Institute for Theoretical Physics
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CRC183
ENTANGLED STATES OF MATTER

Quantum fluctuation in real and complex SYK models

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Nucl. Phys. B 911, 191 (2016)

Nucl. Phys. B 921, 727 (2017)

to be published ...

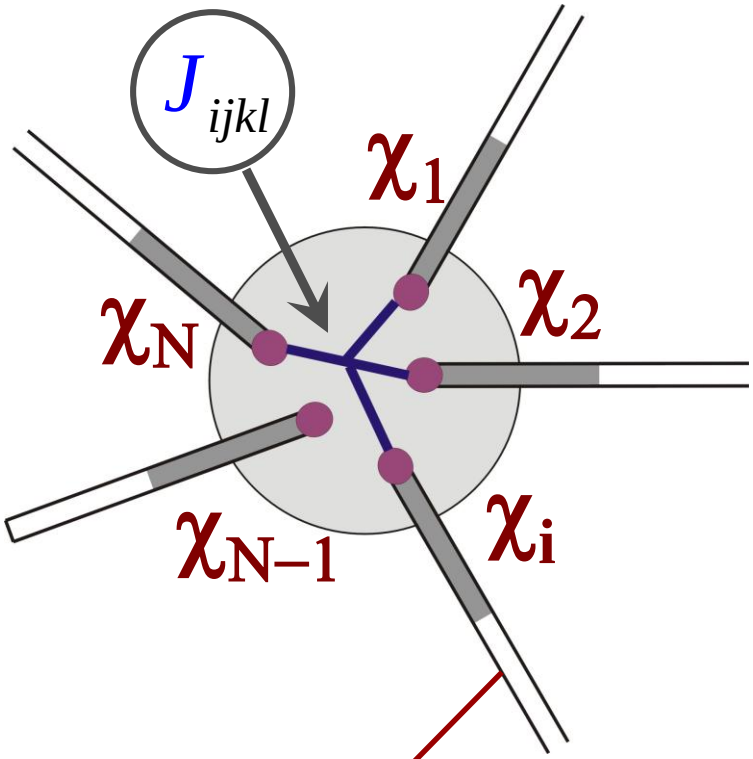
Sachdev-Ye-Kitaev model

S. Sachdev, PRX 5 (2015) 041025
A. Kitaev, talks at KITP, April & May 2015

$$\hat{H} = \frac{1}{4!}\sum_{ijkl}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l$$

Couplings J's are quenched random Gaussian variables,

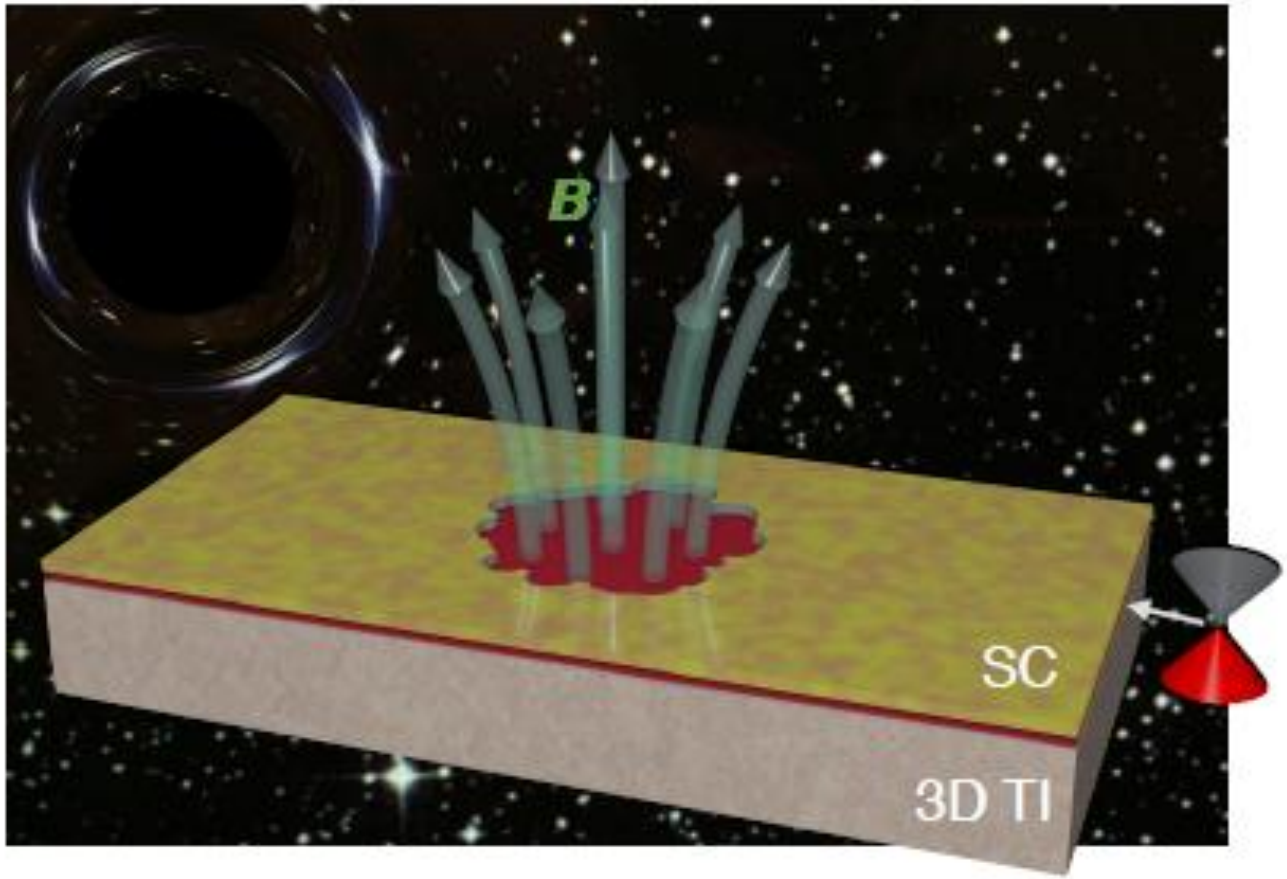
$$\left\langle \left(J_{ijkl} \right)^2 \right\rangle = 3!\, J^2 / N^3$$



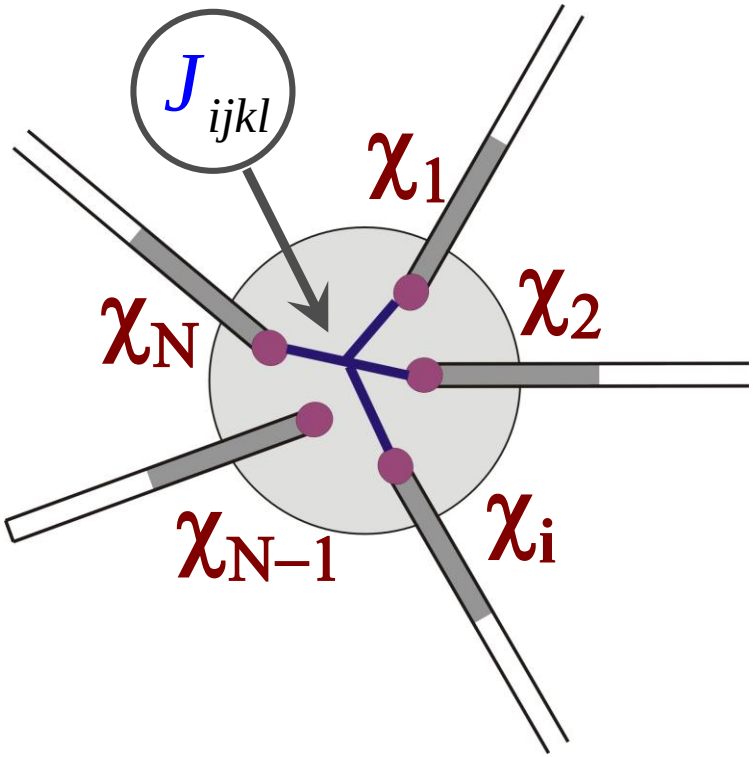
Class D Majorana wire

$\{\chi_i, \chi_j\} = \delta_{ij}$

- **Black hole on a chip: proposal for a physical realization of the SYK model in a solid-state system**



