What is a transport coefficient?

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Bounding Transport and Chaos in Condensed Matter and Holography
Nordita, August 22, 2018

This talk is about thermal physics

Work in 3+1 or 2+1 dimensions

Old-fashioned view

System is in some state, e.g. equilibrium at temperature T

Apply external source, e.g. electric field **E**

Measure the response, e.g. current **J**

Extract transport coefficients by fitting the response to the constitutive relation, e.g. $\mathbf{J}=\sigma\mathbf{E}$, Ohm's law

For DC transport, $\omega \to 0$

Note $\omega \to 0$ and $T \to 0$ do not necessarily commute.

If calculate DC transport at $T=0$, this means $kT \ll \hbar\omega \to 0$.

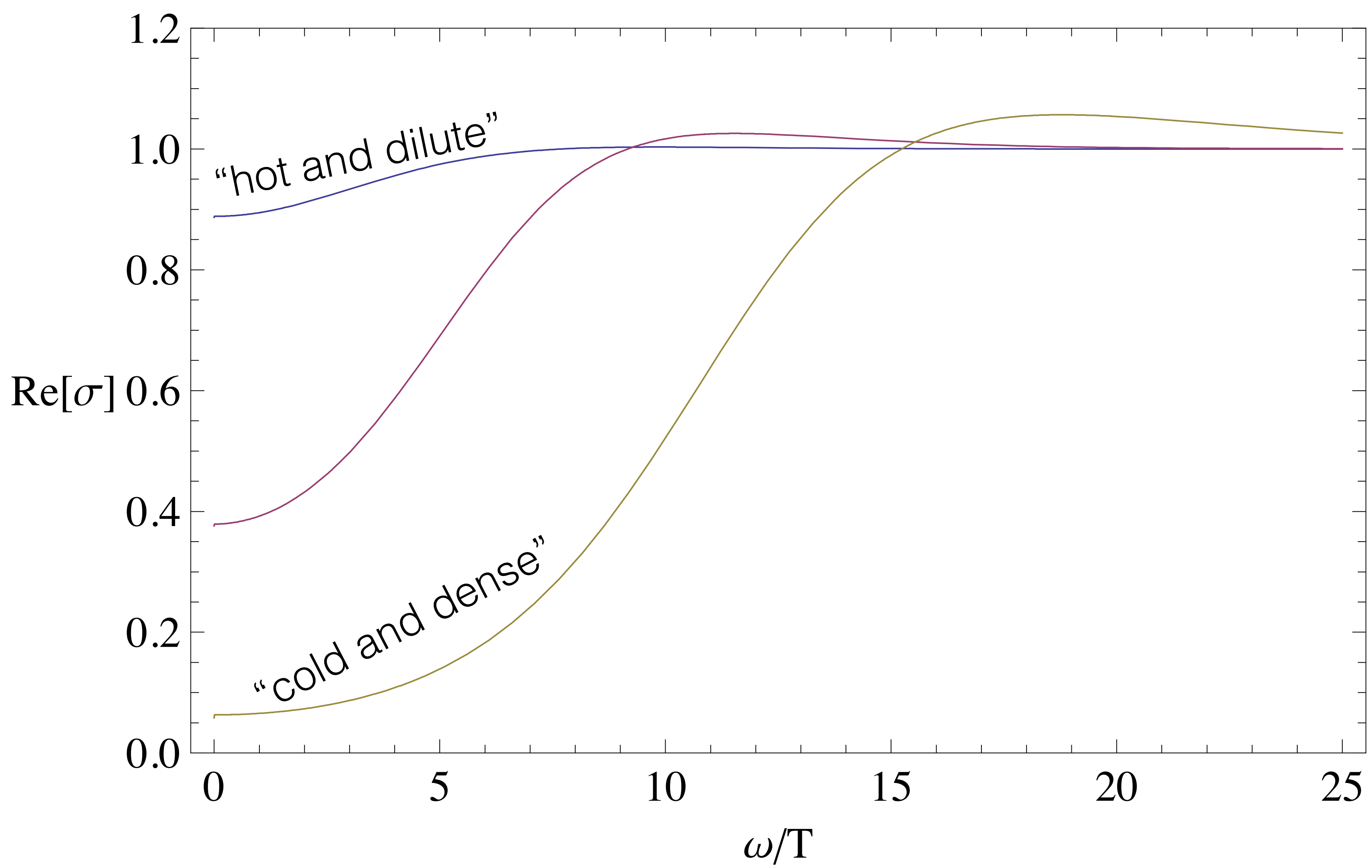
This is “quantum” regime, physics of the ground state.

