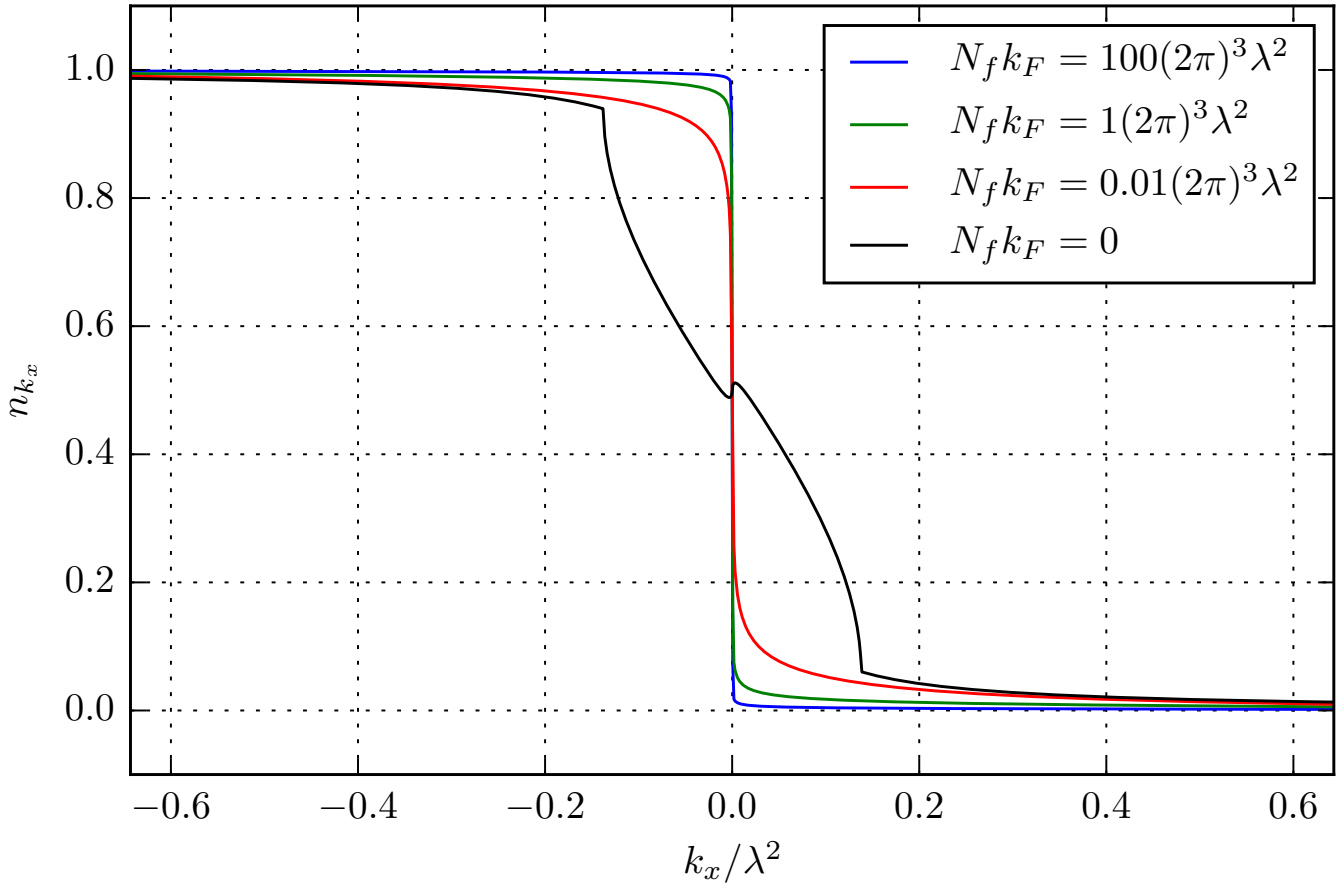
[13] A. Neumayr, W. Metzner, Phys. Rev. B **58** (1998) 15449

