Quantum chaos, black hole scrambling and hydrodynamics

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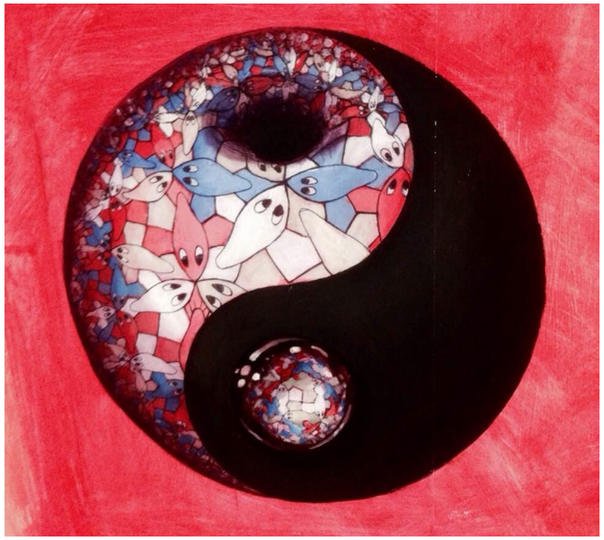


Netherlands Organisation for Scientific Research

Institute for Theoretical Physics

Saso Grozdanov

Vincenzo Scopelliti

Outline

1. Quantum Chaos from an out-of-time-correlation function
2. Chaos and diffusion
3. A bound on chaos = a bound on diffusion?
4. Ultra strongly correlated systems are similar to dilute gases
5. A kinetic equation for Quantum Chaos

Quantum chaos from an out-of-time correlation function

- Classical chaos
  - Initial conditions in classical dynamics

linear/integrable	non-linear/quasiperiodic	non-linear chaotic
Harmonic oscillator	3-body systems: partly	3+body systems: other part
2-body systems	Planetary dynamics	**Ergodicity**
	KAM Theorem	

- Quantum Chaos
  - Incorrect objection: Schrodinger equation is linear:

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$$

  - Correct objection: trajectories are "quantized"

$$\int pdq = 2\pi\hbar n$$

Review:
D'Alessio, Kafri, Polkovikov, Rigol

- Quantum Chaos

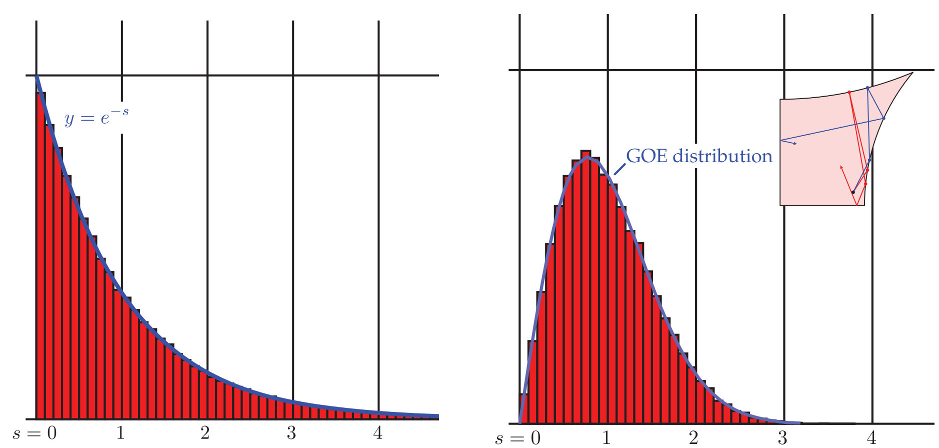
Bohigas, Gianonni, Schmitt

  - Insight: quantized classical chaotic systems have an eigenvalue spectrum in the same class as random matrices (Wigner-Dyson)

dense, interacting spectrum with significant level repulsion

non-interacting harm.osc.

Advances in Physics 257



chaotic cavity

Figure 3. (Left panel) Distribution of 250,000 single-particle energy level spacings in a rectangular two-dimensional box with sides a and b such that $a/b = \sqrt[4]{5}$ and $ab = 4\pi$. (Right panel) Distribution of 50,000 single-particle energy level spacings in a chaotic cavity consisting of two arcs and two line segments (see inset). The solid lines show the Poisson (left panel) and the GOE (right panel) distributions. From Ref. [80].

Berry, Tabor

  - Corrollary: classical integrable systems have an eigenvalue spectrum with Poisson statistics

Caveat: not exact equivalence, there are counterexamples

- In a dynamical setting: when does the dynamics become indistinguishable from RMT?

Ergodicity

- A third way to detect chaos

$$C(t) = -\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle$$

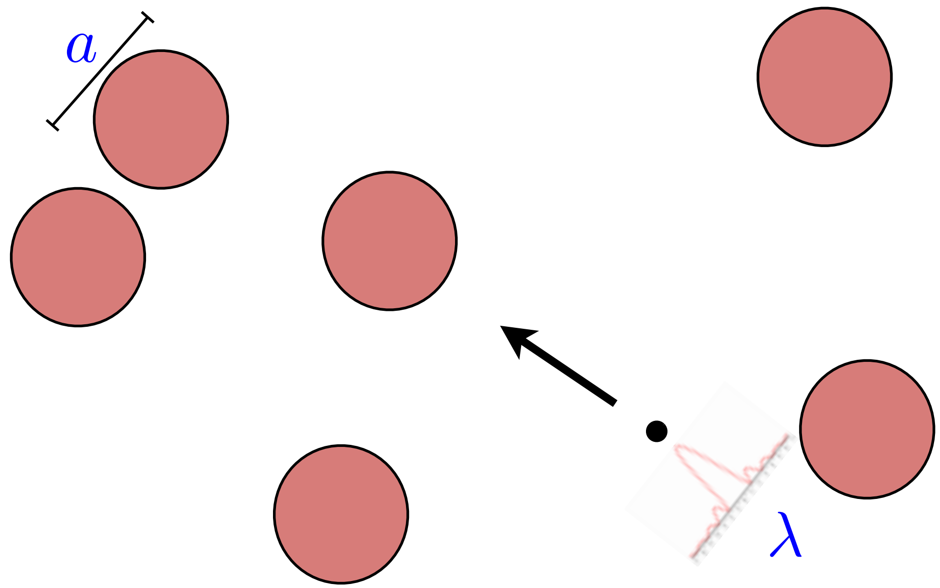
  - Choose

$$W = q(t) \quad V = p(0)$$

$$[W(t), V(0)] = [q(t), p(0)] = i\hbar\{q(t), p(0)\} = i\hbar \frac{\partial q(t)}{\partial q(0)}$$

$$\text{Chaos}: \; q(t) \sim \delta q(0) e^{\lambda_L t} \qquad C(t) \sim \hbar^2 e^{2\lambda t} \text{ with } \lambda = \lambda_{\text{Lya}}$$

- Semi-classical computation of conductivity in weak disorder

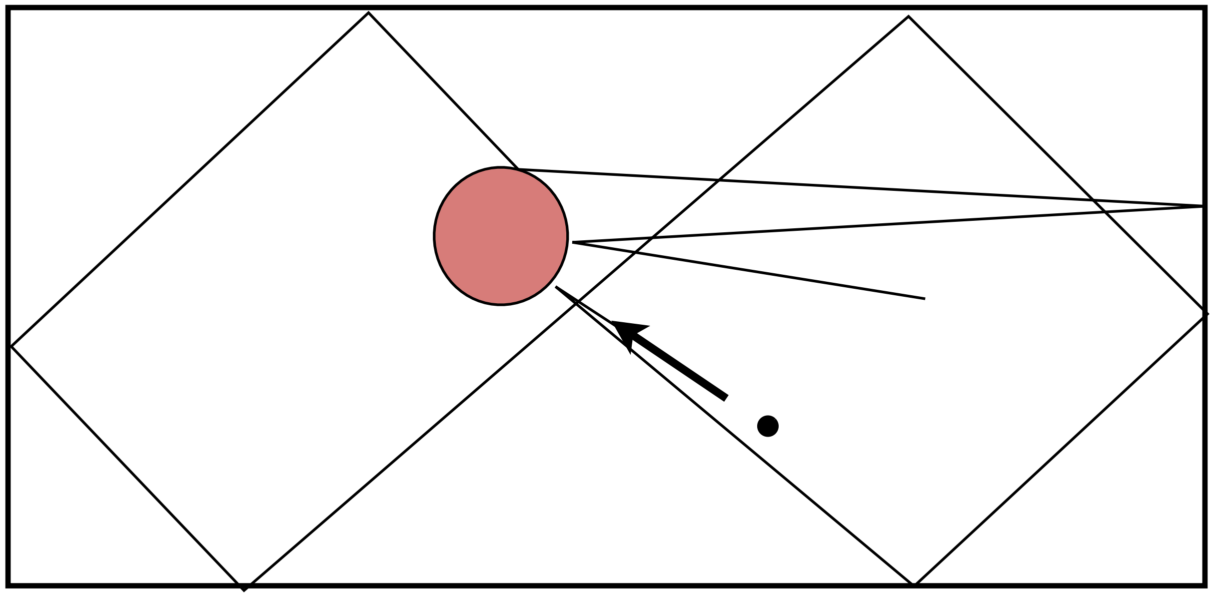


- Semiclassical regime $\lambda \ll a$

Larkin, Ovchinnikov

$$C(t) = -\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle \sim \hbar^2 e^{2\lambda t}$$