D. Pikulin, M. Franz, arXiv:1702.04426



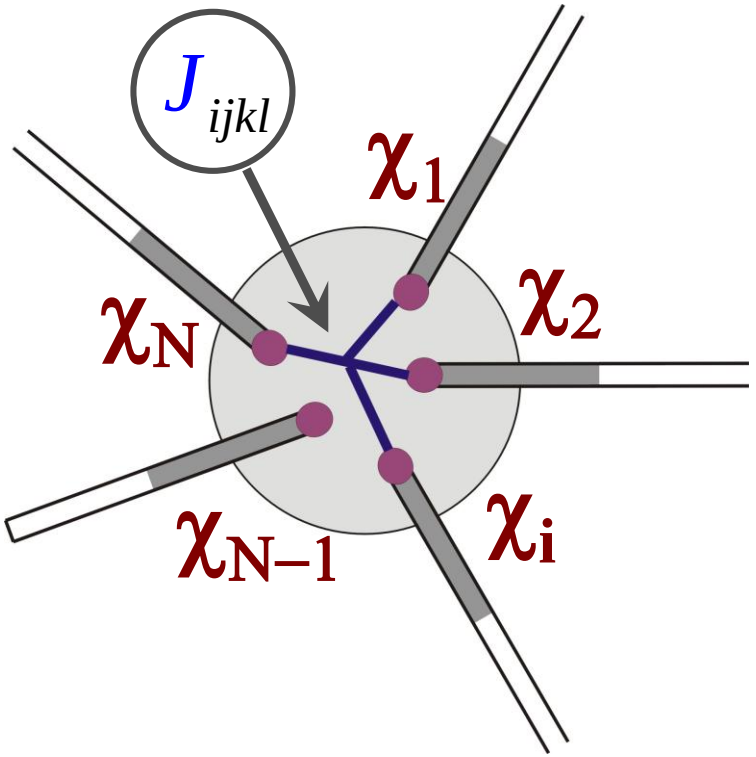
SYK model

- **AdS_2/CFT_1 holography**

  black-hole physics, dilaton gravity, quantum information paradox, gravitational shock-wave scattering, etc.

- **Strong-correlation physics**

  Many-body (de)-localization, RMT-like level statistics, quantum chaos, strange metals, …



This talk

- **Reparametrization ('conformal') symmetry in SYK**
- **Schwarzian action & Liouville quantum mechanics**
- **Complex SYK model: emergent Coulomb blockade**
- **Quantum chaos & OTO correlators**

Infra-red 'conformal' symmetry

A. Kitaev' 2015

- averaging over disorder with the replica trick

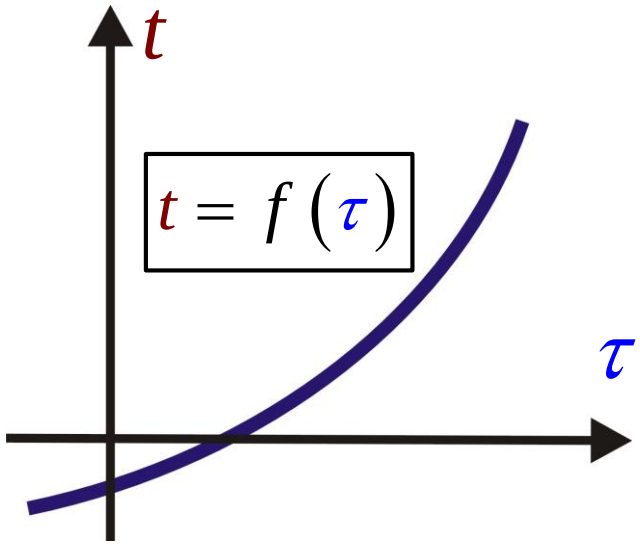
$$\left\langle Z^R = \exp\left(-\sum_{a=1}^{R}\int\left(\dot{\chi}^a\chi^a - H^a\right)dt\right)\right\rangle \qquad \hat{H}^a = \frac{1}{4!}\sum_{ijkl}^{N} J_{ijkl}\chi_i^a\chi_j^a\chi_k^a\chi_l^a$$

Replicated partition sum

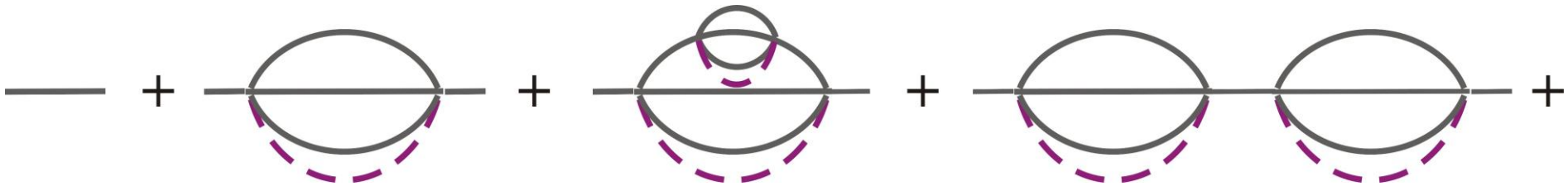
- reparametrizations of the time

$$\tilde{\chi}^a(\tau) = \left[f'(\tau)\right]^{1/4}\chi^a(t)$$

fermion's scaling dimension

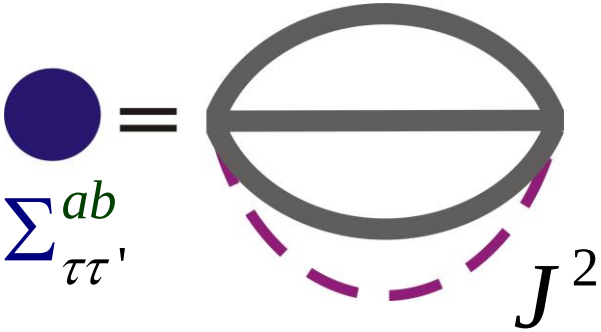


Mean field solution



- **Self-consistent Dyson equation** (S. Sachdev, J. Ye '1993)

$$-\left(\cancel{\partial_\tau} + \Sigma\right) \bullet G = 1, \quad \Sigma^{ab}_{\tau\tau'} = J^2 \left[G^{ab}_{\tau\tau'} \right]^3$$



- **Mean-field solution (T=0)**

$$\bar{G}^{ab}_{t-t'} \propto -\frac{\delta^{ab}}{\sqrt{J}} \frac{1}{|t-t'|^{1/2}}$$

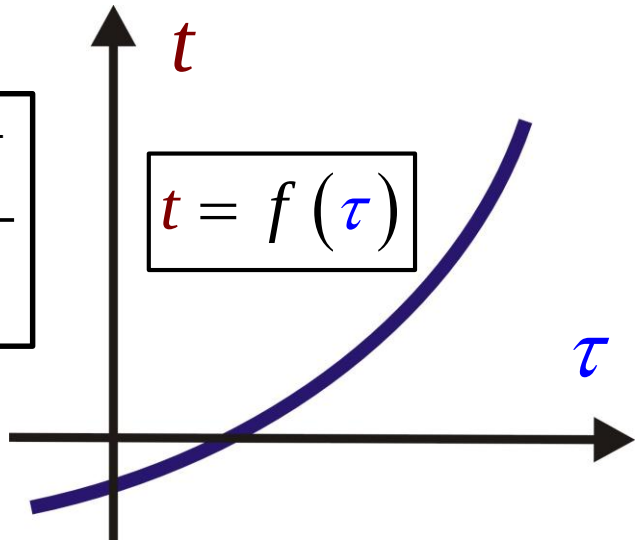
- is of conformal form with scaling dimension $\Delta = 1/4$

Goldstone mode manifold

Kitaev' 2015

- **Mean-field solution is not unique!**
  - in the conformal limit one has infinite set of solutions

$$G(\tau_1,\tau_2) \propto \mp \frac{[f'(\tau_1)]^{1/4}[f'(\tau_2)]^{1/4}}{|f(\tau_1)-f(\tau_2)|^{1/2}}$$



- **Mean-field solution**
  - Invariant under conformal transformation SL(2,R)