If calculate DC transport at $T \neq 0$, this means $kT \gg \hbar\omega \to 0$.

This is “hydrodynamic” regime, physics of the thermal state.

Example: a toy model in 2+1 dimensions

A certain strongly interacting scale-invariant quantum system without quasiparticles, can compute σ(ω) using "holographic" methods:



Hartnoll, 0903.3246

How does one compute $\sigma(\omega)$?

When have quasiparticles and weak coupling $\lambda \to 0$, can use Boltzmann equation.

For some systems with no quasiparticles, with strong coupling $\lambda \to \infty$ <u>and</u> many species $N \to \infty$, can use holographic methods.

Not all physical systems are well described by quasi-particles, and certainly the real world does not have $N = \infty$.

To make things worse, $N \to \infty$ and $\omega \to 0$ do not commute.
In holography one always takes $N \to \infty$ first.

Hydrodynamics

Universal approach at $\hbar\omega \ll kT$, does not care whether λ is small or large, whether N is finite or infinite

Defines what you mean by a transport coefficient, through the constitutive relations such as $\mathbf{J}=\sigma\mathbf{E}$

Formulated as classical partial differential equations, so "easy" to solve: no path integrals, no Monte-Carlo

Transport coefficients in hydrodynamics

Variations with respect to the external source give retarded correlation functions of observable quantities

Transport coefficients can be expressed through Kubo formulas, e.g.

$$\sigma(\omega) = \frac{1}{\omega}\mathrm{Im}\langle J_x J_x\rangle^{\mathrm{ret.}}(\omega, \mathbf{q}{=}0)$$

These may be evaluated from first principles, connecting phenomenological hydrodynamics to microscopic physics

Old-fashioned view

Q: What is a transport coefficient?

A: A parameter in the hydrodynamic equations. The connection with microscopics is through the Kubo formulas.

How to count transport coefficients

Hydrodynamic equations are written as a gradient expansion near local thermal equilibrium.

However simply writing down all possible terms with a given number of derivatives with a given symmetry is not enough.

Not all transport coefficients are independent: constraints come from field redefinitions, Onsager relations, consistency of thermodynamics, positivity of entropy production.

A simple example: diffusion

One degree of freedom φ, for example φ=δT.

$$\partial_t \varphi - D\nabla^2\varphi + [\text{terms with more derivatives of } \varphi] = 0$$

Diffusion at leading order, $\partial_t \sim \nabla^2 \sim \epsilon$ is small. One transport coefficient D at O(ϵ).

A simple example: diffusion

$$\left[\partial_t\varphi - D\nabla^2\varphi\right] + \left[\tau_0\partial_t^2\varphi + D\tau_1\partial_t\nabla^2\varphi + \Gamma\nabla^2\nabla^2\varphi\right] + O(\epsilon^3) = 0$$

Looks like three transport coefficients τ_0,τ_1,Γ at $O(\epsilon^2)$.

Eigenmodes:

$$\omega = -iD\mathbf{k}^2 - i\left[\Gamma + D^2(\tau_0{+}\tau_1)\right]\mathbf{k}^4 + O(\mathbf{k}^6)\,, \qquad \omega = -\frac{i}{\tau_0} + O(\mathbf{k}^2)$$

Naively, must have D>0 and τ_0>0 for stability.

A simple example: diffusion

However, can also work with $\psi \equiv \varphi + \alpha\partial_t\varphi + \beta D\nabla^2\varphi = \varphi + O(\epsilon)$

The field ψ obeys same equation as φ, but with

$$\tau_0 \to \tau_0 - \alpha\,, \qquad \tau_1 \to \tau_1 + \alpha - \beta\,, \qquad \Gamma \to \Gamma + \beta D^2\,.$$

So τ_0 may be redefined away and is not a transport coefficient. The only transport coefficient at $O(\epsilon^2)$ is

$$\Gamma + D^2(\tau_0 + \tau_1)$$

which is invariant under the above redefinitions. The instability at $\tau_0 < 0$ can be removed by a field redefinition.

Another example: relativistic hydrodynamics

Relativistic hydrodynamic equations as written in the text-books by Landau & Lifshitz (*Fluid Mechanics*) or by Weinberg (*Gravitation and Cosmology*)

a) predict that thermal equilibrium does not exist

b) predict that things propagate faster than light

Hiscock, Lindblom, 1984

Hiscock, Lindblom, 1987

Another example: relativistic hydrodynamics

One can mess with the equations by coupling them to extra degrees of freedom that take care of stability and causality (Israel-Stewart theory). This is what is used in practice if you actually want to solve the equations.