- Semi-classical computation of conductivity in weak disorder



- Semiclassical regime $\lambda \ll a$ variation on Sinai billiards

Larkin, Ovchinnikov

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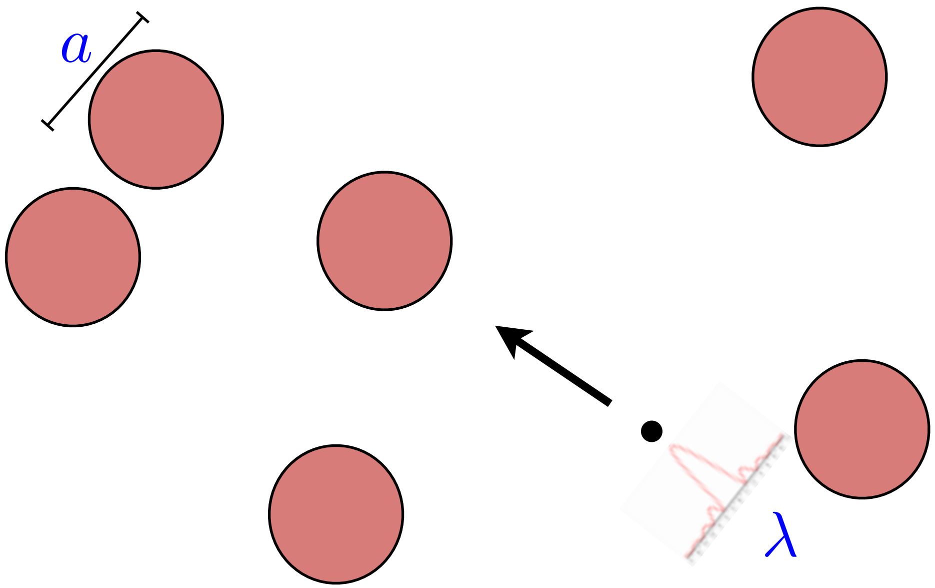

- Semiclassical regime $\lambda \ll a$
- Nevertheless: quantum physics takes over when

Larkin, Ovchinnikov

$$C(t) = -\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle \sim \hbar^2 e^{2\lambda t} \sim 1$$

Ehrenfest time: $t_{Ehr} = \frac{1}{\lambda} \ln \frac{1}{\hbar}$

- Careful:

  In the quantum regime chaotic behavior is hard.

  i.e. most quantum analogues of classical systems with chaos do not exhibit exponential growth in this OTOC correlator.

  - Need a small parameter — e.g. Grozdanov, Kukuljan, Prosen
  - In semi-classical systems $\hbar$ $\quad C(t) \sim \hbar^2 e^{2\lambda t}$
  - In holography: $\frac{1}{N}$ $\quad C(t) \sim \frac{1}{N^2} e^{2\lambda t}$

    Semi-classical single-trace lumps: large N classicalization/ master field

- In a dynamical setting: when does the dynamics become indistinguishable from RMT?

Ergodicity

- In a dynamical setting: when does the dynamics become indistinguishable from RMT?

Ergodicity

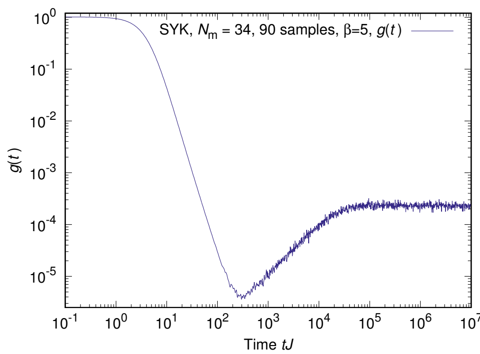

Figure 1: A log-log plot of SYK $g(t;\beta=5)$, plotted against time for $N=34$. Here we use the dimensionless combination tJ for time. Initially the value drops quickly, through a region we call the *slope*, to a minimum, which we call the *dip*. After that the value increases roughly linearly, $\sim t$, until it smoothly connects to a plateau around $tJ = 3\times10^4$. We call this increase the *ramp*, and the time at which the extrapolated linear fit of the ramp in the log-log plot crosses the fitted plateau level the *plateau time*. The data was taken using 90 independent samples, and the disorder average was taken for the numerator and denominator separately.

Cotler et al

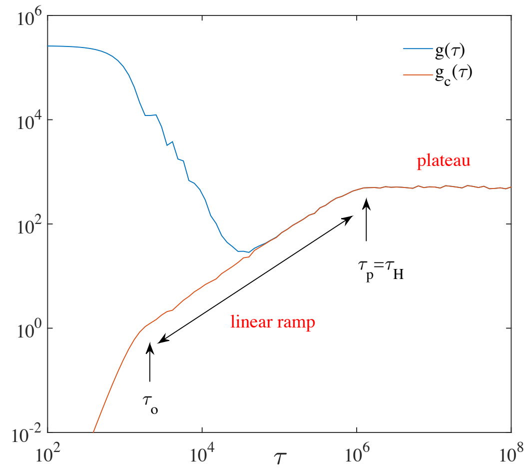

FIG. 1. Typical structure[15] of the linear universal "ramp" in the spectral form factor $g(\tau)$ as well as of the connected spectral form factor $g_c(\tau)$, which exhibits a *longer* "ramp" ranging from a microscopic short time scale τ_0 below which non-universal effects set in, up to the Heisenberg time τ_H (also called plateau time τ_p).

Chen, Ludwig

Chaos and diffusion

- A very special feature of dilute gases

Maxwell

$$\eta = \frac{1}{3} m \rho \ell_{\mathrm{m.f.p.}} \sqrt{\langle v^2 \rangle}$$

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$$\eta = \frac{1}{3} m \sqrt{\langle v^2 \rangle} \frac{1}{\sigma_{2-to-2}}$$

van Zon, van Beijeren, Dellago

$$\lambda = \frac{1}{\tau_{\rm ave}} \langle \frac{1}{2} \ln(\Delta \vec{v})^2 \rangle \simeq \frac{\sqrt{\langle v_{\rm rel}^2 \rangle}}{\ell_{\rm m.f.p.}} \simeq \rho \sqrt{\langle v^2 \rangle} \sigma_{\text{2-to-2}}$$

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- This is behind the Boltzmann equation

$$\frac{d}{dt} f(\mathbf{p}, t) = \int_{\mathbf{k}} (R^{in}(\mathbf{p}, \mathbf{k}) - R^{out}(\mathbf{p}, \mathbf{k})) f(\mathbf{k}, t)$$

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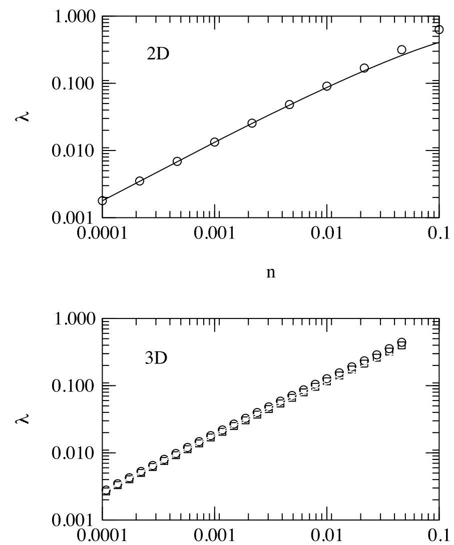
van Zon, van Beijeren, Dorfman; Saarloos

- Can we understand chaos from a kinetic-like equation?



Ad hoc: clock equation

$$\frac{d}{dt} f_k = -f_k + f_{k-1}^2 + 2 f_{k-1} \sum_{\ell=0}^{k-2} f_\ell$$

- Hydro and scrambling are different scales:
  - BBGKY hierarchy from statistical partition function

$$\frac{d}{dt} f_n = \sum_{i=1}^{n} \int dq_{n+1} dp_{n+1} \left\{ U, f_{n+1} \right\}_{q_{n+1}, p_{n+1}}$$

Early time controlled by f_1

scrambling, chaos

Ergodicity

Late time controlled by f_n

Transport

relaxation to equil.

- Dilute approximation (truncates hierarchy)

$$f_2 \sim f_1^2$$

gives Boltzmann equation

$\text{scrambling=chaos=ergodicity}$ is very different from $\text{local therm.=equilibration}$

There is a connection:

In classical thermalization chaos is the source of ergodicity

In special situations (weakly coupled dilute gas) they are set by the same physics

A bound on chaos = a bound on diffusion?

- A bound on chaos

Maldacena, Shenker, Stanford

  - Related regulated function:

$$F(t) = \langle W(t) y V(0) y W(t) y V(0) y \rangle \sim 1 - e^{2\lambda t}$$

$$y^4 = \frac{e^{-\beta H}}{Z}$$

  - *Not time ordered:* but $|TFD\rangle = \sum_n e^{-\frac{\beta}{2}E} |n\rangle |n\rangle$

$$F(t) = \sum \langle TFD | (W(t) V(0) \otimes \mathbb{1})(1 \otimes W(t) V(0)) | TFD \rangle$$

$$F(t) \sim \sum \langle W(t) V(0) \rangle^\dagger \langle W(t) V(0) \rangle$$

  - Analyticity in QFT demands

$$\lambda \leq 2\pi T$$

- Black holes saturate this bound: maximal chaos

$$\lambda_{BH} = 2\pi T$$

  - This observation is the driving force behind SYK

Kitaev
e.g. Stanford@Strings'16

It would be nice to have a solvable model of holography.

theory	bulk dual	anom. dim.	chaos	solvable in $1/N$
SYM	Einstein grav.	large	maximal	no
$O(N)$	Vasiliev	$1/N$	$1/N$	yes
SYK	"$\ell_s \sim \ell_{AdS}$"	$O(1)$	maximal	yes

- OTOCs in finite N SYK

Bagrets, Altland, Kamenev

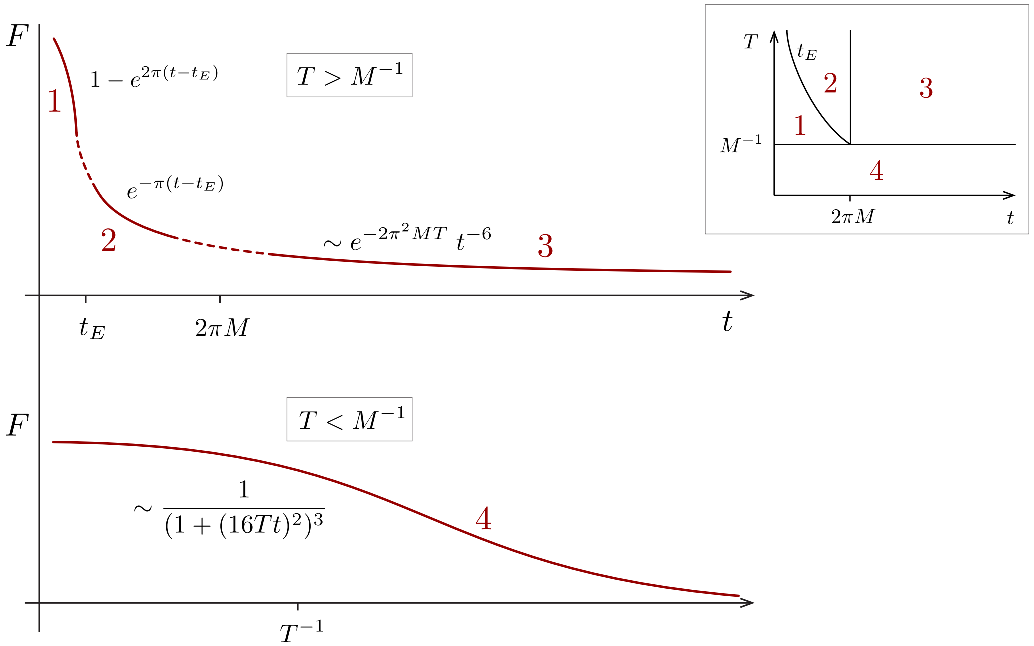


Figure 1: Results for the OTO correlation function. Top: At high temperatures, $T > M^{-1}$ and large times, $t > 2\pi M$, the function crosses over from exponential to power-law decay with an exponent t^{-6}. Bottom: at low temperatures, $T < M^{-1}$ the function is nowhere exponential. At large times $t > T^{-1} > M^{-1}$ it again shows t^{-6} power-law behavior. The inset shows the parametric extension of the four regimes in a $t - T$ plane.