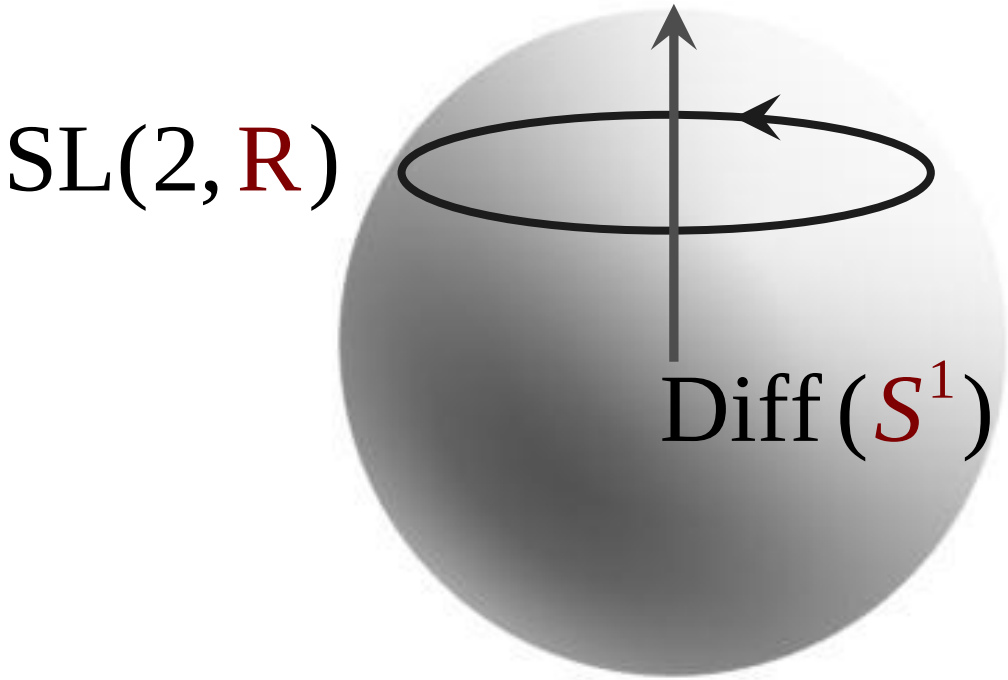
$$\bar{G}^{ab}_{t-t'} \propto -\frac{\delta^{ab}}{\sqrt{J}}\frac{1}{|t-t'|^{1/2}}$$

$$f(\tau) = \frac{a\tau+b}{c\tau+d} \in \mathrm{SL}(2,\mathrm{R})$$

Goldstone mode manifold

- emergence of infinite dimensional soft-mode manifold

$$\boxed{\mathrm{Diff}(S^1)/\mathrm{SL}(2,\mathrm{R})}$$

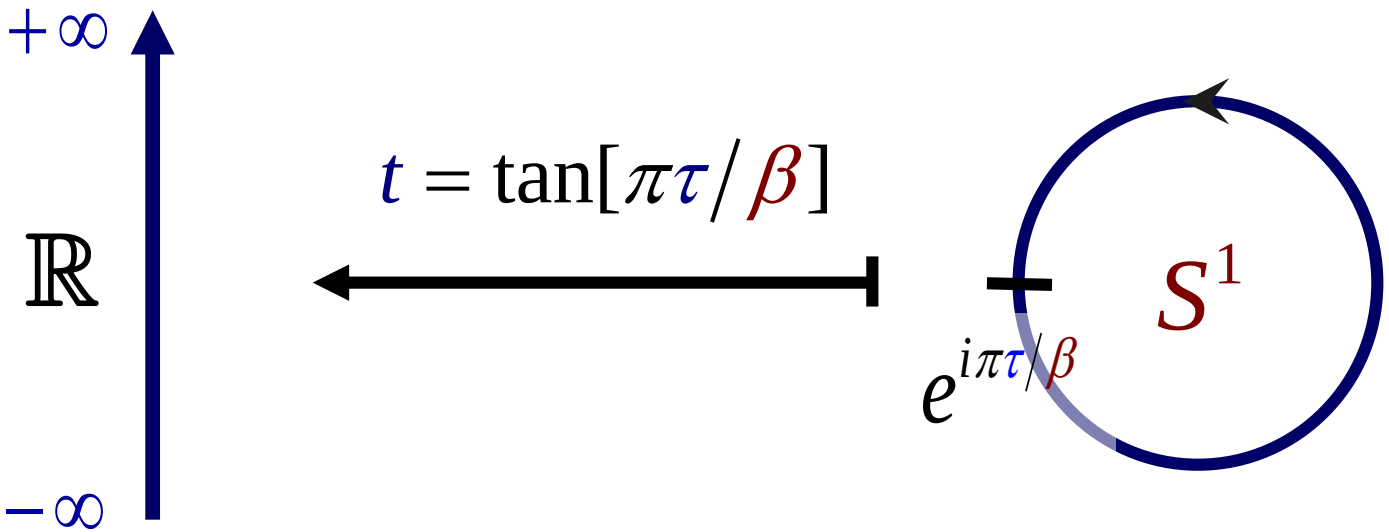


- **Heisenberg magnet:** $\mathrm{SU}(2)/\mathrm{U}(1) \sim S^2$

Finite-T saddle point

Sachdev & Ye '1993, Kitaev' 2015

- **Q: what is the mean field two-point function ?**
  - creative use of the reparametrization symmetry:



$$\bar{G}_t^{ab} \propto -\frac{\delta^{ab}}{\sqrt{J}} \frac{1}{|t|^{1/2}} \longrightarrow \bar{G}_\tau^{ab} \propto -\frac{\delta^{ab}}{\sqrt{\beta J}} \frac{\sqrt{\pi}}{\sin^{1/2}[\pi|\tau|/\beta]}$$

T->0 limit

finite T=1/β

Schwarzian action &
Liouville quantum mechanics

Goldstone action

Kitaev' 2015; J. Maldacena & D. Stanford '2015

- **Schwarzian action of reparametrizations**

$$\boxed{S_0[f] = -M \int_{-\infty}^{+\infty} \{f,\tau\} d\tau}$$

$$\mathrm{tr}\left(\partial_{t_1} \bigcirc \partial_{t_2}\right)$$

$\bar{G}_{12}[f]$

$\bar{G}_{21}[f]$

at t>M fluctuations become strong

- the Schwarzian derivative is defined by

$$\{f,\tau\} \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2, \qquad M \propto N \ln N / J$$

- and respects the coset structure versus H=SL(2,R)

$$\{h \circ f, \tau\} = \{f,\tau\} \quad \text{if} \quad h(t) = \frac{at+b}{ct+d} \in \mathrm{SL}(2,\mathrm{R})$$

Green's function

Q: What is the IR limit of Green's function?

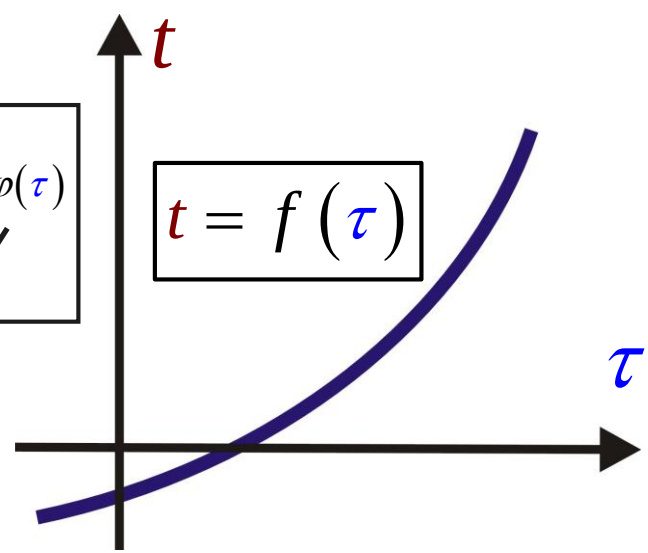
$$G(\tau_1-\tau_2)\propto\mp\int_{G/H}Df(\tau)\times\frac{\left[f'(\tau_1)\right]^{1/4}\left[f'(\tau_2)\right]^{1/4}}{\left|f(\tau_1)-f(\tau_2)\right|^{1/2}}\times e^{-S_0[f]}$$

- average the mean-field result over Goldstone modes

- **Phase representation (measure is flat!)**

$$S_0[\varphi]=\frac{M}{2}\int_{-\infty}^{+\infty}\left[\varphi'(\tau)\right]^2 d\tau,\quad f'(\tau)=e^{\varphi(\tau)}$$

non-compact phase



Green's function

Q: What is the IR limit of Green's function?

$$G(\tau_1-\tau_2)\propto\int_0^{+\infty}\frac{d\alpha}{\sqrt{\alpha}}\left\langle e^{\frac{1}{4}\varphi(\tau_1)}e^{\frac{1}{4}\varphi(\tau_2)}\exp\left[-\alpha\int_{\tau_1}^{\tau_2}e^{\varphi(\tau)}d\tau\right]\right\rangle_\varphi$$