Note however that in hydrodynamics the quantities T, **v**, μ have no microscopic definitions out of equilibrium and can be redefined, just like in the above example of diffusion.

Landau & Lifshitz choose one definition, Eckart/Weinberg choose another. One can adopt other definitions such that the problems with causality and instability go away, just like in the above example of diffusion.

Freistuhler, Temple, 2014
Bemfica, Disconzi, Noronha, 1708.06255

Another example: relativistic hydrodynamics

For fluids whose only conserved quantites are energy and momentum in 3+1 dim, there are

2 transport coefficients at $O(\partial)$

10 transport coefficients at $O(\partial^2)$

more transport coefficients at $O(\partial^3)$

Hydrodynamics is great and has been around for many years. However, there are issues.

Hydrodynamics contains its own demise

Conventional hydrodynamics assumes locality

Hence the derivative expansion is local, e.g.
$T^{ij} = O(1) + O(\partial) + O(\partial^2) + \ldots$

ABCDEFG… eq-s

Euler eq-s   Navier-Stokes eq-s

But hydrodynamic equations predict gapless modes, e.g. sound waves with $\omega = v_s k + \ldots$

Gapless modes mediate long-range correlations and may lead to a breakdown of locality.

Hydrodynamics contains its own demise

Thus the existence of sound may imply non-local correlations.

Correlations give rise to transport coefficients, through the Kubo formulas.

Thus the existence of sound may imply non-existence of transport coefficients.

This is indeed what happens.

THE NONEXISTENCE OF THE LINEAR DIFFUSION EQUATION BEYOND FICK'S LAW

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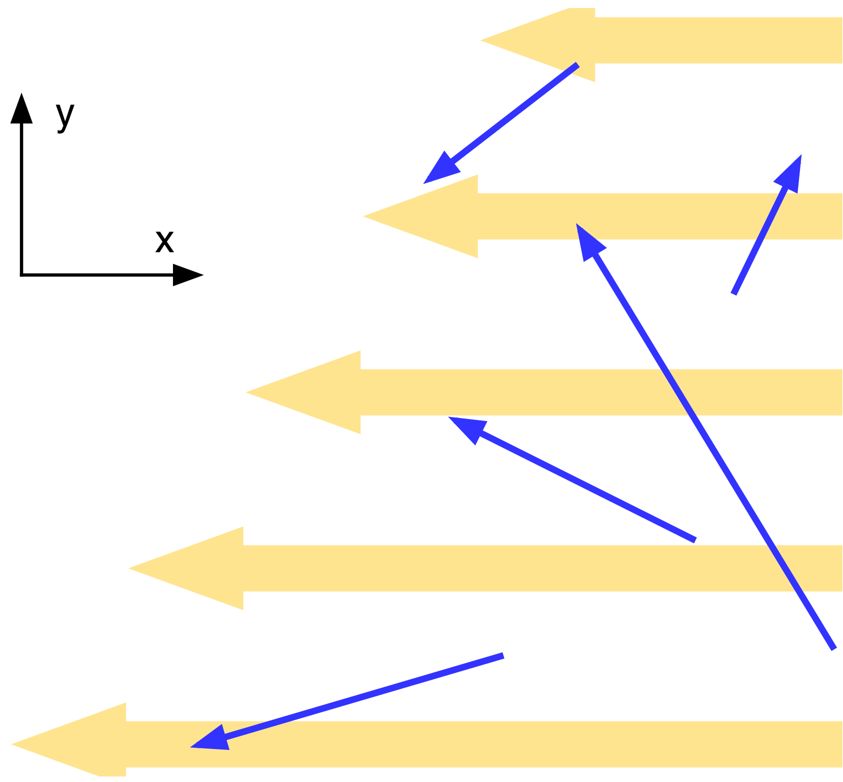
Physica **75** (1974) 1–36

So let us look at transport coefficients in hydrodynamics in more detail

Example: viscosity



Momentum transfer
between layers of fluid,

$T_{xy} = \eta\, \partial_y v_x + O(\partial^2)$

$$\langle T^{xy} T^{xy} \rangle^{\mathrm{ret.}} = p - i\omega\eta + O(\omega^2)$$