$$t_E = \frac{\ln(MT)}{2\pi T}$$

$$M = \frac{N\ln(N)}{64\sqrt{\pi}J}$$

- A refined version

$$C(t,x) = -\langle [W(t,x), V(0)]^\dagger [W(t,x), V(0)] \rangle \sim \hbar^2 e^{\xi(x - v_{LR} t)}$$

gives you a "scrambling" velocity

$$\xi v_{LR} = 2\lambda$$

  - First pioneered in 1+1 dimension systems
  - Lieb-Robinson proved:

    The velocity v_{LR} is an absolute upper bound on information spreading.
  - v_{LR} acts as en emergent lightcone.

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    The velocity v_{LR} is an absolute upper bound on information spreading.
  - v_{LR} acts as en emergent lightcone.
- Idea: also in other systems this butterfly/Lieb-Robinson velocity is the maximum "speed" at which information spreads

- Diffusion is characterized by a velocity

$$D \sim \frac{v^2}{T} \sim \frac{v^2}{\lambda}$$

- Long sought goal: a fundamental quantum bound on diffusion

$$\frac{\eta}{s} \geq \frac{1}{4\pi}$$

Kovtun, Son, Starinets

$$D \geq \frac{v_{inc}^2}{T}$$

Hartnoll
Hartman, Hartnoll, Mahajan

- (Unstated) Hypothesis: v_{LR} provides this fundamental velocity

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- Long sought goal: a fundamental quantum bound on diffusion

$$\frac{\eta}{s} \geq \frac{1}{4\pi}$$

Kovtun, Son, Starinets

$$D \geq \frac{v_{inc}^2}{T} \quad \text{or} \quad D \leq \frac{v_{inc}^2}{T}$$

Hartnoll
Hartman, Hartnoll, Mahajan
Lucas,
......

- (Unstated) Hypothesis: v_{LR} provides this fundamental velocity

- Scrambling rate/Chaos is a microscopic “particle” property

- Diffusion is a macroscopic collective property

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A priori these are determined by very different physics.

Khemani, Viswanath, Huse

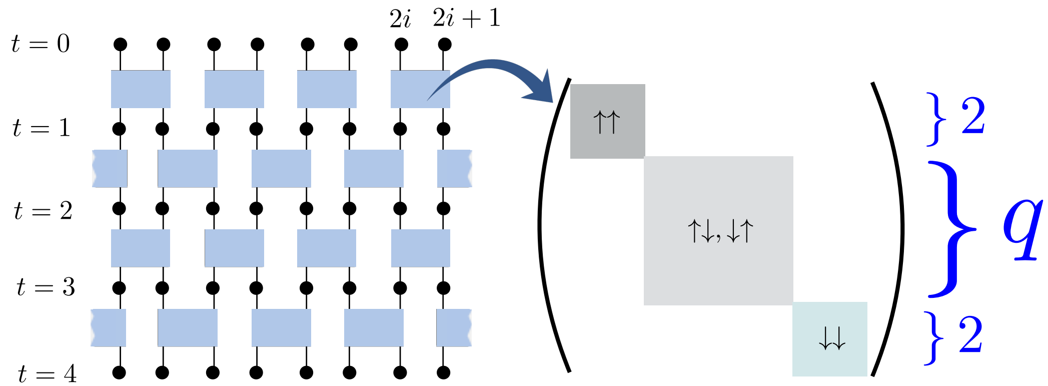
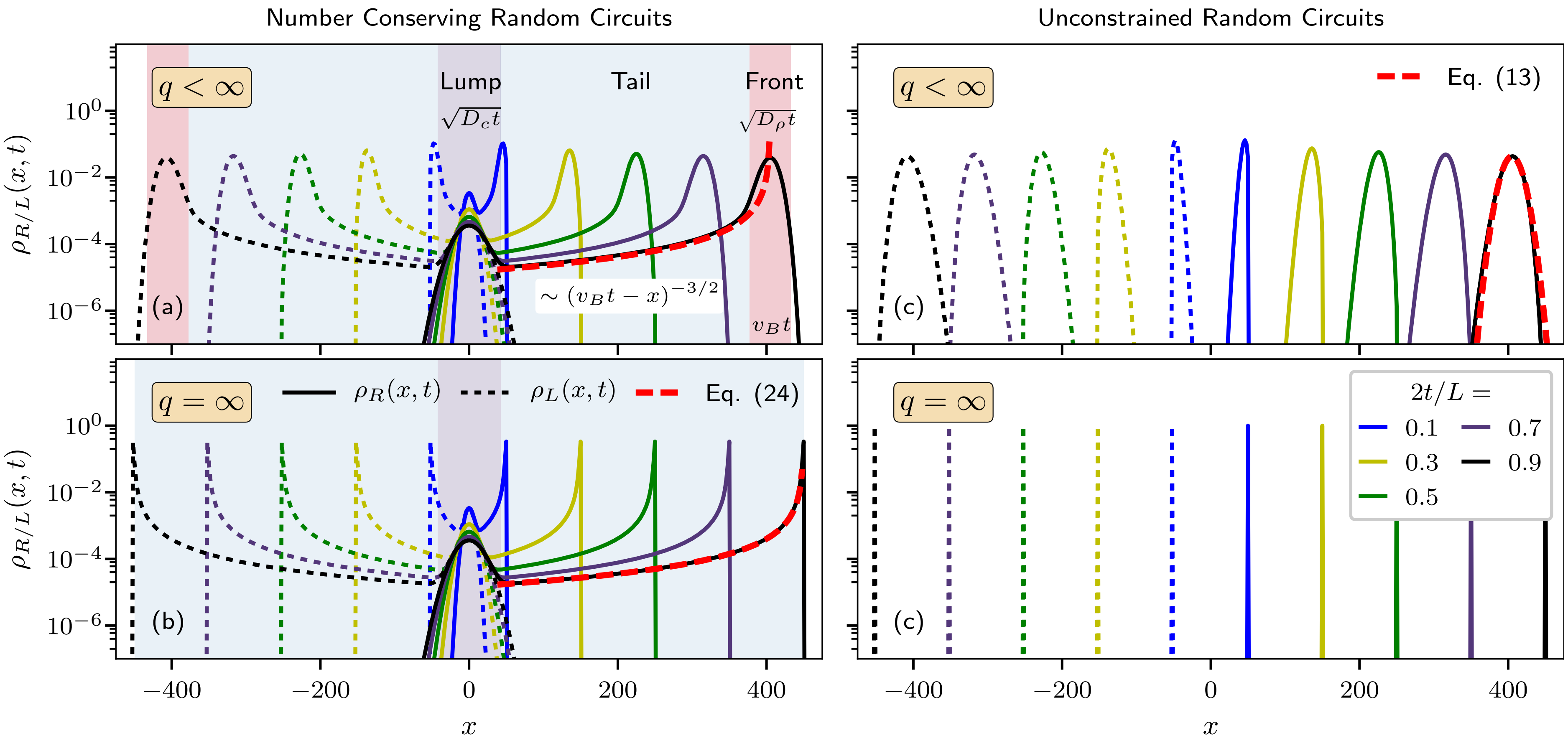


FIG. 1. Left: a diagram of the unitary circuit. Each site (black dot) is the direct product of a two-state qubit and a q-state qudit. Each gate (blue box) locally conserves S_z^{tot}, the total z component of the two qubits it acts upon, and is thus a block-diagonal unitary of the form shown on the right, with each block of each gate independently Haar-random. The smaller blocks do not flip the qubits and thus operate only on the two qudits, while the larger block also produces S_z^{tot}-conserving qubit "flip-flops".