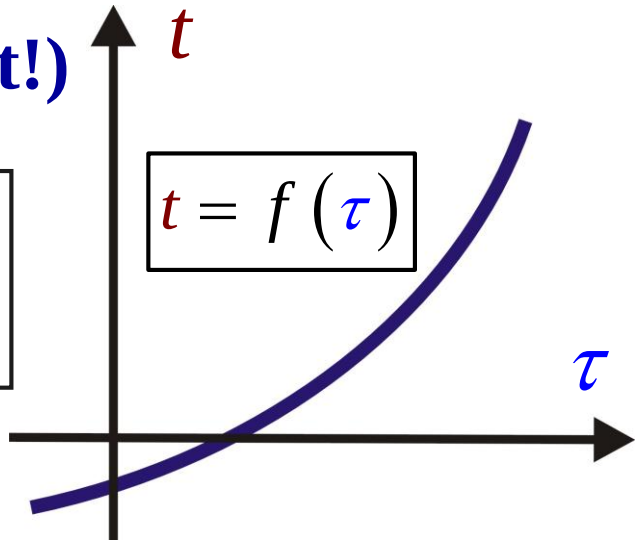
Vertex operators

Liouville potential

- **Phase representation (measure is flat!)**

$$S_0[\varphi]=\frac{M}{2}\int_{-\infty}^{+\infty}\left[\varphi'(\tau)\right]^2d\tau,\quad f'(\tau)=e^{\varphi(\tau)}$$

non-compact phase



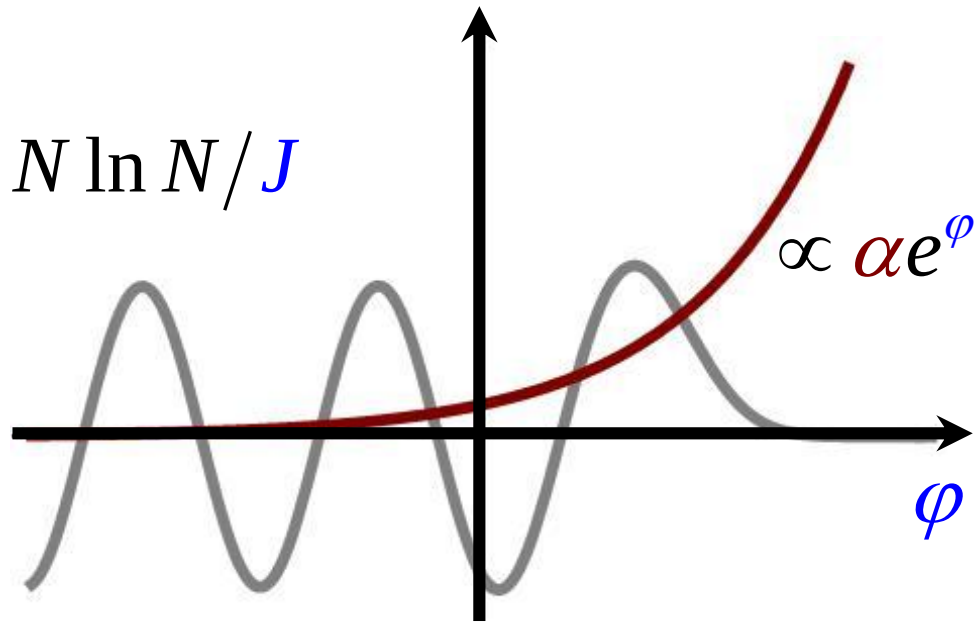
Liouville QM

$$\langle \varphi | k \rangle \propto K_{2ik}\left(\sqrt{8M\alpha}\, e^{\varphi/2}\right)$$

- **Effective Hamiltonian**

$$\hat{H} = -\frac{\partial_\varphi^2}{2M} + \alpha e^{\varphi}, \quad M \sim N \ln N / J$$

"effective mass"

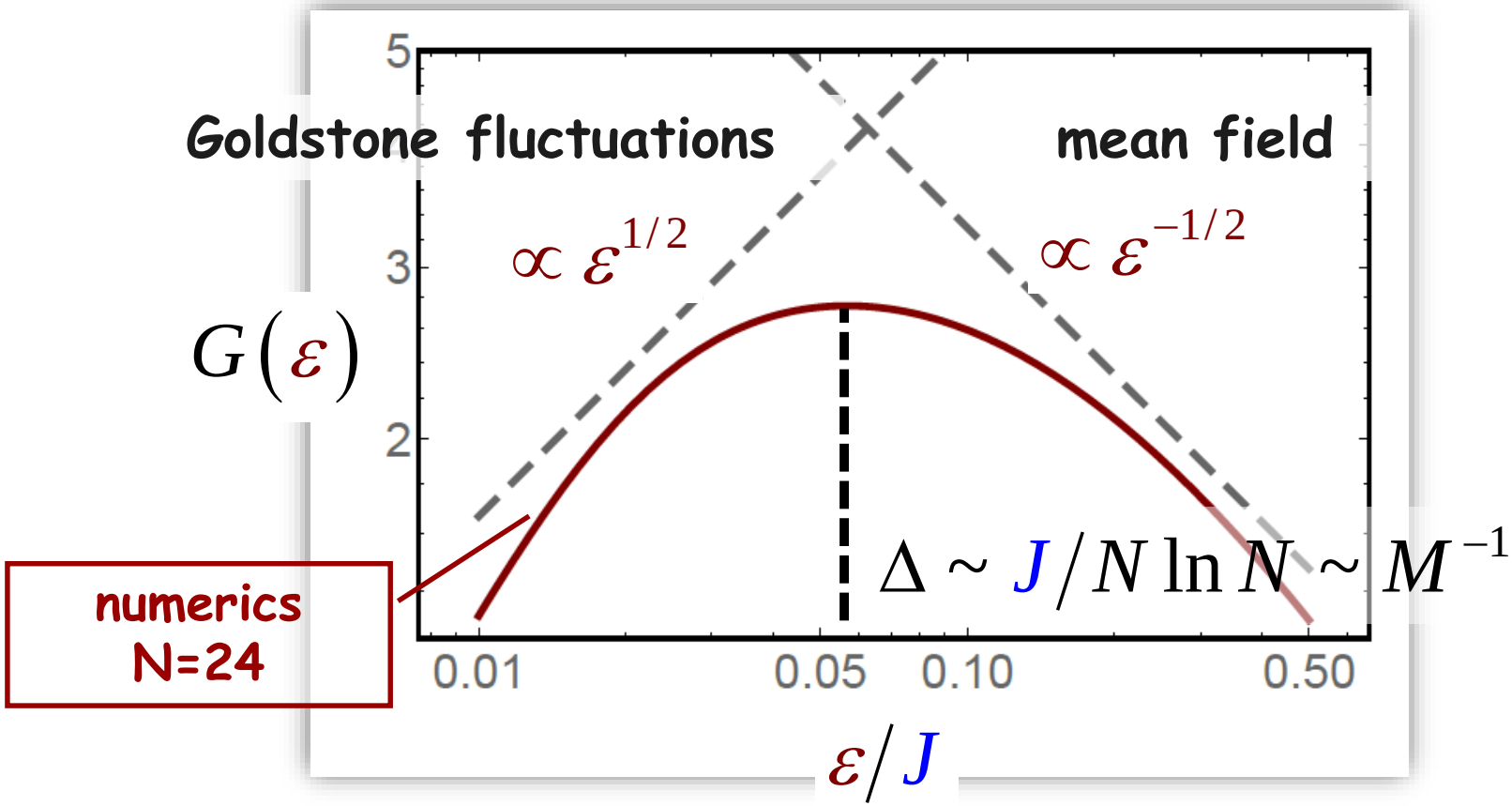


$$k \in \mathbb{R}^+$$

- **Spectral decomposition of the Green's function**

$$G(\tau) \propto \mp \int_0^{+\infty} \frac{d\alpha}{\sqrt{\alpha}} \sum_k \langle 0 | e^{\frac{1}{4}\varphi} | k_\alpha \rangle e^{-|\tau| k^2/2M} \langle k_\alpha | e^{\frac{1}{4}\varphi} | 0 \rangle$$