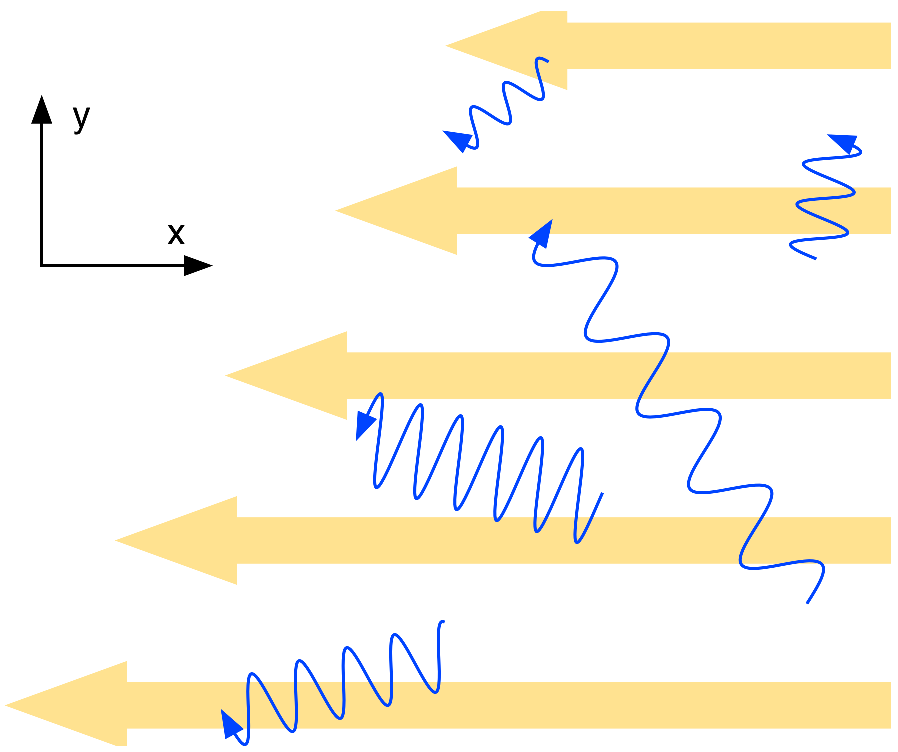
In a gas ($\lambda \to 0$)

$\eta = \rho\, v_{th}\, \ell_{mfp} \sim 1/\lambda^2 \to \infty$

In holography ($N \to \infty$, $\lambda \to \infty$)

$\eta = s/4\pi \sim N^2 \to \infty$

Example: viscosity



Momentum can also be transfered by collective excitations.

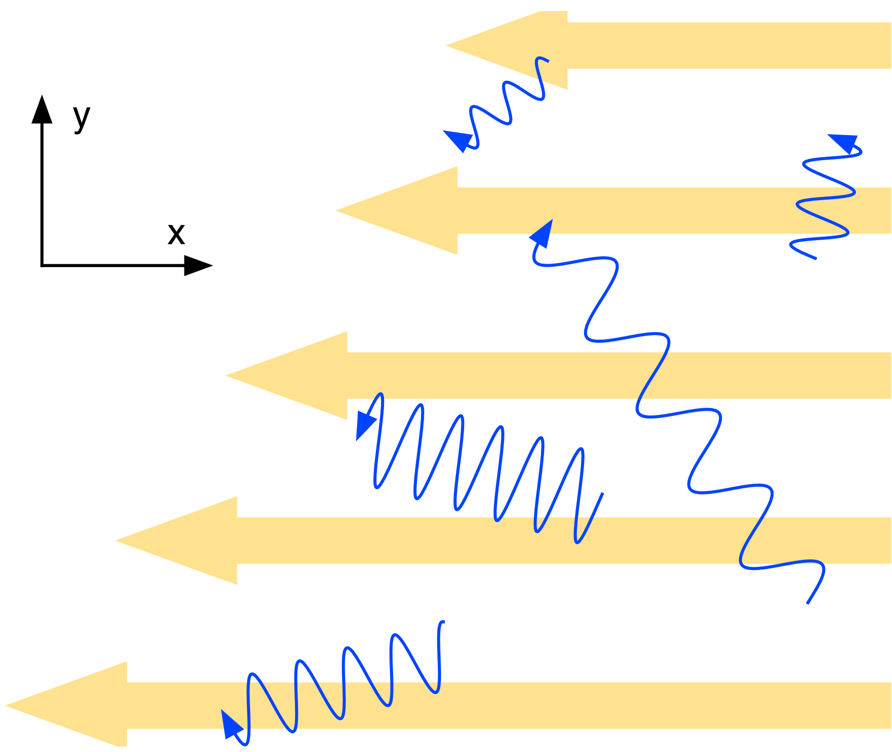
In a gas of sound/shear waves:

$$\ell_{\mathrm{mfp}} \sim \frac{1}{\frac{\eta}{e+p}\mathbf{k}^2}$$

Contribution to viscosity:

$$\int^{\Lambda} d^D k \, \frac{T}{\frac{\eta}{e+p}\mathbf{k}^2} \quad \begin{array}{l} \nearrow \; \dfrac{\Lambda T^2}{\eta/s} \quad \text{in D=3} \\ \searrow \; \text{IR divergent in D=2} \end{array}$$