Khemani, Viswanath, Huse



$\frac{1}{q}$ acts as the coupling constant

Khemani, Viswanath, Huse

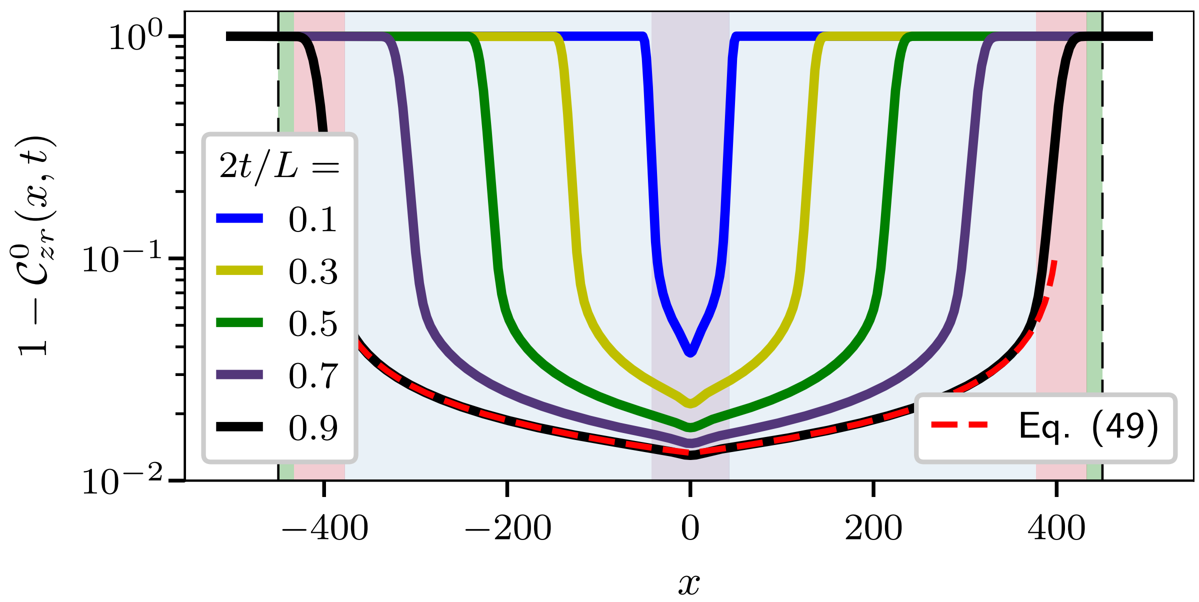


FIG. 4. One minus the out-of-time-order commutator (OTOC) between $z_0(t)$ and r_x at zero chemical potential, $\mathcal{C}^0_{zr}$, plotted against x for a system of length $L = 1000$ at different times t showing the different regimes discussed in the text. For $|x| > t$ (outside the dashed vertical lines), the OTOC is strictly zero due to the locality of the circuit. In the region $v_B t < |x| < t$, which is inside the causal light cone but before the leading front arrives, the OTOC is exponentially small (green shaded area for the latest time). The arrival of the ballistic operator front ($|x| \sim v_B t$) leads to a strong increase in the OTOC from a value exponentially small to an $O(1)$ value (shaded red area for the latest time). However, diffusive tails in the operator shape or internal structure lead to diffusive power-law tails in space and time $\sim (x - v_B t)^{-1/2}$ in the late-time approach of the OTOC to its final value of 1 (shaded blue area for the latest time). By contrast, for an unconstrained random circuit (not shown), the OTOC at a given site approaches one exponentially quickly after the leading front passes[35,36]. The diffusive region near the origin $|x| \lesssim \sqrt{D_c t}$ (shaded purple) receives a subleading $1/t$ contribution from the conserved charges which shows up as a "dimple" in the curves at early times which becomes weaker at late times. All curves are obtained via a simulation using $q = 3$ and taking into account all processes to order $1/q^2$. The dashed red curve is the $q = \infty$ prediction for the functional form of the tail (49).

Late time behavior: $(x - v_B t)^{-1/2}$, no small parameter: $\frac{1}{q} = \frac{1}{3}$

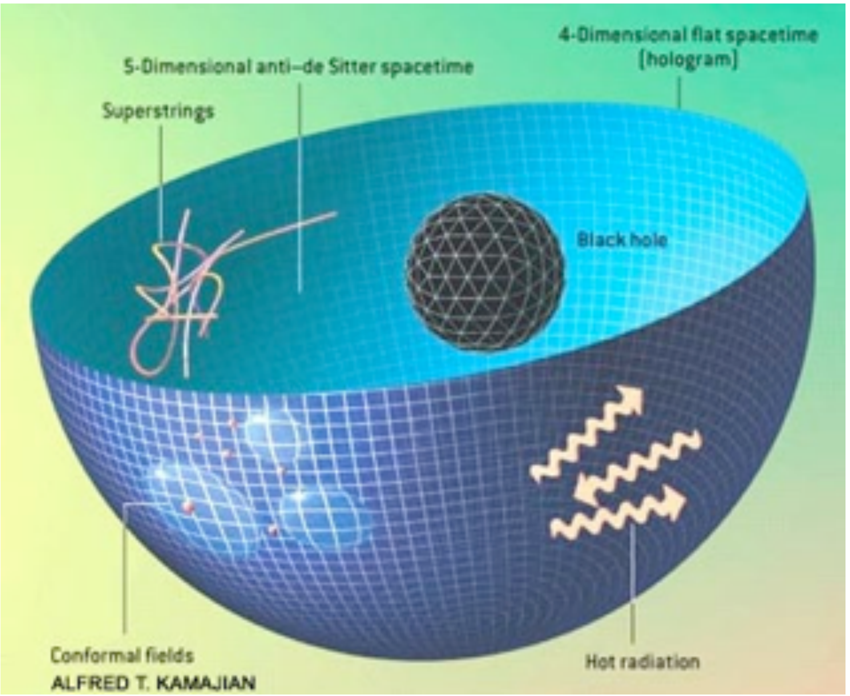
- Is scrambling rate related to diffusion?

$$D \sim \frac{v^2}{T} \sim \frac{v_{\mathrm{LR}}^2}{\lambda}$$

Ultra strongly correlated systems are similar to dilute gases

AdS-CFT duality

strongly coupled field theories without an energy scale (CFT) have a dual description as a weakly coupled string theory in negatively curved space time (AdS).

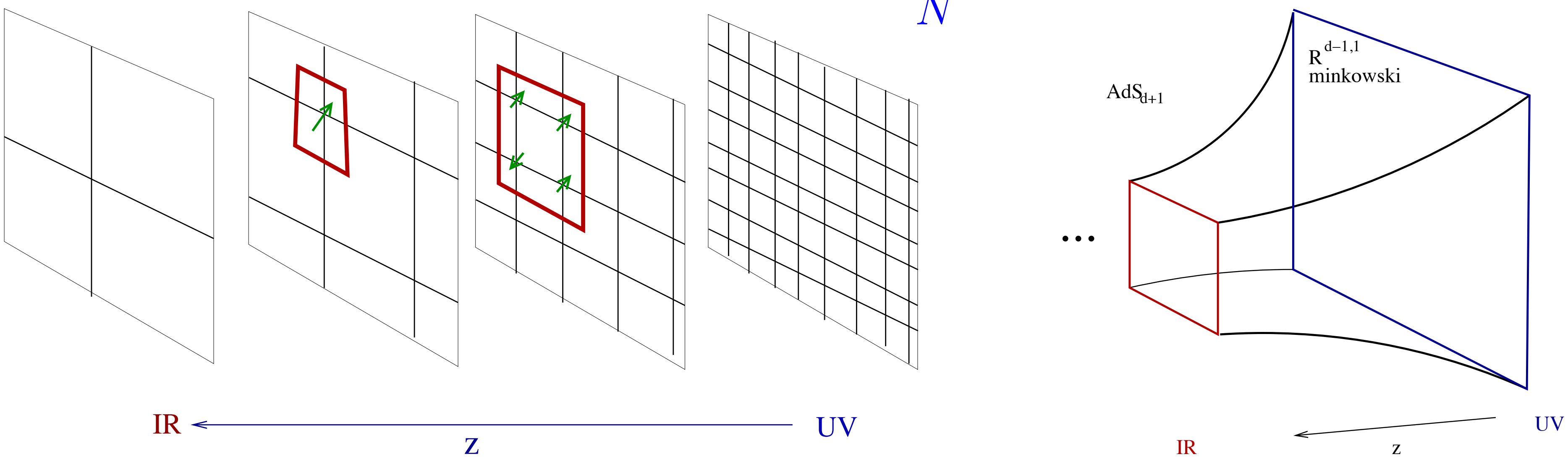


Maldacena ATMP**2**, 231 (1998); Witten ATMP**2**, 253 (1998); Gubser, Klebanov, Polyakov, PL**B428**,105 (1998)

Holography for Strongly coupled systems

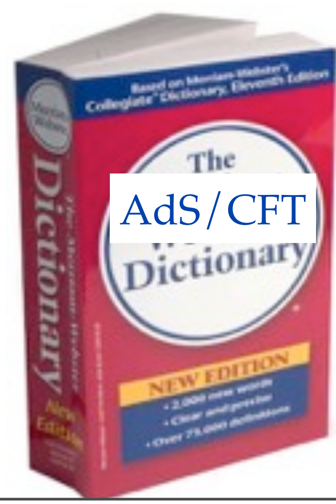
works best when d.o.f. are matrices Φ_{ij} $i, j = 1 \dots N$ with $N \gg 1$

semi-classical limit $\frac{1}{N} \to 0$



$$Z_{CFT}(J) = \exp i S_{AdS}^{\mathrm{on-shell}}(\phi(\phi_{\partial AdS} = J))$$

Quantum numbers
Finite Temp
Finite Density
Conserved Current
Energy dynamics



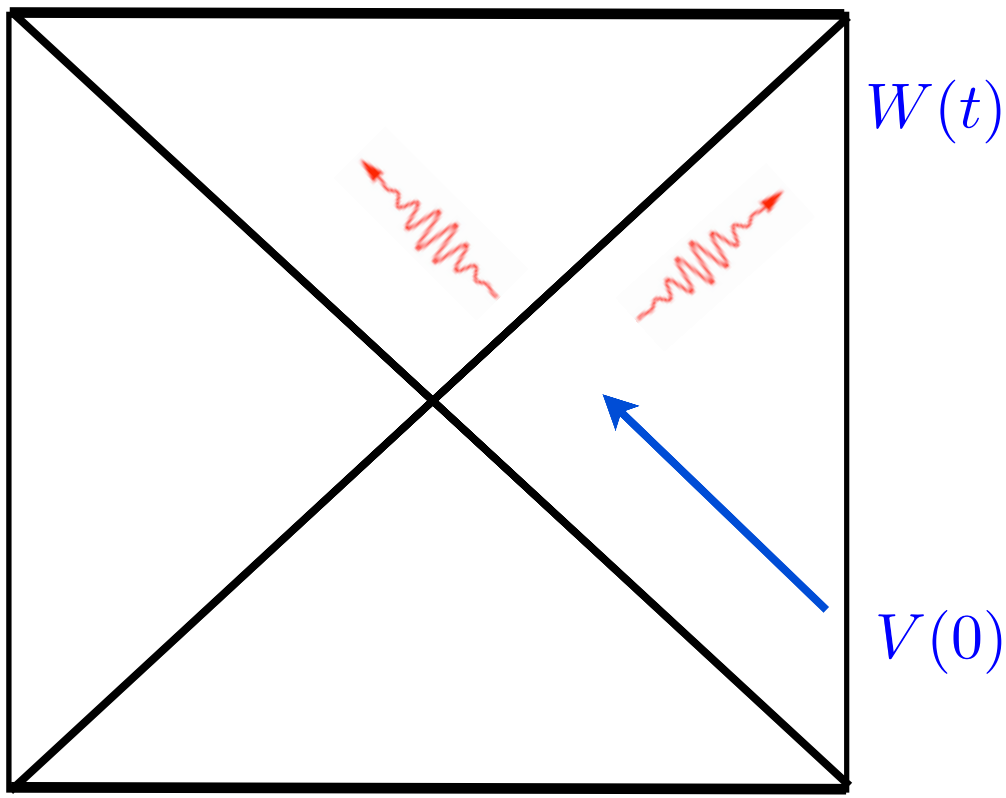
Quantum numbers
AdS Black hole
Extremal AdS black hole
Gauge field
Gravity dynamics