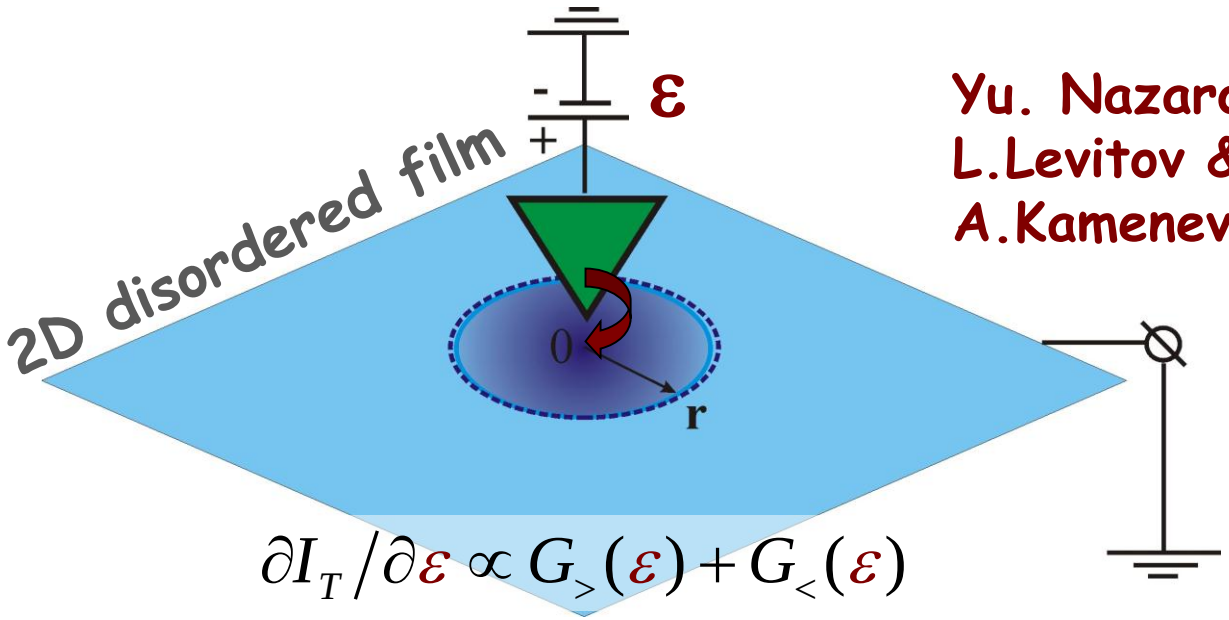
Green's function



- **Time domain:**

$$G(\tau) \propto \pm \frac{1}{\sqrt{J}} \begin{cases} |\tau|^{-1/2}, & \tau < 1/\Delta \\ \Delta^{-1} |\tau|^{-3/2}, & \tau > 1/\Delta \end{cases}$$

Zero-bias anomaly



Yu. Nazarov, JETP '89;
L.Levitov & A.Shytov, JETP '97
A.Kamenev & A.Andreev, PRB '99

$$\partial I_T / \partial \varepsilon \propto G_>(\varepsilon) + G_<(\varepsilon)$$

tunneling conductance

$$G_>(\varepsilon) = \int t^{-1} e^{-S(t) + i\varepsilon t} dt \sim \nu_0 \exp\left(-\frac{1}{8g\pi^2} \ln^2 \frac{1}{\varepsilon} \right)$$

$$\boxed{S(t) = \frac{1}{8g\pi^2} \ln^2 t}$$

- electron's action of tunnelling under "Coulomb barrier"

Four-point Green's function

$$G_4(\tau_1,\tau_2,\tau_3,\tau_4) = \frac{1}{N^2}\sum_{i,j}^{N}\left\langle \chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\right\rangle$$

- **Time ordering:**

$\tau_1,\tau_4 \approx 0^{\pm}$ $\qquad$ $\tau_2,\tau_3 \approx \tau^{\pm}$

$$G_4(\tau) \propto \begin{cases} |\tau|^{-1}, & \tau < 1/\Delta \\ \Delta^{-1/2}|\tau|^{-3/2}, & \tau > 1/\Delta \end{cases}$$

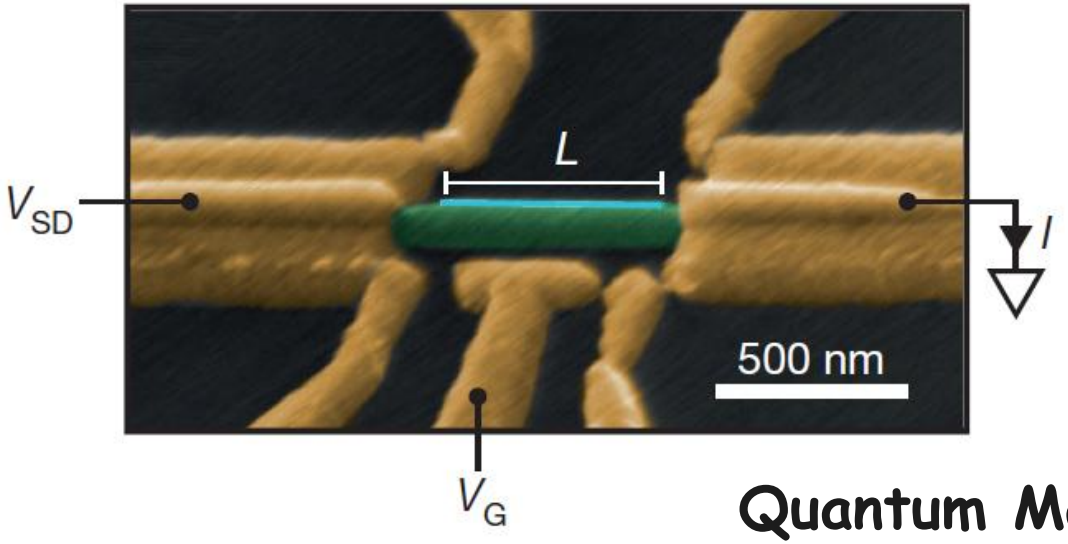
$\Delta \sim J/N\ln N$

single-partilce level spasing

universal long-time decay

Random mass Dirac model

L. Balents & M.Fisher '97, D. Shelton & A. Tsvelik '98



$$\hat{H} = \begin{pmatrix} -iu\partial_x & m(x) \\ m(x) & iu\partial_x \end{pmatrix}$$

$$\langle m(x)m(x')\rangle_{\text{dis}} \propto \delta(x-x')$$

Quantum Majorana wire at criticality

- **Statistics of zero-energy wave functions**

$$\boxed{\left\langle \left|\psi_0(x)\psi_0(0)\right|^p \right\rangle_{\text{dis}} \sim L^{-1}\left|x\right|^{-3/2}}$$

universal (p-independent) decay

Summary

- **Conformal symmetry breaking in SYK model leads to large Goldstone mode fluctuations**

- **Fluctuations qualitatively affect physics at large time scales and low energies,**

$$t > N \ln N / J$$

- **… and modify correlation functions**

- **Complex SYK model shows emergent Coulomb blockade at low T**

Additional slides

Chaos & and OTO correlators

OTO correlation function

Larkin & Ovchinikov '69

$$\boxed{F(t) = Z^{-1}\mathrm{tr}\left(e^{-\beta\hat{H}}\,\hat{X}\,\hat{Y}(t)\hat{X}\,\hat{Y}(t)\right)}$$

$\hat{X}\,,\hat{Y}$ - one body operators in many body context

- **1st interpretation**
  - up to inessential terms, $F(t) = \left\langle [\hat{X}\,,\hat{Y}(t)]^2 \right\rangle$
  - for single particle system,

Lyapunov exponent

$$F(t) = \left\langle \left(i\hbar\{p, q(t)\}\right)^2 \right\rangle \propto \hbar^2 \left\langle \left(\{\partial_q q(t)\}\right)^2 \right\rangle \propto \hbar^2 e^{2\lambda t}$$

  - correlations are build up at the scale $\boxed{t_E = \lambda^{-1}\ln\hbar}$
- **2nd interpretation**
  - for many (qubit) system, $\hat{X} = \sigma_i^z\,,\hat{Y} = \sigma_j^z$, non-vanishing [,] builds up at times sufficiently large to entagle sites i & j.