Example: viscosity



- This is the physics of thermal fluctuations. It is invisible if hydrodynamics is viewed just as a collection of partial differential equations.
- If you think η/s can be arbitrarily small, think again.
- Classical hydrodynamics may be irrelevant to physics in 2+1 dimensions.

How does one describe these effects quantitatively?

Stochastic hydrodynamics

A toy model for thermal fluctuations:

Landau & Lifshitz, *Statistical Physics Part II*

$$T^{ij} = T^{ij}_{\text{classical}}(T, v, \partial T, \partial v, \dots) + \tau_{ij}$$

Gaussian noise $\langle \tau_{ij}(x) \tau_{kl}(y) \rangle = 2T G_{ijkl}\, \delta(x-y)$

appropriate projector with viscosities, to satisfy FDT in equilibrium

Energy-momentum conservation ⇒ $\delta v^i = \delta v^i[\tau],\ \delta T = \delta T[\tau]$

Get $\langle \delta T\, \delta T \rangle$, $\langle \delta v^i\, \delta v^j \rangle$, $\langle T^{ij}\, T^{kl} \rangle$, Kubo formulas

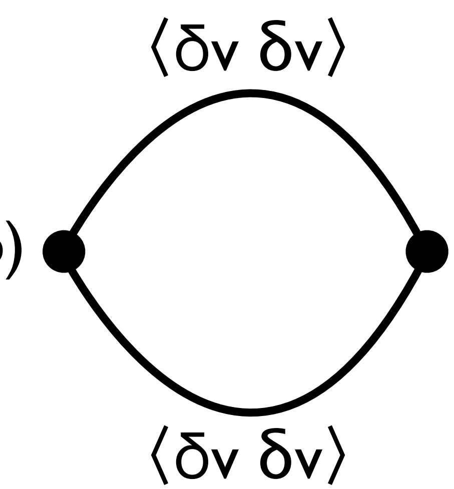
Stochastic hydrodynamics

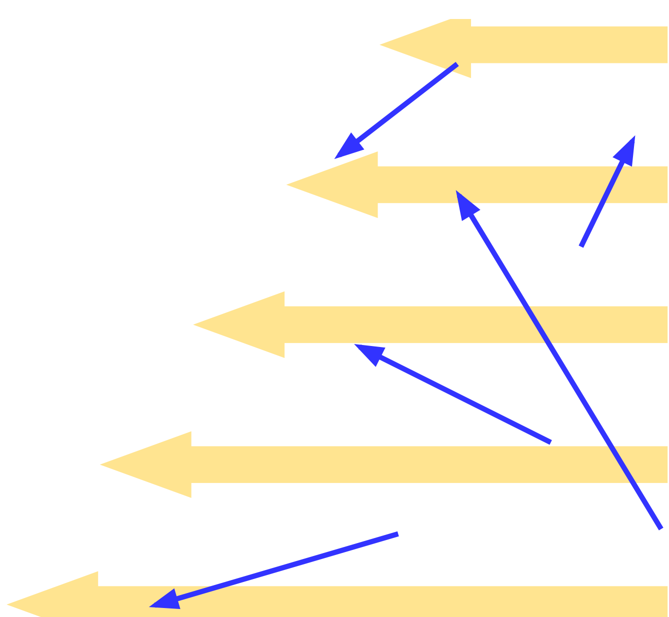
$$T^{ij} = \cdots + (\epsilon+p)v^i v^j + \cdots + \tau^{ij}$$