OTOC in holography

- Shockwave calculation in AdS BH

Roberts, Stanford, Susskind

$$F(t) = \sum \langle TFD|(W(t)V(0) \otimes \mathbb{1})(1 \otimes W(t)V(0))|TFD\rangle$$



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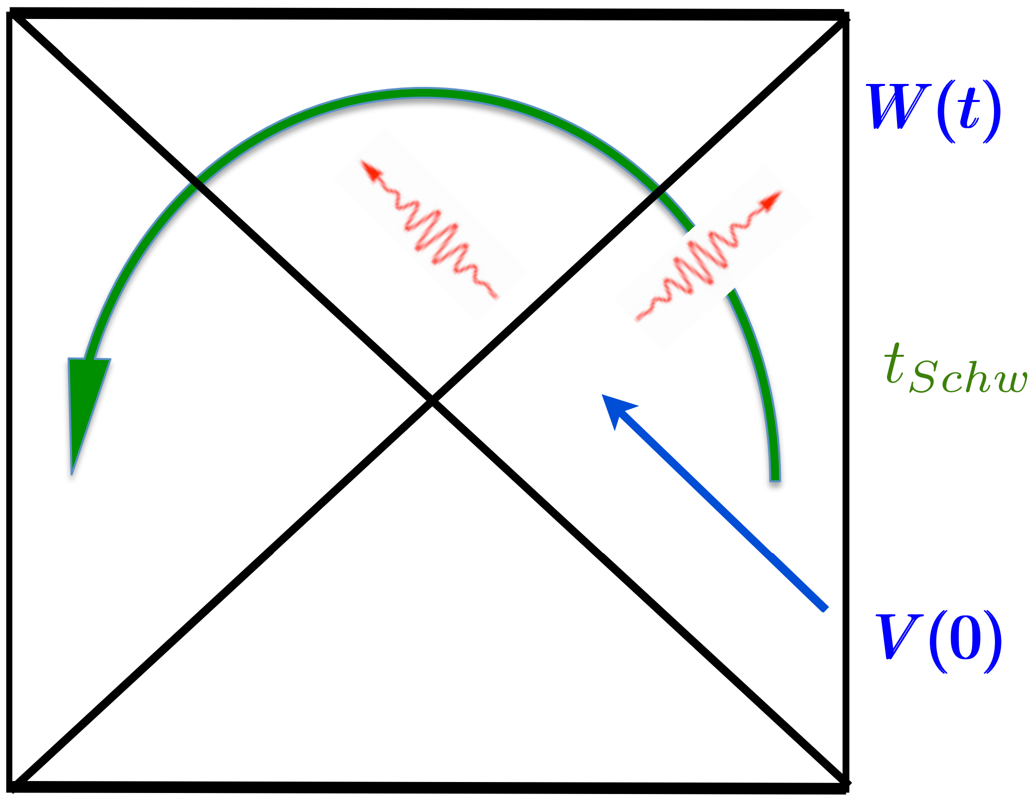


- Is scrambling rate related to diffusion?

Blake;
Davison, Fu, Georges, Gu, Jensen, Sachdev.

For "relevant diffusion" (=irrelevant suscep)

$$D = \frac{d-\theta}{\Delta_\chi}\frac{v_{LR}^2}{2\pi T}$$

$$\Delta_\chi \equiv [\rho] - [\mu] > 0$$

..similar results for massive gravity (mean-field disorder), but fails in general

Lucas, Steinberg;
Gu, Lucas, Qi

- Refinement: charged systems with mean-field disorder
  - Thermal diffusivity set by horizon properties only (cf. $D_P = \eta/sT$)

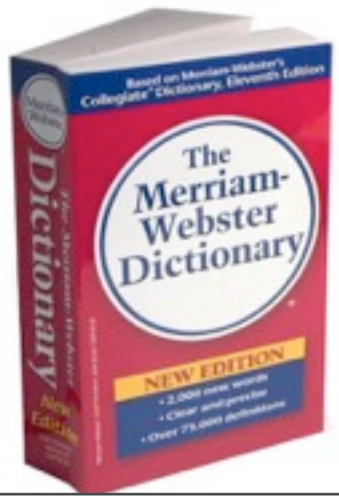
$$D_T = \frac{z}{2z-2}\frac{v_{LR}^2}{\lambda_L}$$

Blake, Davison, Sachdev

- From a physics perspective these are puzzling results:

$$Z_{CFT}(J) = \exp i S_{AdS}^{\text{on-shell}}(\phi(\phi_{\partial AdS} = J))$$

Quantum numbers
Finite Temp
Finite Density
Conserved Current
Energy dynamics



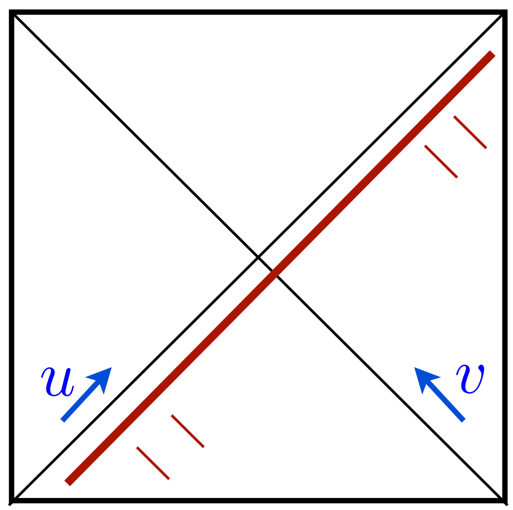
Quantum numbers
AdS Black hole
Extremal AdS black hole
Gauge field
Gravity dynamics

- Shock waves are sound
  - General metric

$$ds^2_{d+2} = A(UV)dUdV + B(UV)g_{ij}dx^i dx^j - A(U,V)h(U,\vec{x})dUdU$$

  - Shock wave equation

$$\delta(U)\left(\Delta_g h - d\frac{B'}{A}h\right) = 32\pi E A \delta^d(\vec{x})\delta(U)$$



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  - Shock wave equation

$$\delta(U)\left(\Delta_g h - d\frac{B'}{A}h\right) = 32\pi E A \delta^d(\vec{x})\delta(U)$$

  - Sound perturbation from AdS/CFT

$$\Delta_g h(U,\vec{x}) - 2d\frac{B}{A}h(U,\vec{x}) - d\frac{B'}{A}U\frac{\partial}{\partial U}h(U,\vec{x}) = 0$$

$$\text{for } h(U,\vec{x}) \sim \delta(U)h(\vec{x}) \text{ reduces to shock}$$

- The shockwave is in Kruskal coordinates.
  - Using Poincare coordinates

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{x}^2 - e^{ikz}\left(f(r)H_1(t,r)dt^2 - 2H_2(t,r)dtdr + H_3(t,r)\frac{dr^2}{f(r)}\right).$$

  - Solution to Einstein's Eqns:

$$H_1(t,r) = H_3(t,r) = \left(C_1\, e^{\frac{k^2 t}{3r_+}} + C_2\, e^{-\frac{k^2 t}{3r_+}}\right) e^{-\frac{k^2+12r_+^2}{3r_+}\int^r dr'\, f(r')^{-1}},$$

$$H_2(t,r) = \left(C_1\, e^{\frac{k^2 t}{3r_+}} - C_2\, e^{-\frac{k^2 t}{3r_+}}\right) e^{-\frac{k^2+12r_+^2}{3r_+}\int^r dr'\, f(r')^{-1}}.$$

- Write as a sound wave.
  - Obeys a diffusion relation

$$\omega_o = \frac{ik^2}{3r_+}, \ \omega_i = -\frac{ik^2}{3r_+},$$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{x}^2 - C_1\, e^{-i\omega_o t + ikz} e^{(i\omega_o - 4r_+)r_*(r)} f(r) \left(dt - \frac{dr}{f(r)}\right)^2$$
$$- C_2\, e^{-i\omega_i t + ikz} e^{-(i\omega_i + 4r_+)r_*(r)} f(r) \left(dt + \frac{dr}{f(r)}\right)^2 .$$

- For the sound wave to be regular (on the horizon)

$$\omega_o = -2ir_+ = -2i\pi T\,, \quad \omega_i = 2ir_+ = 2i\pi T\,,$$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{x}^2 - C_1\, e^{-i\omega_o(t+r_*(r))+ikz} f(r)\left(dt - \frac{dr}{f(r)}\right)^2$$
$$- C_2\, e^{-i\omega_i(t-r_*(r))+ikz} f(r)\left(dt + \frac{dr}{f(r)}\right)^2 .$$

- This regularity condition also means

$$k^2 + \mu^2 = 0, \text{ with } \mu^2 = 6r_+^2 = 6\pi^2 T^2\,,$$

- This is the shock wave equation

$$\left(\partial_i \partial_i - \mu^2\right) h(x) = 0$$

- More precisely:
  - Sound is the physical (gauge-invariant) mode of h_{tt}
  - In radial gauge

$$Z_3 = h_{tt} + \left(\frac{k^2 f' - 2\omega^2 r}{2k^2 r}\right)(h_{xx} + h_{yy}) + \frac{2\omega}{k} h_{tz} + \frac{\omega^2}{k^2} h_{zz}$$

  - In a different gauge

$$Z_3 = h_{tt} - \frac{2i\omega f}{f'} h_{tr} + \frac{f^2}{f'^2}\left(2\omega^2 + f'^2\right) h_{rr}.$$