OTO correlation function

Maldacena, Shenker & Stanford '2015

$$F(t) = Z^{-1}\mathrm{tr}\left(e^{-\beta\hat{H}/4}\hat{X}\, e^{-\beta\hat{H}/4}\hat{Y}(t)\, e^{-\beta\hat{H}/4}\hat{X}\, e^{-\beta\hat{H}/4}\hat{Y}(t)\right)$$

- **Upper bound on exponential growth in non-integrable many-body system**

$$F(t) = F_0 - \varepsilon e^{\lambda_L t} + O(\varepsilon^2), \quad \lambda_L \leq 2\pi T$$

- this is a short time behaviour of the OTO correlator

- **SYK model is maximally chaotic in the above sense**

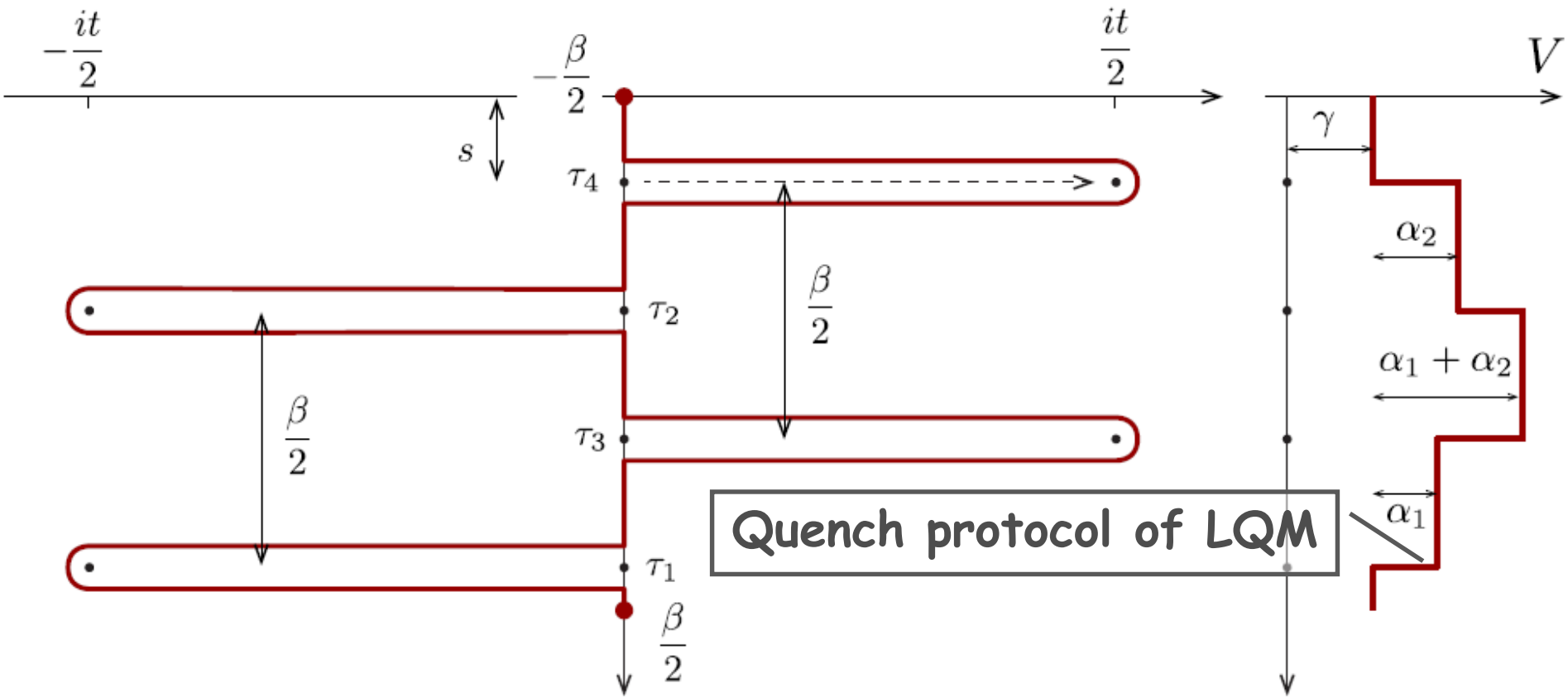
Kitaev '2015

OTO correlation function

can be obtained from contour-ordered 4-point function

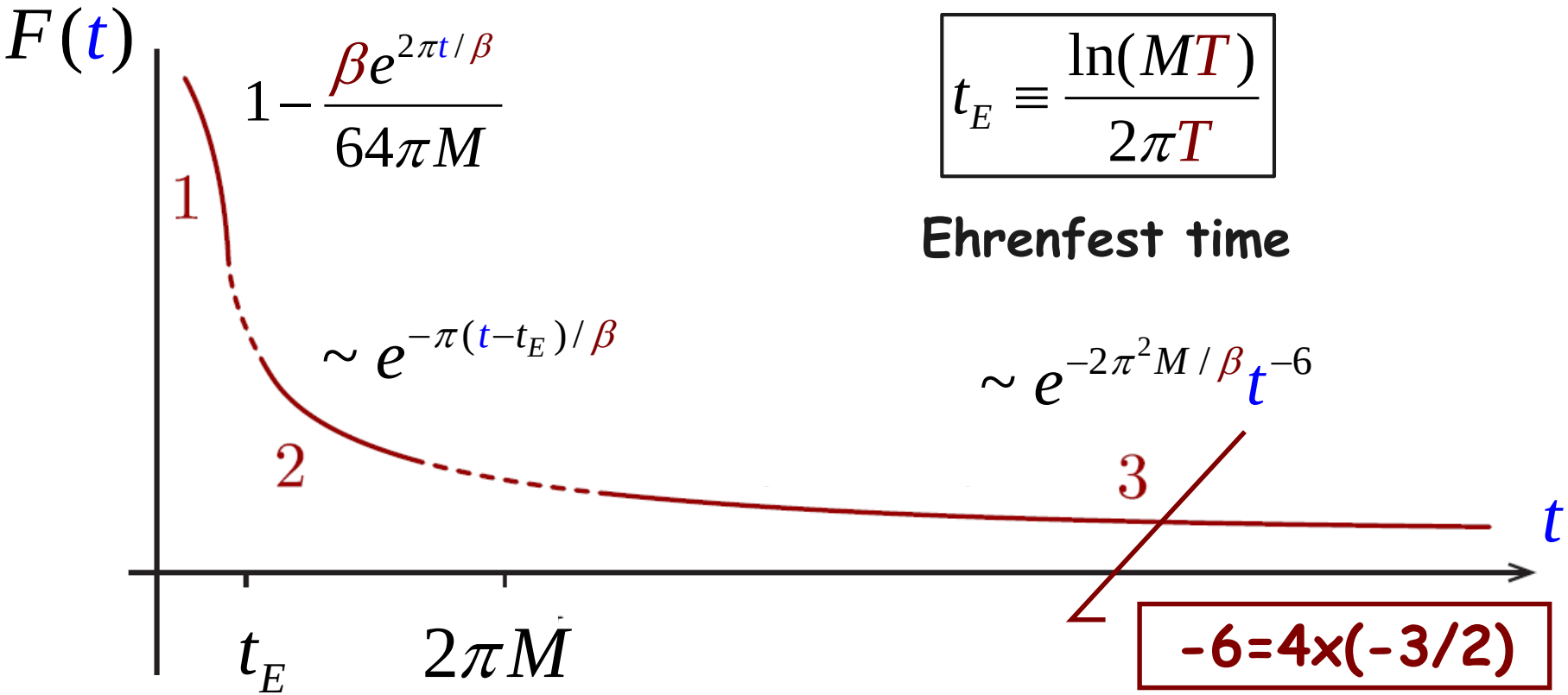
$$G_4(\tau_1,\tau_2,\tau_3,\tau_4) = \frac{1}{N^2}\sum_{i,j}^{N}\left\langle T_\tau \chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\right\rangle$$