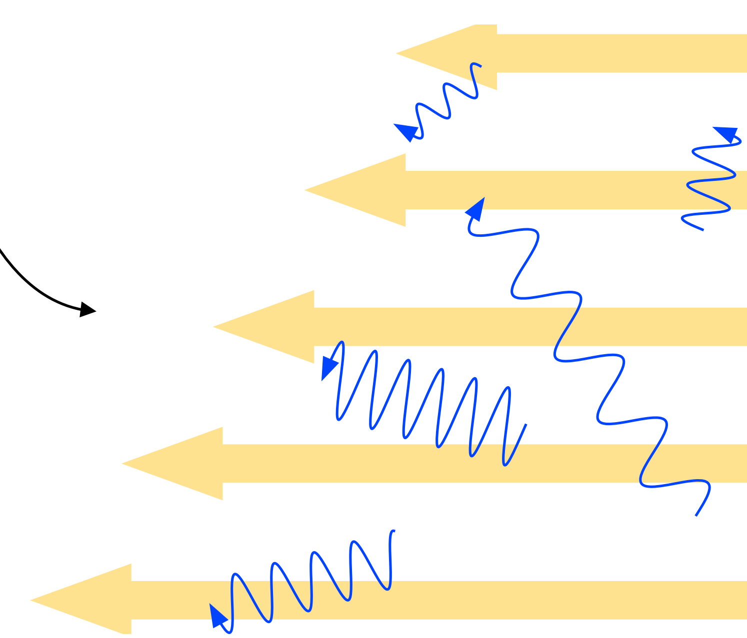
linear

leading non-linear,
no derivatives term

higher order

$\langle T_{xy} T_{xy} \rangle = 2T\eta \;+\; (\varepsilon+p)$  $(\varepsilon+p)$





Example: viscosity in 3+1 dimensions

$$\langle T_{xy}T_{xy}\rangle^R = p + O(\Lambda^3 T) - i\omega\left(\eta + \frac{17T^2\Lambda}{120\pi^2\eta/s}\right) + O\left(\frac{\omega^{3/2}}{(\eta/s)^{3/2}}\right) + O(\omega^2)$$

0-th order classical

I-st order classical

2-nd order classical

correction to p

correction to η

cutoff-independent

PK, Yaffe, hep-th/0303010

PK, Moore, Romatschke, 1104.1586

This is “one-loop” fluctuation correction to $\langle T_{xy}\, T_{xy}\rangle$

Actual physical viscosity includes all such corrections, but no way to compute them all in practice

Holography is useless here because it takes $N \to \infty$

Example: viscosity in 3+1 dimensions

$$\langle T_{xy} T_{xy} \rangle^R = p + O(\Lambda^3 T) - i\omega \left(\eta + \frac{17 T^2 \Lambda}{120 \pi^2 \eta/s} \right) + O\left(\frac{\omega^{3/2}}{(\eta/s)^{3/2}} \right) + O(\omega^2)$$

As expected, small η/s implies large corrections to η/s.
So fluctuations are *mandatory* for small-viscosity physics.

Relevant for the quark-gluon plasma and unitary Fermi gases.

PK, Moore, Romatschke 1104.1586
Chafin, Schaefer, 1209.1006
Romatschke, Young, 1209.1604

Function (η + 1/η) has a minimum, implies a lower bound on η. Hydro of QGP with η/s = 1/4π appears inconsistent, regardless of the experiment.

PK, Moore, Romatschke 1104.1586

Fluctuations are more important than 2-nd order hydro.

Analogy with quantum gravity

Thermal fluctuations are more important than higher-derivative hydrodynamics:

$$T^{\mu\nu}_{\rm cl} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + \dots$$

$$G_{xy,xy}(\omega) = O(1) + O(\omega) + O(\omega^{3/2}) + O(\omega^2) + \dots$$

Quantum fluctuations are more important than higher-derivative gravity:

$$S = \int d^4x \left[\frac{1}{16\pi G} R + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \dots \right]$$

$$V(r) = -\frac{G m_1 m_2}{r} \left[1 + O\left(\frac{Gm}{r}\right) + O\left(\frac{G\hbar}{r^2}\right) + O(e^{-m_0 r}) \right]$$

$$m_0 \sim (c_i G)^{-1/2}$$

Classical: Stelle 1978
Quantum: Bjerrum-Bohr, Donoghue, Holstein, 2002

Hydrodynamics in 2+1 dimensions

Same one-loop diagram gives: $\eta=\eta_0+O(\ln\omega)$, $\sigma=\sigma_0+O(\ln\omega)$
(scale-invariant theory, $\mu=0$)

σ = charge conductivity,
related to diffusion constant $D=\sigma\chi$,
χ = charge susceptibility

Scale-dependent
transport coef-s: $\eta(\lambda)\equiv\eta(\omega{=}\lambda)\,,\ \sigma(\lambda)\equiv\sigma(\omega{=}\lambda)\,,\ g_\eta\equiv\eta/s\,,\ g_\sigma\equiv\sigma T/\chi$