[14] B. Meszena, PS, K. Schalm, arXiv:1612.05326

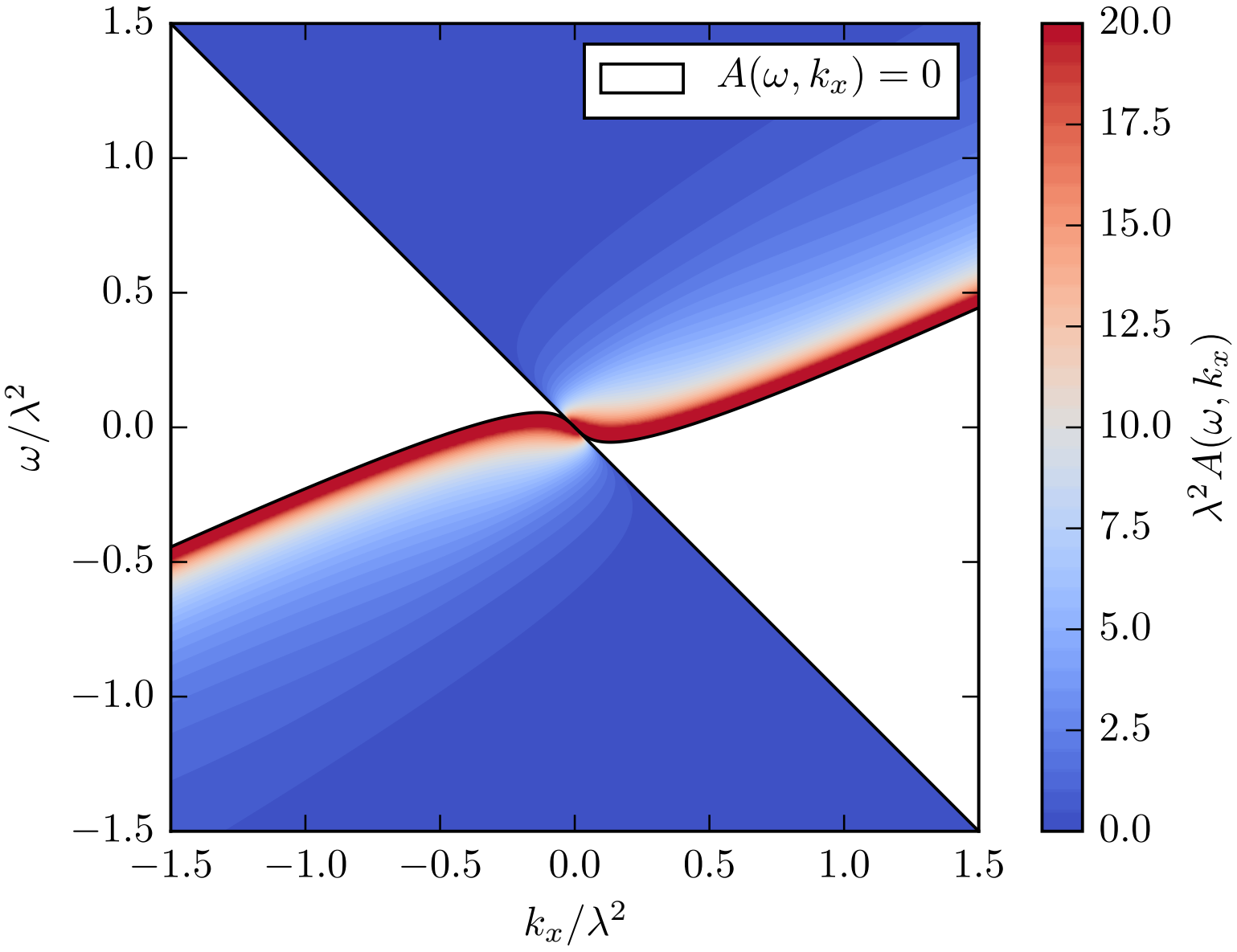
Momentum distribution function

$$n_{k_x} = \int_{-\infty}^{0} \mathrm{d}\omega