after analytic continuation into compex plane



OTO correlator in SYK model

High temperature limit, T>1/M



$$t_E \equiv \frac{\ln(MT)}{2\pi T}$$

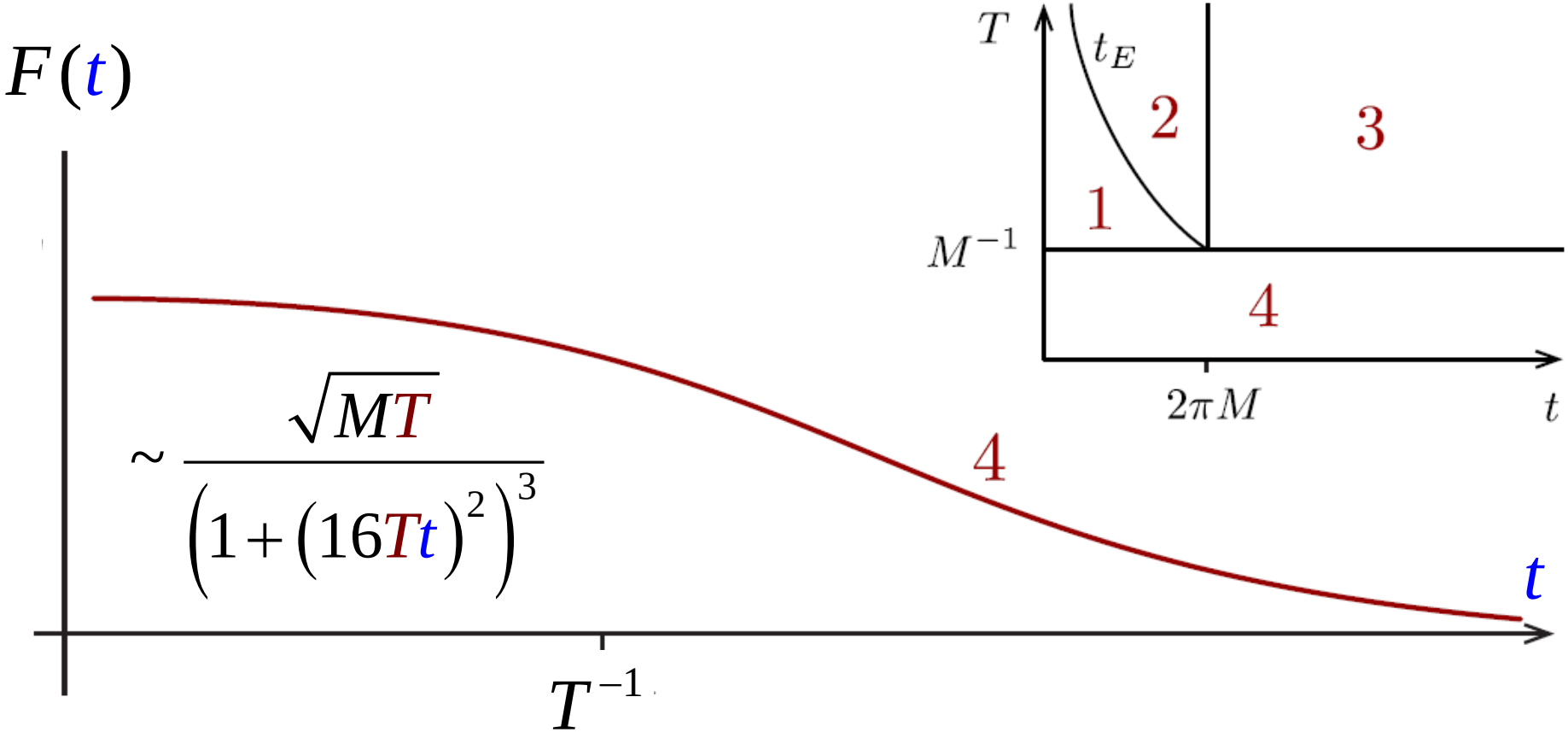
Ehrenfest time

-6=4x(-3/2)

- **Liouville universality:** $\langle O(\tau)O(\tau')\rangle \sim |\tau-\tau'|^{-3/2}$
  - O(t) is any local operator

OTO correlator in SYK model

Low temperature limit, T<1/M



- at small T and/or large times the system looses semiclassical interpretation

Partition sum & many-body spectrum

Finite-T Goldstone action

- **Schwarzian action of reparametrizations**

$$S_\beta[f] = -M \int_{-\infty}^{+\infty} \left\{ \tan\left[\pi f / \beta\right], \tau \right\} d\tau$$

- {h,τ} is the Schwarzian defined on the 'right' coset H\G

  $f : S^1 \mapsto S^1$ is a reparametrization of the termal circle

- **Path integral measure (is not flat, left-invariant)**

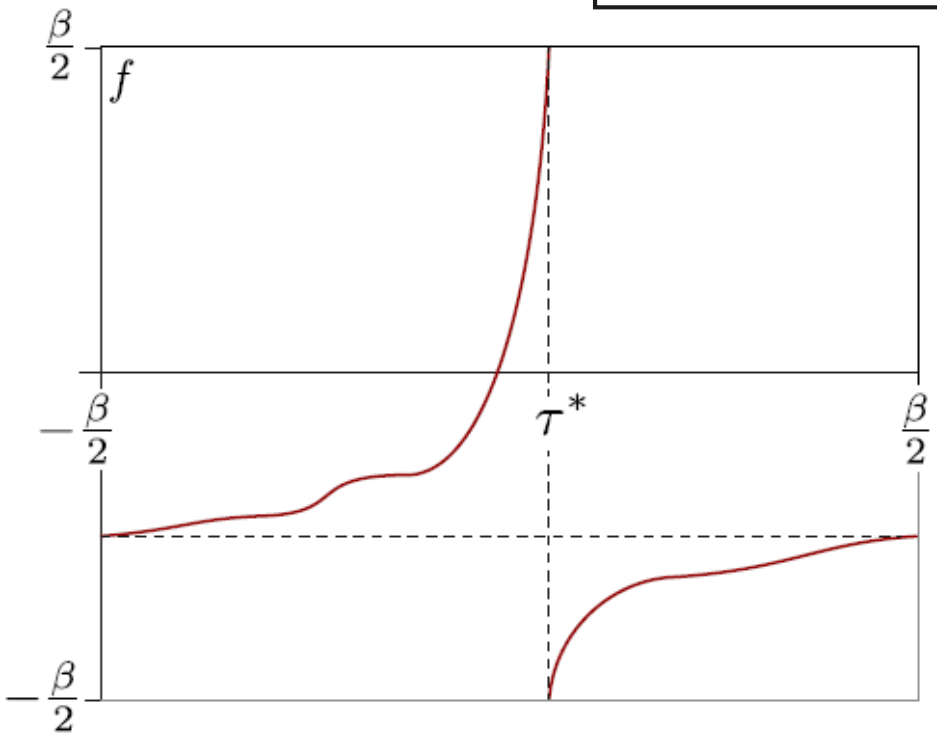
$$\mu[f] := \prod_\tau \frac{df_\tau}{f'} \qquad \mu[g \circ f] = \mu[f]$$