  - The latter reduces on the horizon to the previous calculation

Support is $1/U$ instead of $\delta(U)$

- Sound at *imaginary* values of frequency and momentum

$$\omega = 2\pi i T = i\lambda \quad , \quad k^2 = -\mu^2 = -6\pi^2 T^2 = -\frac{\lambda^2}{v_B^2}$$

- Hydrodynamical sound (known up to 3rd order analytically)

$$\omega(k) = \pm\frac{1}{\sqrt{3}}k - \frac{i}{6\pi T}k^2 + \ldots$$

  - Relaxational modes: real momentum, complex/imaginary frequency

    measures relaxation time

  - Penetration depth: real frequency, complex/imaginary momentum

    measures relaxation length (penetration depth)

  - Doubly imaginary: “temporal response” to “spatial profile”

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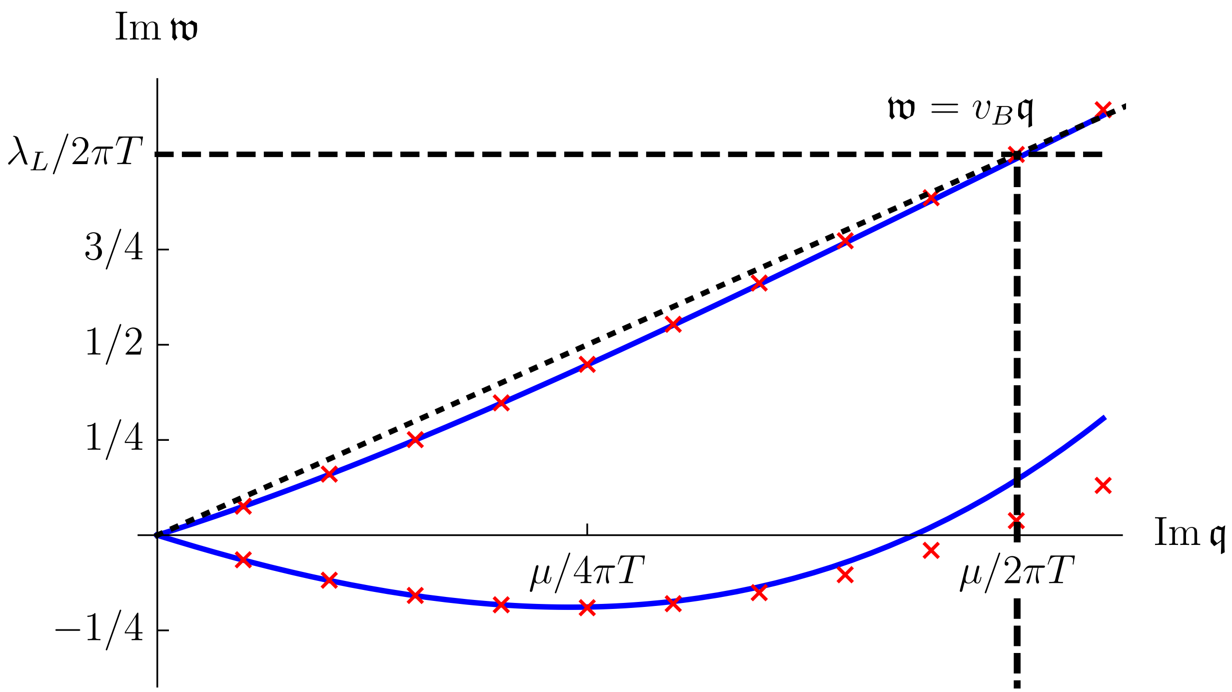

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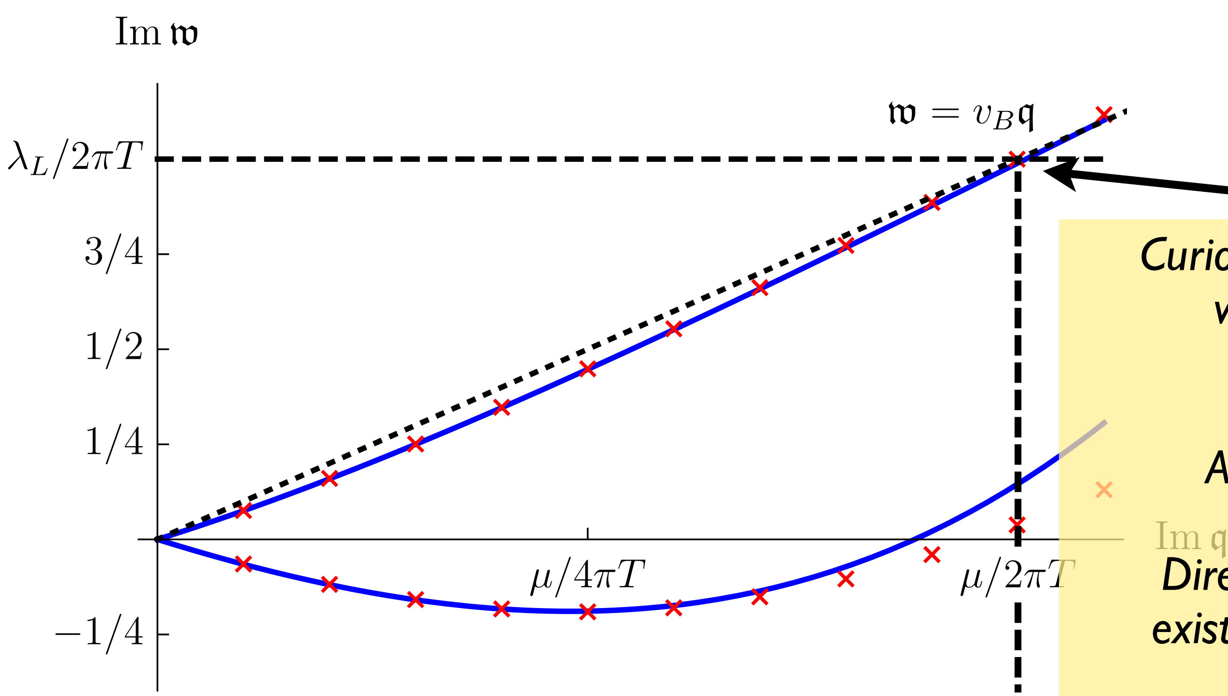

Curious: QNM mode residue vanishes precisely at

$$\omega = 2\pi i T$$

Also happens in SYK.
[Gu, Qi, Stanford]
Direct consequence of the existence of the shockwave solution
[Blake, Lee, Liu]

- In generality

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left[R + \frac{12}{L^2} + \mathcal{L}_{matter}\right]$$

$$ds^2 = -f(r)dt^2 + \frac{g(r)dr^2}{f(r)} + b(r)\left(dx^2 + dy^2 + dz^2\right) - \left[f(r)C_{\pm}W_{\pm}(dt \pm \frac{1}{f(r)}dr)^2\right]$$

$$W_{\pm}(t,z,r) = e^{-i\omega\left[t\pm\int^r \frac{dr'}{f(r')}\right]+ikz}\, h_{\pm}(r)$$

$$\partial_t W \pm |_{r_h} = \mp\mathfrak{D}\,\partial_z^2 W_I|_{rh} \quad tr\text{-Einstein Eq.}$$

$$\mathfrak{D} = \frac{v_{LR}^2}{\lambda_L}$$

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$$D_{CFT} = \frac{\eta}{sT} = \frac{3}{4} D_{hor} \qquad \frac{D}{\mathfrak{D}} = \frac{3\, b'(r_h)}{8\pi T}\,,$$

  - Even though this also computed on the horizon (special to momentum diffusion)

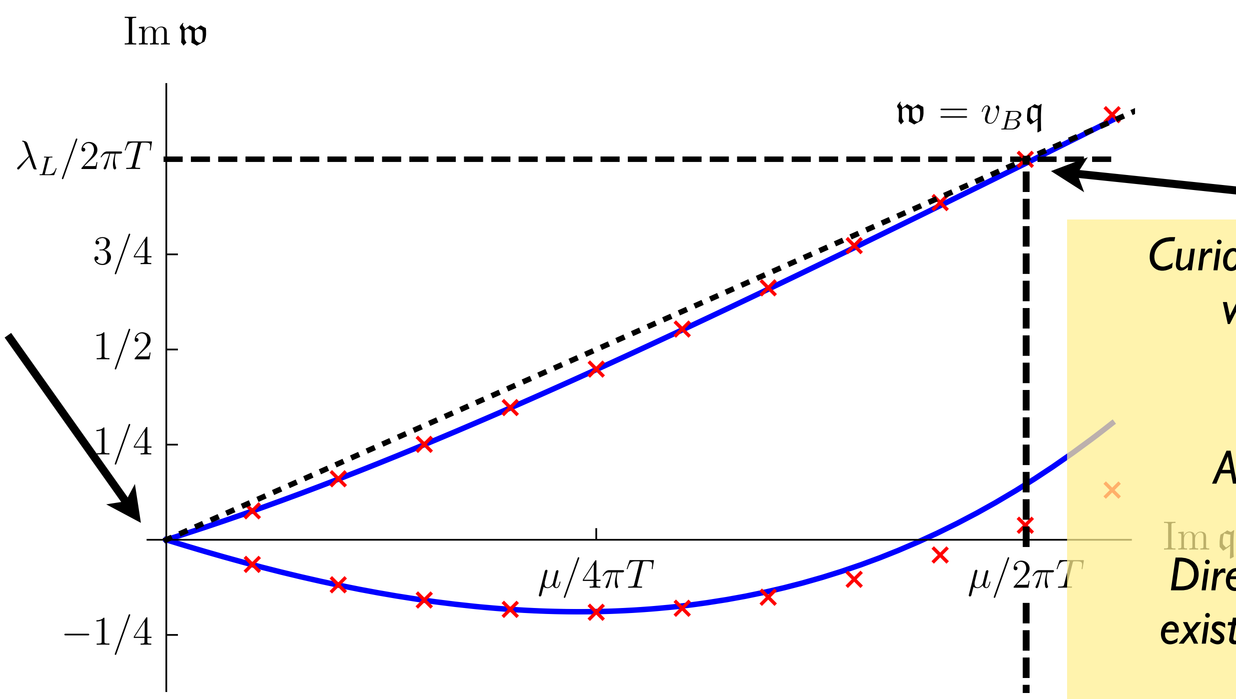
Davison, Fu, Georges, Gu, Jensen, Sachdev.
Blake, Davison, Sachdev