A(\omega, k_x)$$

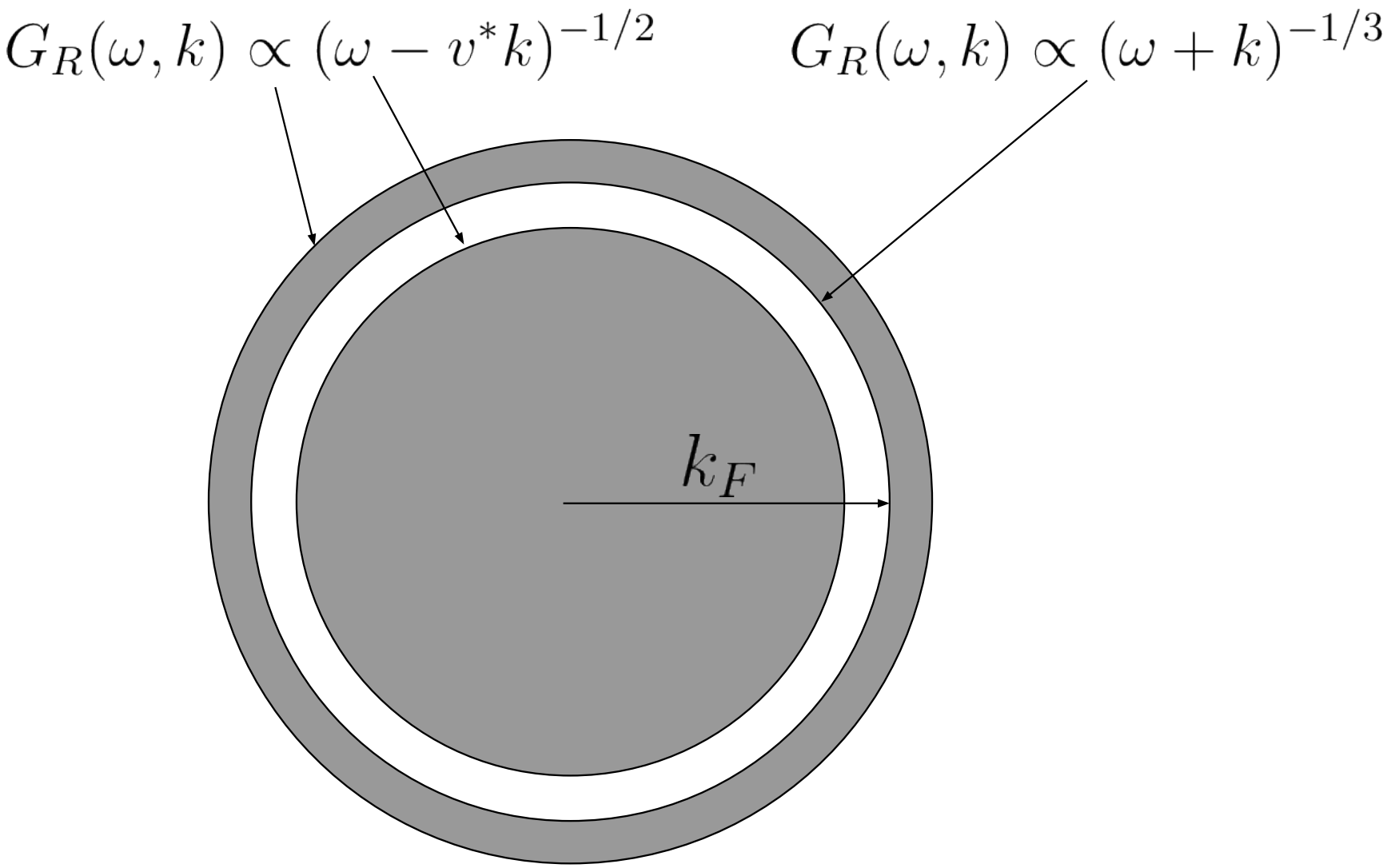
Fermi-liquids have an n_{k_x}-discontinuity at the fermi surface.