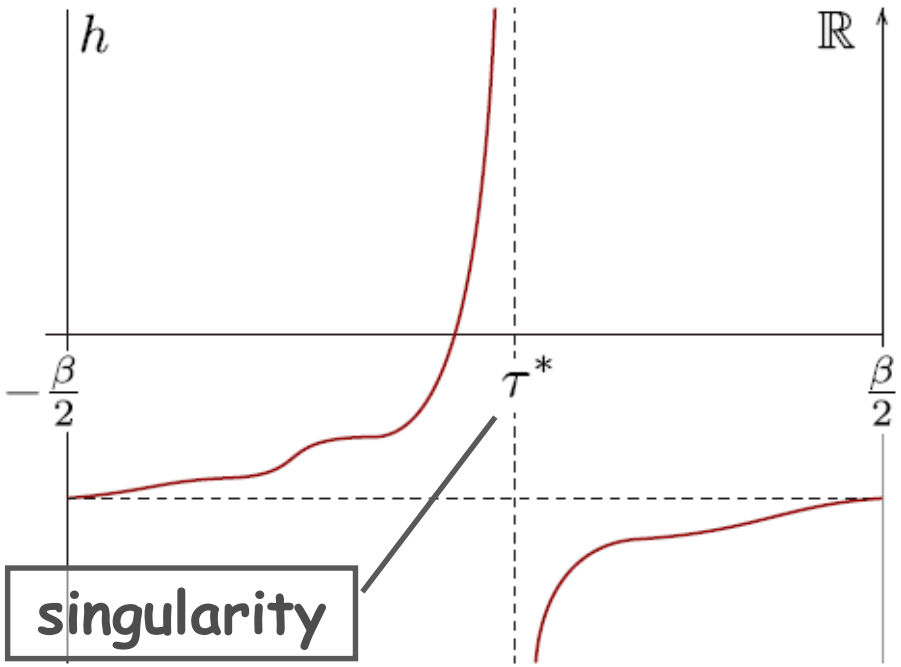
Cf. Witten & Stanford '2017

Phase represenation

- use the left-invariance of the measure to integrate over

$$h(\tau) := \tan\left[\pi f(\tau)/\beta\right]$$





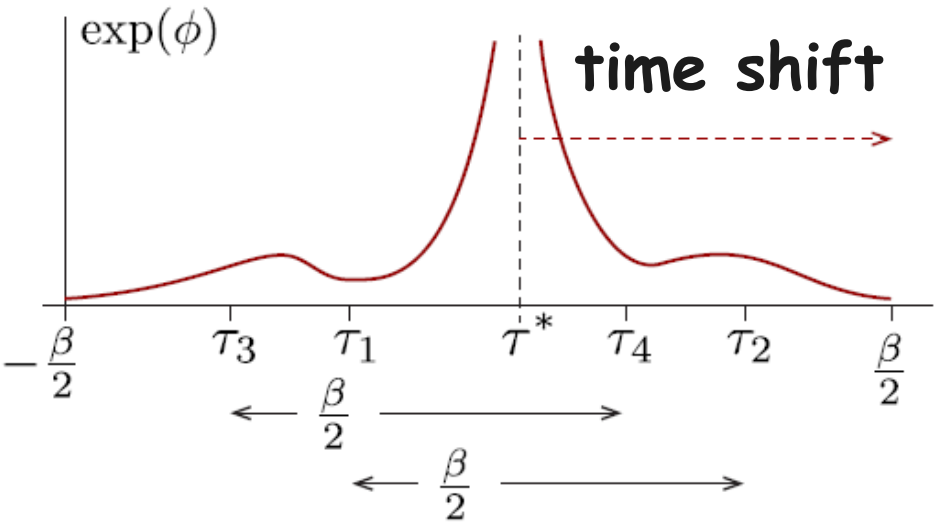
$f : S^1 \mapsto S^1$

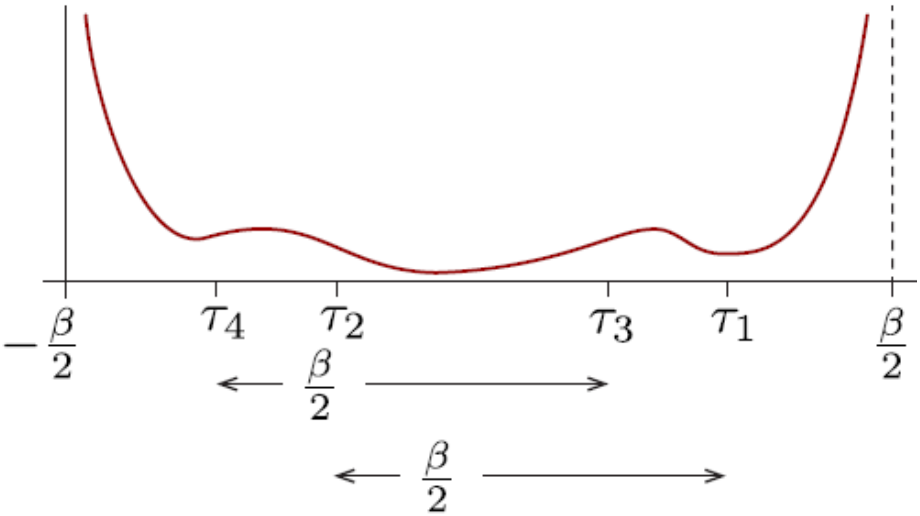
$h : S^1 \mapsto \mathbb{R}$

Phase represenation

& introduce the non-compact phase variable of the Liouville quantum mechanics as

$$\boxed{h'(\tau) := \exp\left[\varphi(\tau)\right]}$$





- plausible trajectories which contribute, $\varphi\left(\pm\beta/2\right) \to +\infty$

Liouville QM

- **Finite temperature action (measure is flat!)**

$$S_\beta[\varphi] = \frac{M}{2}\int_0^\beta \left[\varphi'(\tau)\right]^2 d\tau + J\left[\int_0^\beta e^{\varphi(\tau)} d\tau - t_H\right],$$

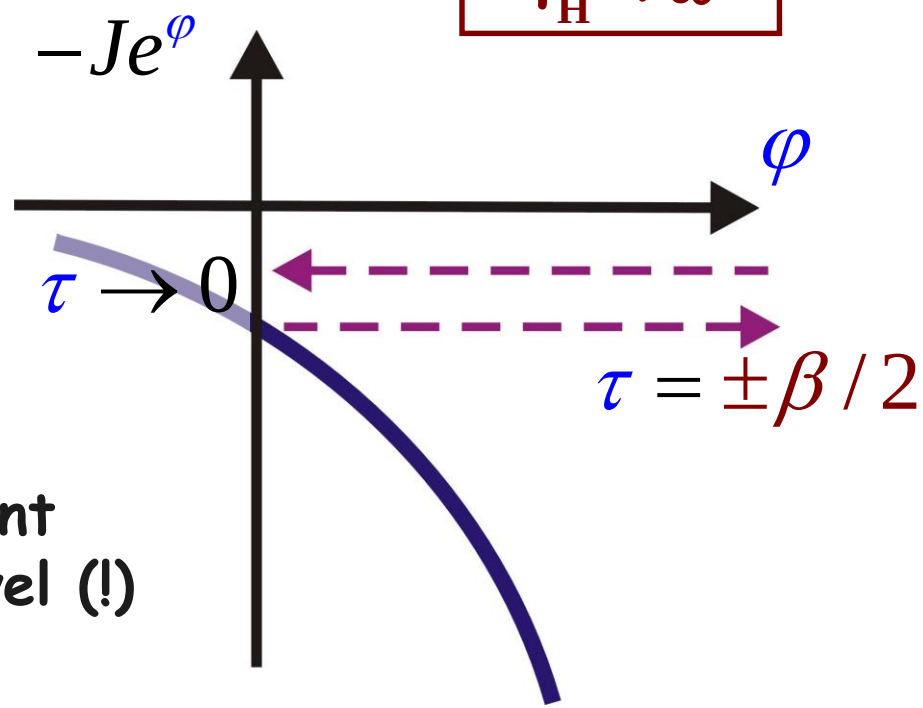
Lagrange multiplier, ~J

$t_H \to \infty$

- **Saddle point trajectory**
  - semiclassical tunneling:

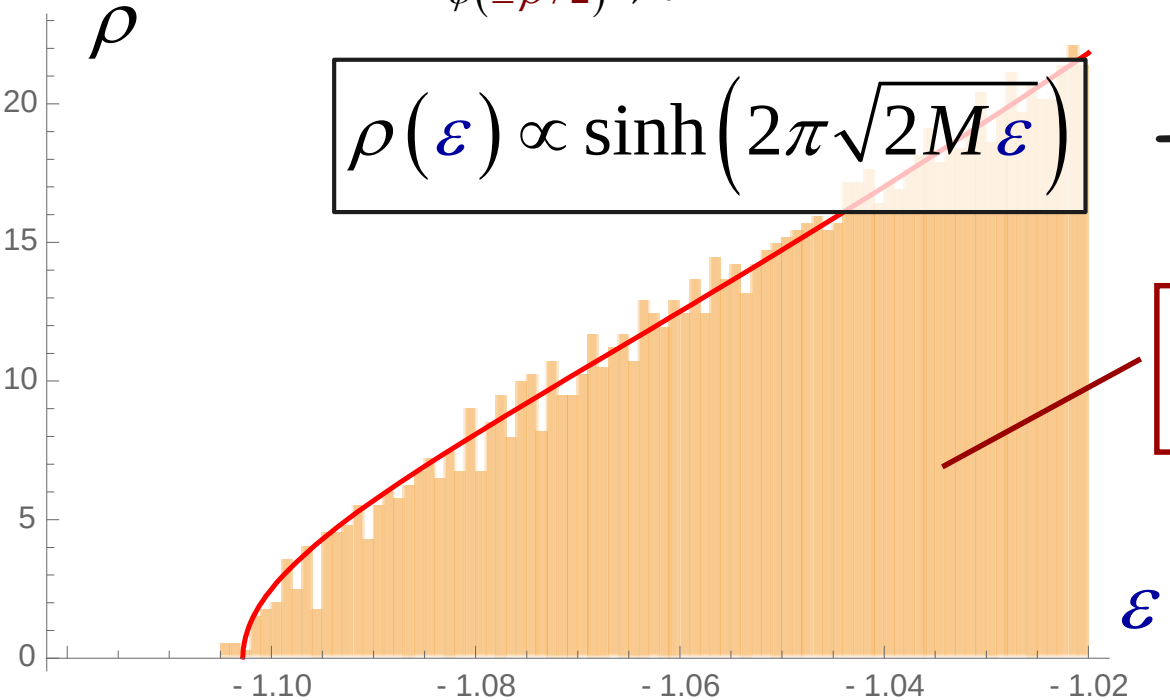
$$\varphi_0(\tau) = -\ln\left(\cos^2\left[\pi\tau/\beta\right]\right)$$

  - reproduces finite-T two-point function on the mean-field level (!)



SYK partition sum

$$Z(\beta) = \int_{\varphi(\pm\beta/2)\to\infty} D\varphi \exp\left(-S_\beta[\varphi]\right) \propto \int_0^{+\infty} \rho(\varepsilon)\, e^{-\beta\varepsilon} d\varepsilon$$



$$\rho(\varepsilon) \propto \sinh\left(2\pi\sqrt{2M\varepsilon}\right)$$

- many-body DoS

N=34, data courtesy Garcia-Garcia

- combinatorial method (Verbaarschot, Garcia-Garcia '2016)
- (q>>2)-body interaction (Cotler at al. '2016)
- Z is semiclassically exact (Witten & Stanford '2017)