Physical diffusion is given by the behavior near

$\omega \ll 1$

by now verified in many models

[Blake, Davison, Grozdanov,Liu]



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Two important conclusions

- Diffusion is characterized by a velocity

$$D \sim \frac{v^2}{T} \sim \frac{v_{\mathrm{LR}}^2}{\lambda}$$

- Long sought goal: a fundamental quantum bound on diffusion

$$\frac{\eta}{s} \geq \frac{1}{4\pi}$$

Kovtun, Son, Starinets

$$D \geq \frac{v_{inc}^2}{T} \quad \text{or} \quad D \leq \frac{v_{inc}^2}{T}$$

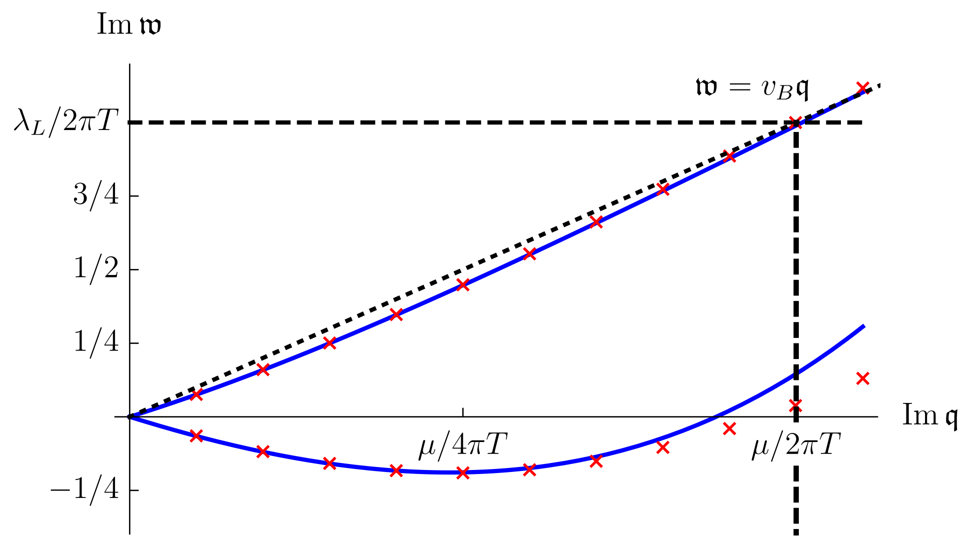
Hartnoll
Hartman, Hartnoll, Mahajan

- (Unstated) Hypothesis: v_{LR} provides this fundamental velocity

- Can v_{LR} give rise to a fundamental diffusion bound?
  - It appears that quantitatively there is no firm relation between late-time diffusion and scrambling

$$\frac{D}{\mathfrak{D}} = \frac{3\,b'(r_h)}{8\pi T}\,,$$

  - The butterfly velocity does not appear to be a speed limit.



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    - Except: a weakly coupled dilute gas.

Maxwell

$$\eta = \frac{1}{3} m \rho \ell_{\mathrm{m.f.p.}} \sqrt{\langle v^2 \rangle}$$

Famous "first" result of molecular kinetic theory

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van Zon, van Beijeren, Dellago

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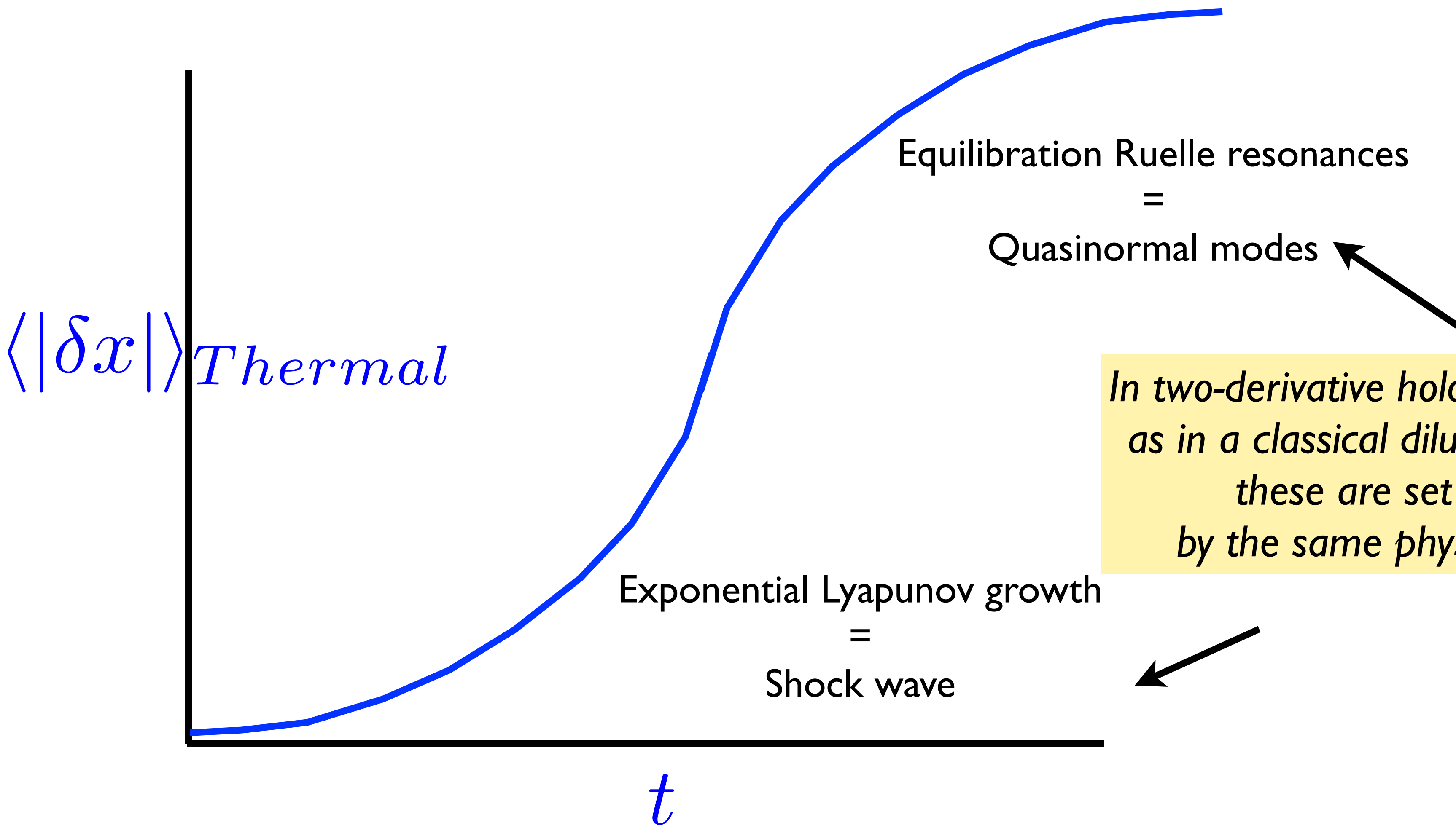
  - Except: two-derivative holography

but now it is the macroscopic properties that set ergodicity

- “Chaos [scrambling] in the black hole S-matrix” Polchinski



In two-derivative holography as in a classical dilute gas these are set by the same physics

Ultra strongly correlated systems are similar to dilute gases

- Quantum chaos in weakly coupled systems

“Surprisingly a relation of the form $D \sim v_{LR}^2 \tau$ shows up in a number of non-holographic contexts”

- Most of these are weakly coupled zero density field theory results.

  This should not be a surprise. This is the classical dilute gas computation.

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  From the point of view what you compute it is a *surprise*

Scrambling in weakly coupled QFT is classical dilute gas

- Object of interest for λ, v_{LR}

$$C(t) = -\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle \quad \sim e^{2\lambda(t - \frac{x}{v_{LR}})}$$

growing mode

- Object of interest for $D = \dfrac{\eta}{\chi}$

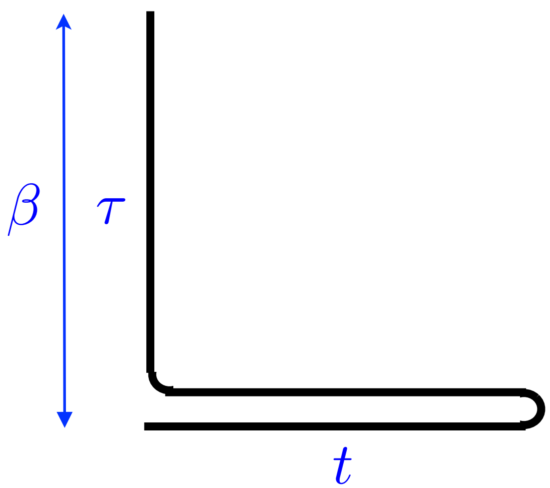
$$\eta = \lim_{\omega \to 0} \frac{1}{i\omega} \mathrm{Im} \langle T_{xy}(\omega), T_{xy}(-\omega) \rangle_R$$

G_R *only supports decaying modes*

- Transport
- Scrambling/Chaos

$$G_R(t) \sim p_x p_y q_x q_y \langle [\Phi^{ab}\Phi^{ab}, \Phi^{cd}\Phi_{cd}] \rangle_\beta$$

Schwinger-Keldysh contour



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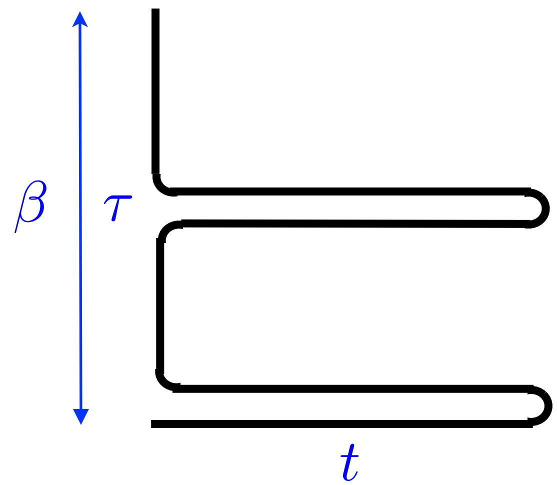
OTOC contour



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OTOC contour

  - In free field theory

$$C(t) \sim G_R(t) = -2G_R^{\Phi\Phi}(t) + \mathcal{O}(\lambda)$$

Stanford, Jeon

  - In perturbation theory Transport and Scrambling sum the same ladder diagrams

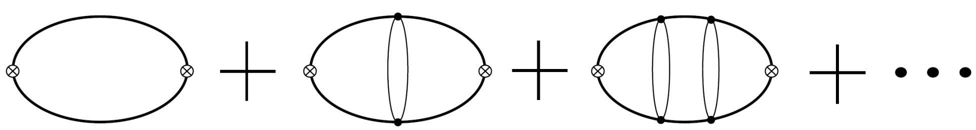

FIG. 2: Resummation of ladder diagrams. The insertions of the energy-momentum tensor operator $\hat{T}^{xy}$ is denoted by the crossed dots and black dots are the vertices with the coupling constant λ.