Quenched Limit Results



Quenched Limit Results - Fermi surface topology



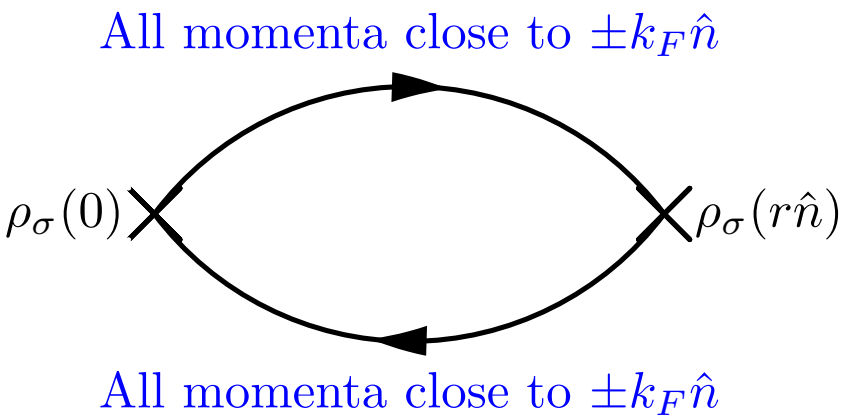
Large separation composite two-point functions

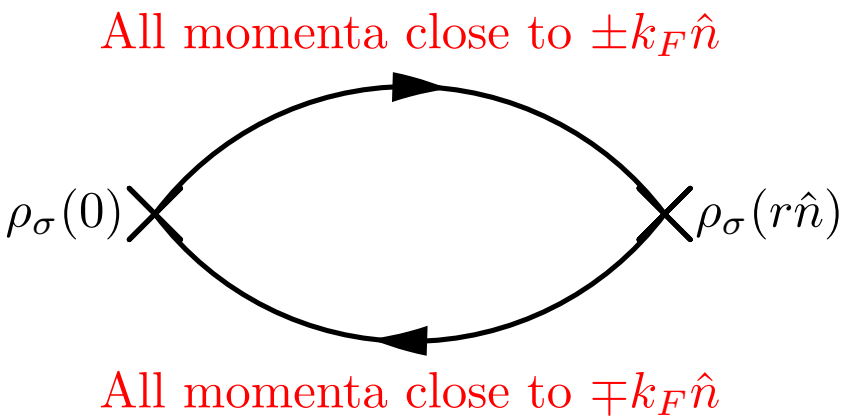
$$\langle \rho(0)\rho(\tau, r\hat{n})\rangle = N_f k_F \frac{\left(\tau^2 - r^2\right)}{4\pi^3 r \left(r^2 + \tau^2\right)^2} + N_f k_F \frac{\sin(2k_F r) \exp\left(-\left(\lambda_-^2 + \lambda_+^2 + 6\lambda_-\lambda_+\right) h(r)\right)}{4\pi^3 r \left(r^2 + \tau^2\right)}$$

$$\langle b^\dagger(0) b(\tau, r\hat{n})\rangle = - N_f k_F i \frac{\exp\left(-4\lambda_-^2 h(r) - 2ik_F r\right)}{8\pi^3 r (r - i\tau)^2} + N_f k_F i \frac{\exp\left(-4\lambda_+^2 h(r) + 2ik_F r\right)}{8\pi^3 r (r + i\tau)^2} + N_f k_F \frac{\exp\left(-\left(\lambda_-^2 + \lambda_+^2 - 6\lambda_-\lambda_+\right) h(r)\right)}{4\pi^3 r \left(r^2 + \tau^2\right)}$$

Same patch and antipodal patch contributions

$$\langle \rho(0)\rho(\tau,r)\rangle_{k_F^1} = N_f k_F \frac{\tau^2 - r^2 + \left(\tau^2 + r^2\right)\sin(2k_F r)e^{-\frac{\lambda^2\left(\tau^2+2r^2\right)}{3\pi\sqrt{\tau^2+r^2}}}}{4\pi^3 r\left(\tau^2+r^2\right)^2}$$





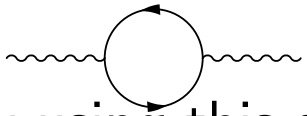
For the same patch contribution, all interacting diagrams cancel out at leading order for large separations. This is expected [15] [16].
For the antipodal patch contribution, all diagrams cancel at length scales larger than λ^{-2}.