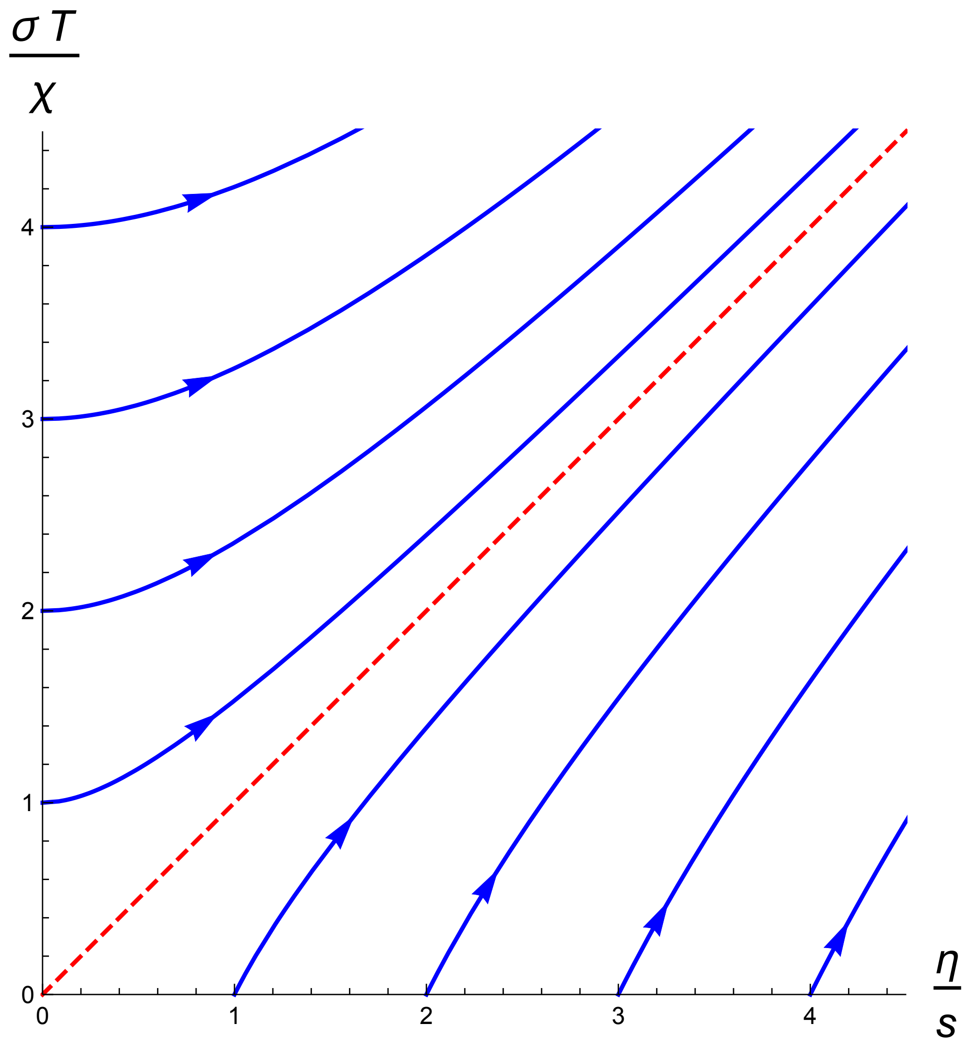
Physical objects (correlation functions) can not depend on the arbitrary scale λ, hence RG equations:

$$\lambda\frac{\partial g_\eta}{\partial\lambda}=-\frac{1}{16\pi c}\frac{1}{g_\eta}\,,$$

$$\lambda\frac{\partial g_\sigma}{\partial\lambda}=-\frac{1}{8\pi c}\frac{1}{g_\sigma+g_\eta}$$

$c=s/T^2$, counts d.o.f.

Hydrodynamics in 2+1 dimensions



- Both η/s and σT/χ become large as ω→0.
- "fixed line" as ω→0:

$$\boxed{\frac{\eta}{s} = \frac{\sigma T}{\chi}}$$

PK, 1205.5040

Viscosity and charge conductivity are not independent transport coefficients in 2+1 dimensions

What is the small parameter?

Simplest case: scale-invariant uncharged relativistic fluid.

The only dimensionful quantities are T, s, η.

The only dimensionless combination in natural units c=1 is

$$g_{\text{eff}} \equiv \frac{T/s^{1/D}}{\eta/s}$$

Fluctuation correction to η is proportional to g_{eff}.