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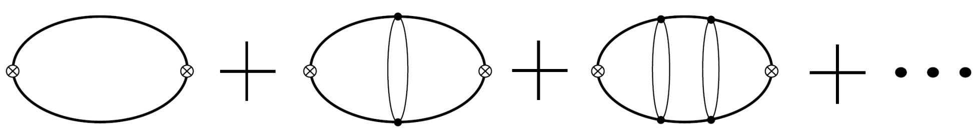

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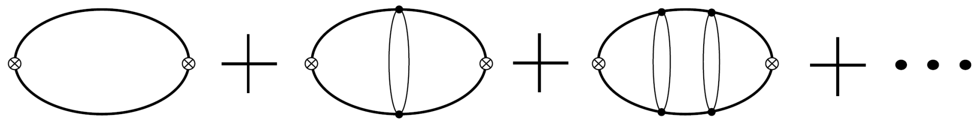
This Bethe-Salpeter eqn is the QFT version of the Boltzmann equation



$$\widetilde{G}(p|k) = \frac{\pi}{E_{\mathbf{p}}} \frac{\delta(p_0^2 - E_{\mathbf{p}}^2)}{-i\omega + 2\Gamma_{\mathbf{p}}} \left[1 + \int \frac{d^4\ell}{(2\pi)^4} R(\ell - p) \widetilde{G}(\ell|k) \right].$$

- Ansatz

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gives

$$\frac{d}{dt} f(\mathbf{p}, t) = \int_{\mathbf{k}} (R^{in}(\mathbf{p}, \mathbf{k}) - R^{out}(\mathbf{p}, \mathbf{k})) f(\mathbf{k}, t)$$

Schwinger Keldysh vsOTOC Contour

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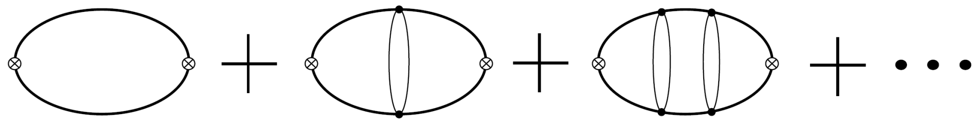
- SchwKeld 

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- OTOC

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Grozdanov, Schalm, Scopelliti,

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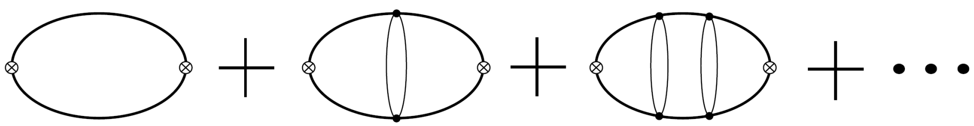


Boltzmann equation (net density)

Kinetic equation (gross collisions)*

$$\frac{d}{dt} f(\mathbf{p},t) = \int_{\mathbf{k}} (R^{in}(\mathbf{p},\mathbf{k}) - R^{out}(\mathbf{p},\mathbf{k})) f(\mathbf{k},t)$$

$$\frac{d}{dt} \mathsf{f}(\mathbf{p},t) = \int_{\mathbf{k}} \frac{\epsilon(\mathbf{p})}{\epsilon(\mathbf{k})} (R^{in}(\mathbf{p},\mathbf{k}) + \widehat{R^{out}}(\mathbf{p},\mathbf{k})) \mathsf{f}(\mathbf{k})$$

purely relaxational

front propagation into unstable states

$$f(\mathbf{p},t) \sim e^{\lambda t} \text{ with } \lambda \leq 0$$

$$\mathsf{f}(\mathbf{p},t) \sim e^{\lambda t} \text{ with } \lambda \leq \lambda_{max} > 0$$

Saarloos, vBeijeren,
Aleiner, Faoro, Ioffe

$$* : \; \widehat{R^{out}}(\mathbf{p},\mathbf{k}) = R^{out}(\mathbf{p},\mathbf{k}) - 2\delta(\mathbf{p}-\mathbf{k}) R^{out}(\mathbf{k},\mathbf{k})$$

- Chaos follows from kinetic equation for *gross* energy exchange

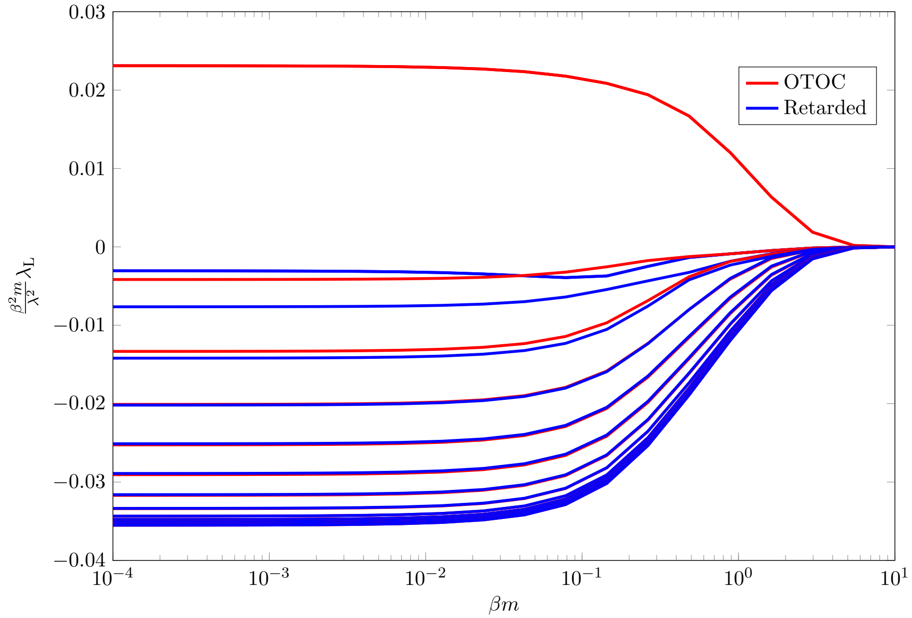
$$\frac{d}{dt}\mathsf{f}(\mathbf{p},t) = \int_{\mathbf{k}} \frac{\epsilon(\mathbf{p})}{\epsilon(\mathbf{k})} \left(R^{in}(\mathbf{p},\mathbf{k}) + R^{out}(\mathbf{p},\mathbf{k}) - 2\delta(\mathbf{p}-\mathbf{k})R^{out}(\mathbf{k},\mathbf{k})\right) \mathsf{f}(\mathbf{k})$$

  - This is derived as opposed to ad hoc clock model

$$\frac{d}{dt}f_k = -f_k + f_{k-1}^2 + 2f_{k-1}\sum_{\ell=0}^{k-2} f_\ell$$

Qualitatively physics is similar (unstable front dynamics) blue: eigenvalues λ for SchwKeld/Boltzmann

red: eigenvalues λ for OTOC/Energy-exchange



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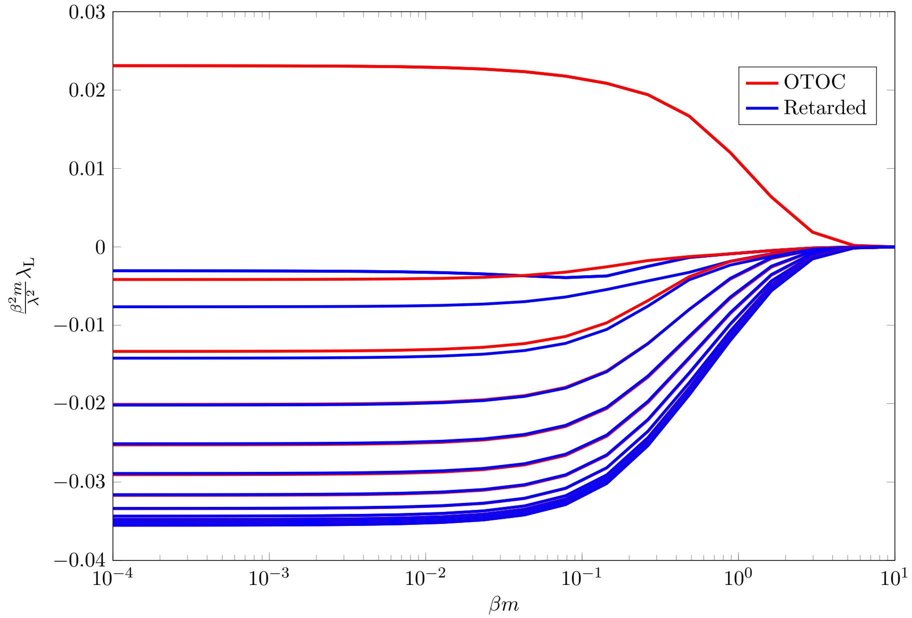

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Conclusion

1. Quantum Chaos from an out-of-time-correlation function

$$C(t) = -\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle \; \sim \hbar^2 e^{2\lambda t} \sim 1$$

2. Chaos and diffusion

different time scales: exception dilute gas

3. A bound on chaos = a bound on diffusion?

No, here, or trivial, or ...

4. Ultra strongly correlated systems are similar dilute gases

Scrambling and diffusion are set by the same physics

5. A kinetic equation for Quantum Chaos

Grozdanov, Schalm, Scopelliti,

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in graphene: Klug, Scheurer, Schmalian

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Thank you