[15] J. Feldman, H. Knörrer, R. Sinclair, and E. Trubowitz, in *Singularities* edited by G. M. Greuel (Birkhauser, 1998)
[16] A. Neumayr, W. Metzner, Phys. Rev. B **58** (1998) 15449

Finite $N_f k_F$ effective theory

Integrating out the fermions only keeping 2 vertex fermion loops gives correction to boson kinetic term:

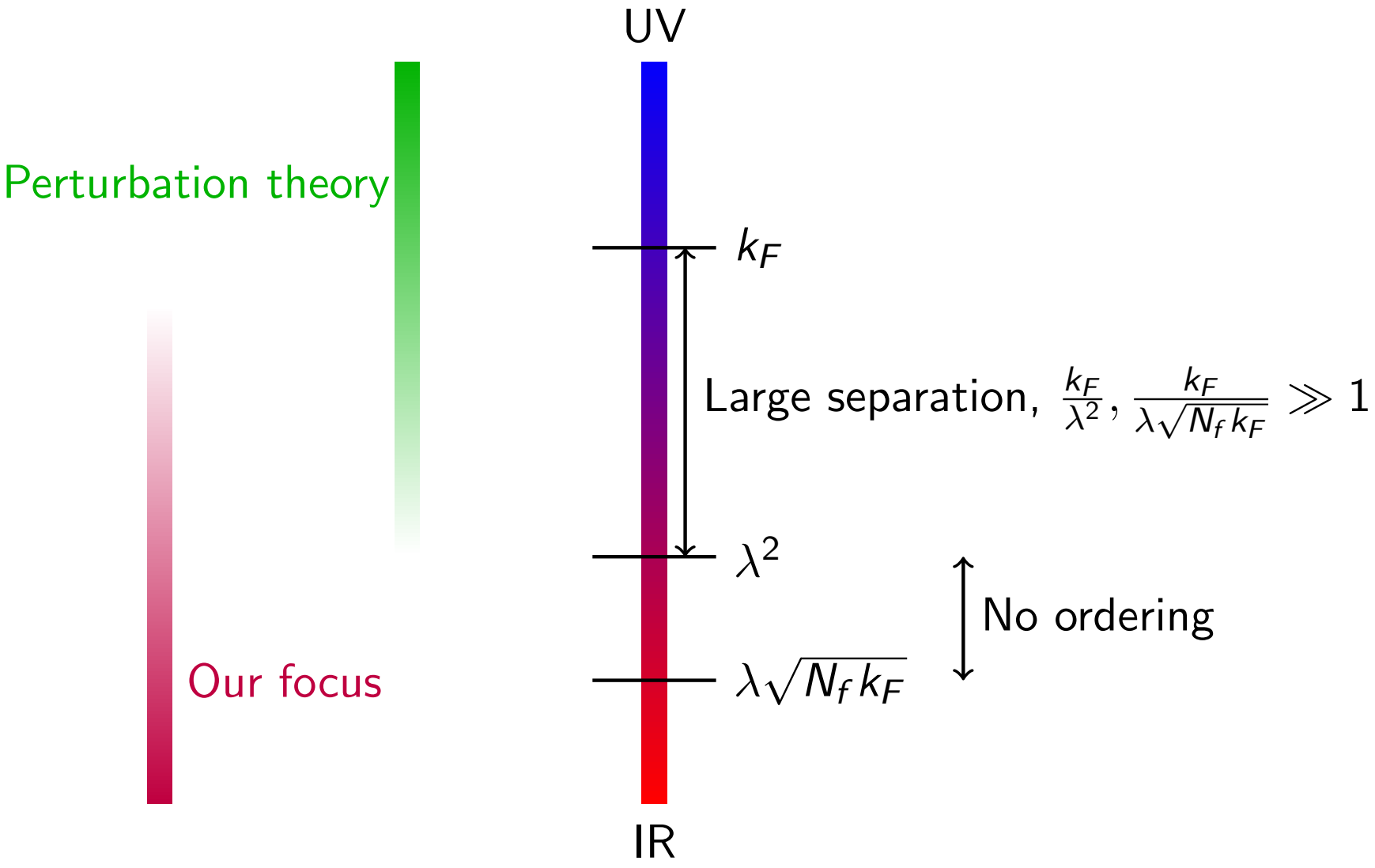
$$S_{b,\text{eff}} = \int \frac{\mathrm{d}\omega \mathrm{d}k^2}{(2\pi)^3} \frac{1}{2} \phi \Big(\omega^2 + k_x^2 + k_y^2 \underbrace{+ \frac{\lambda^2 N_f k_F |\omega|}{2\pi v \sqrt{v^2(k_x^2 + k_y^2) + \omega^2}}}_{} \Big) \phi$$