"Coupling constant" g_{eff} is small at weak coupling or large N.

"Coupling constant" g_{eff} is large when η/s is small.

This was hydrodynamics with conserved momentum.
Same thing happens for purely diffusive transport.

Particle and heat diffusion

Conservation of energy and particle number:

$$\partial_t \epsilon + \boldsymbol{\nabla}\cdot\mathbf{j}_\epsilon = 0\,, \qquad \partial_t n + \boldsymbol{\nabla}\cdot\mathbf{j}_n = 0$$

Constitutive relations:

$$\mathbf{j}_\epsilon = -\Pi_{11}\boldsymbol{\nabla}T - \Pi_{12}\boldsymbol{\nabla}\mu + \ldots\,, \qquad \mathbf{j}_n = -\Pi_{21}\boldsymbol{\nabla}T - \Pi_{22}\boldsymbol{\nabla}\mu + \ldots$$

Onsager relation gives one constraint: $\Pi_{12} = T\,\Pi_{21} + \mu\,\Pi_{22}$

4−1=3 transport coef-s: 2 diffusion constants, 1 “Hall-like”

Particle and heat diffusion

Transport coef-s depend on temperature and density, $\Pi_{AB}(T,\mu)$

Hence non-linear coupling of fluctuations, e.g.

$$\delta \mathbf{j}_\epsilon = \ldots \nabla \delta T + \ldots \nabla \delta \mu + \ldots \delta T \, \nabla \delta T + \ldots \delta \mu \, \nabla \delta T + \ldots \delta T \, \nabla \delta \mu + \ldots \delta \mu \, \nabla \delta \mu$$

One-loop statistical fluctuations renormalize naive particle and heat diffusion constants D_1, D_2 of linear response:

$$(D_1 + D_2) = (D_1^b + D_2^b) + \frac{\Lambda^D C_D}{(D_1^b + D_2^b)}$$

thermodynamic derivatives of $\Pi_{AB}(T,\mu)$, positive constant

naive diffusion constants of linear response

PK, 1407.0690

Particle and heat diffusion

Small (D_1+D_2) implies large corrections to (D_1+D_2). Thus fluctuations are *mandatory* for transport with small diffusion constants.

Gradient coupling of fluctuations between energy and particle density softens the IR behavior, but the derivative expansion of diffusion still breaks down.

Function ($D + 1/D$) has a minimum, depends on Λ. Implies a lower bound on D.

A better way to treat fluctuations

Stochastic hydrodynamics is a phenomenological model

Can be recast as a field theory (MSR) with an action

Can do in a relativistic covariant way, couple to gravity

PK, Moore, Romatschke, 1405.3967

Still have to answer:

- Go beyond Gaussian noise in a systematic way?
- Derivative expansion and field redefinitions?
- Full set of transport coefficients?
- Access all possible n-point correlation functions?
- How to calculate in practice (discretization ambiguity)?

A better way to treat fluctuations

A modern version of the MSR theory is being developed, using the Schwinger-Keldysh formalism from the Wilsonian effective field theory point of view

Haehl, Loganayagam, Rangamani
1511.07809, 1701.07896, 1803.11155

Crossley, Glorioso, Liu 1511.03646,
1701.07817, 1805.09331

Jensen, Pinzani-Fokeeva, Yarom
1701.07436, 1804.04654

Can couple to covariant Schwinger-Keldysh sources, do the derivative expansion, classify transport coefficients in the effective action

Stochastic hydro emerges as a certain limit of the SK action

A more modern view

Q: What is a transport coefficient?

A: A parameter in the Schwinger-Keldysh hydrodynamic effective action. The SK action sits in the path integral over the effective degrees of freedom which describes both thermal and quantum fluctuations.

Main question

Small transport coefficients such as small η/s imply that the effective theory is strongly coupled.

Euclidean strongly coupled theories can be defined on the lattice, and the path integral can be evaluated numerically. Real-time effective theories - not clear.

Holography throws away relevant physics by taking $N \to \infty$.

Unless we are at $N=\infty$, it is not even clear what it means to talk about "small transport coefficients" because there is no quantitative framework to describe their contribution to transport.

My view

Q: What is a transport coefficient?

A: If it is a DC transport coefficient which is “small” or close to a “quantum bound”, I have no idea what the question even means.

Conclusions

When transport coefficients are small, thermal fluctuation effects are large.

The quantitative framework to describe hydrodynamic transport with small viscosity and/or diffusion constant is missing.