We can study this "n=2 fermion loop" theory by using this scalar effective action coupled to fermions in quenched limit, $N_f \to 0$.

$$\text{“} \lim_{\substack{N_f \to 0, \\ N_f k_F = \text{const}}} [S_b + S_f + S_{\text{int}}] \text{”} = \text{“} S_{b,\text{eff}} + \lim_{N_f \to 0} [S_f + S_{\text{int}}] \text{”}$$

New energy scale



Finite $N_f k_F$

The two-vertex fermion loop corrected boson polarization is given by

$$I(\tau, x) = \lambda^2 \int \frac{\mathrm{d}\omega \mathrm{d}k_x \mathrm{d}k_y}{(2\pi)^3} \frac{\cos(\tau\omega - xk_x) - 1}{(i\omega - k_x v)^2 \left(\omega^2 + k_x^2 + k_y^2 + \frac{\lambda^2 N_f k_F |\omega|}{2\pi v \sqrt{v^2(k_x^2+k_y^2)+\omega^2}}\right)}$$

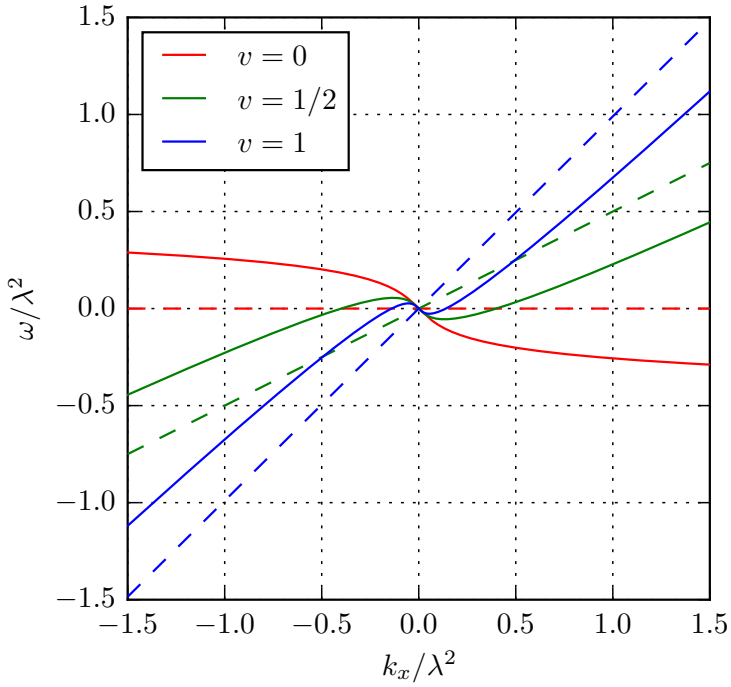
$$G(\tau, x) = G_0(\tau, x) \exp(I(\tau, x))$$

$v_F = 1$

v_F is the fermion dispersion at the free fermi momentum, k_F.

$N_f k_F = 0$: All finite v_F qualitatively the same
$N_f k_F = \infty$: All v_F same up to rescaling of coordinates

The $v_F = 1$ case is representative of all $0 < v_F < 1$ in the $N_f k_F$ extremes
$v_F = 1$ gives a "patch Lorentz symmetry"



$\Longrightarrow$ We only study this case

Real-space solution as a series

$$I(r,\theta) = \frac{\lambda f\left(r\lambda\sqrt{N_f k_F/(2\pi)^3},\theta\right)}{\sqrt{N_f k_F/(2\pi)^3}}, \qquad f(\tilde{r},\theta) = \sum_{n=1}^{\infty} f_n \tilde{r}^n$$

$$\begin{aligned} f_n =& \frac{\pi^{n-2} e^{i\theta}(-1)^n \Gamma\left(\frac{n+1}{4}\right) |\sin(\theta)|^{\frac{1+3n}{2}}}{72(3n-1)\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{1+3n}{4}\right)} \cdot \\ & \cdot \Bigg({}_2F_1\left(\frac{n+3}{2}, \frac{n+5}{4}; \frac{5}{2}; \cos^2(\theta)\right)(n+1) \cdot \\ & \quad \cdot \cos^2(\theta)((1-3n)\cos(\theta) - i(n+1)\sin(\theta)) \\ & + {}_2F_1\left(\frac{n+1}{4}, \frac{n+1}{2}; \frac{3}{2}; \cos^2(\theta)\right) 6((1-2n)\cos(\theta) - i\sin(\phi)) \Bigg) \end{aligned}$$

Limits

IR expansion:

$$G_{\mathrm{IR}}(\omega, k_x) = \mathrm{e}^{\frac{\lambda}{3\sqrt{2\pi N_f k_F}}} \left[\frac{1}{i\omega - k_x} \cos\left(\frac{\omega}{l_0^{1/2}(\omega + ik_x)^{3/2}} \right) \right.$$

$$+ \frac{6\sqrt{3} i \Gamma\left(\frac{1}{3}\right) \omega^{2/3}}{8\pi l_0^{1/3}(\omega + ik_x)^2} {}_1F_2\left(1; \frac{5}{6}, \frac{4}{3}; -\frac{\omega^2}{4 l_0 (\omega + ik_x)^3}\right) +$$

$$\left. + \frac{3\sqrt{3} i \Gamma\left(-\frac{1}{3}\right) \omega^{4/3}}{8\pi l_0^{2/3}(\omega + ik_x)^3} {}_1F_2\left(1; \frac{7}{6}, \frac{5}{3}; -\frac{\omega^2}{4 l_0 (\omega + ik_x)^3}\right) \right]$$

$$\approx \mathrm{e}^{\frac{\lambda}{3\sqrt{2\pi N_f k_F}}} \frac{1}{i\omega - k_x - \Sigma_{\mathrm{RPA}}} \quad \text{for small } \omega \text{ and fixed } k_x$$

$$\approx \mathrm{e}^{\frac{\lambda}{3\sqrt{2\pi N_f k_F}}} \frac{1}{i\omega - k_x - \frac{4\pi}{3\sqrt{3}} \Sigma_{\mathrm{RPA}}} \quad \text{for small } \omega \text{ and fixed } k_x/\omega$$

Large $N_f k_F$-limit:

$$\Sigma_{\mathrm{RPA}}(\omega, k_x) = \frac{-i\lambda^{4/3} \mathrm{sgn}(\omega) |\omega|^{2/3}}{(2\pi)^{2/3}\sqrt{3}(N_f k_F)^{1/3}}$$