Hydrodynamic Modes of Incoherent Black Holes

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Transport in inhomogeneous media

- Conductivity matrix characterizes linear response of system to external sources.
- Apply *constant, time-dependent* electric field E_j and thermal gradient $\zeta_j \leftrightarrow \left(\nabla_j T\right)/T$ and read off $U(1)$ and heat currents:

$$\begin{pmatrix} J^i \\ Q^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T\alpha^{ij} \\ T\bar{\alpha}^{ij} & T\bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} E_j \\ \zeta_j \end{pmatrix}$$

- Translation invariance $\Rightarrow$ divergent DC conductivities $\sim \left(\delta(\omega) + \frac{i}{\omega}\right)$ as $\omega \to 0$.
- For physical, finite conductivities, we need a mechanism for momentum to dissipate $\to$ Explicit breaking of translational invariance by spatially dependent sources.
- Spontaneous breaking of translational invariance is also physically relevant (e.g. charge density waves).

Breaking translational invariance in holography

Explicitly, by sources at infinity ("holographic lattice").[1]

- Analytic results for perturbative lattices, (hard) numerical constructions of strong lattices.
- Easier if we retain homogeneity of metric but break translations in the scalar sector (linear axions,[2] Q-lattices[3]).
- Hydrodynamics dominated by energy and charge diffusion.

[1] [Horowitz, Santos, Tong '12, ...]
[2] [Andrade, Withers '14]
[3] [Donos, Gauntlett '14]

Breaking translational invariance in holography

Spontaneously, if there is an instablility towards a broken phase with spatial modulation.[4]

- The boundary is homogeneous but the bulk (and the horizon) is *not*.
- Numerical constructions in Einstein-Maxwell-Dilaton theories, possibly with Chern-Simons(-like) terms (holographic charge density waves,[5] helical black holes[6]).
- Energy and charge diffusive modes + sound modes corresponding to the Goldstone modes + ...

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[4] [Nakamura, Ooguri, Park '10, . . .]
[5] [Donos, Gauntlett '13]
[6] [Donos, Gauntlett '12]

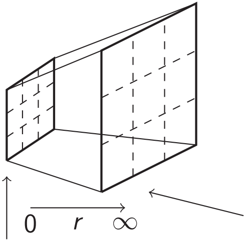
Approach

- Study diffusive modes in a framework accounting for both cases of breaking translations
- Construct a "thermodynamic" zero mode perturbation
- Construct the diffusive modes in long wavelength perturbative expansion
- Obtain dispersion relations by focusing on the near horizon region $\rightarrow$ no need for explicit bulk solutions
- For holographic lattices, connect with DC conductivities $\rightarrow$ obtain generalised form of Einstein relations

Outline

Holographic setup

Bulk action: $S = \int d^D x \sqrt{-g}\,\left(R - V(\phi) - \frac{Z(\phi)}{4} F^2 - \frac{1}{2}(\partial\phi)^2\right)$



Regular horizon

$$g_{ij} = g_{ij}^{(0)}(x) + \dots$$
$$U = 4\pi T\, r + \dots$$
$$a_t = r\, a_t^{(0)}(x) + \dots$$
$$\phi = \phi^{(0)}(x) + \dots$$

Background static black holes:

$$ds^2 = -U(r) G\, dt^2 + \frac{F}{U(r)}\, dr^2 + g_{ij} dx^i dx^j$$
$$A = a_t dt$$

AdS boundary asymptotics

$$g_{ij} = r^2 \bar{g}_{ij}(x) + \dots$$
$$U = r^2 + \dots$$
$$a_t = \mu(x) + \dots$$
$$\phi = r^{\Delta - d - 1}\phi^{(\infty)}(x) + \dots$$

Constraints on the horizon

We consider a generic *sourceless* perturbation $\delta\psi(t,r,x^i)$ with time dependence $\sim e^{-i\omega v_{EF}}$.

Infalling conditions at the horizon imply that

$$\delta g_{ri}^{(0)} = \delta g_{ti}^{(0)} \equiv -v_i(x)\,, \quad \delta g_{tr}^{(0)} = \frac{1}{2}\left(\delta g_{rr}^{(0)} + \delta g_{rr}^{(0)}\right) \equiv -p(x)/(4\pi T)$$

$$\delta a_r^{(0)} = \delta a_t^{(0)} \equiv w(x)$$

Local "currents" on the horizon

$$Q_{(0)}^i = 4\pi T\sqrt{g^{(0)}}v^i\,, \qquad J_{(0)}^i = \sqrt{g_{(0)}}Z^{(0)}\left(\partial^i w + a_t^{(0)}v^i + i\omega\delta a^{(0)i}\right)$$

Constraints on the horizon

The Hamiltonian constraints on the (stretched) horizon have the form

$$\partial_i Q^i_{(0)} = \omega\,(\dots)\ , \qquad \partial_i J^i_{(0)} = \omega\,(\dots)\ ,$$

$$-\,2\,\nabla^i\nabla_{(i}\,v_{j)} + \nabla_j\,p - Z^{(0)} a_t^{(0)}\,\nabla_i w + \nabla_i\phi^{(0)} v^j \nabla_j\phi^{(0)} = \omega\,(\dots)$$

with the brackets involving various other fields.

The above set of equations is *not* closed.

Diffusive mode

Next, we note that there is a zero mode solution corresponding to perturbing the temperature T by a constant amount δT and the zero lattice mode of the chemical potential $\bar{\mu} \equiv \oint \mu(x)$ by a constant $\delta\bar{\mu}$

$$\delta\psi^{RT} = \frac{\partial\psi}{\partial T}\,\delta T + \frac{\partial\psi}{\partial\bar{\mu}}\,\delta\bar{\mu}\,,$$

Can be brought to a form that

- does not introduce boundary sources,
- is regular at horizon,
- satisfies horizon constraints.

Diffusive mode

To construct a bulk diffusive mode, we need to consider a time-dependent sourceless perturbation. We take[7]

$$\delta\psi = e^{-i\omega v_{EF}} e^{i\varepsilon k_i x^i} \left[\delta\psi^{RT} + \varepsilon\delta\psi_{[1]} + \dots\right]$$

The functions $\delta\psi_{[\alpha]}(x)$ are periodic on the lattice $\Rightarrow$ Bloch decomposition. In particular

$$\omega = \varepsilon\,\omega_{[1]} + \varepsilon^2\,\omega_{[2]} + \dots\,, \qquad p = e^{i\varepsilon k_i x^i}\,\left(4\pi\delta T + \varepsilon\,p_{[1]} + \dots\right)\,,$$

$$v^i = e^{i\varepsilon k_i x^i}\,\left(\varepsilon\,v^i_{[1]} + \varepsilon^2\,v^i_{[2]} + \dots\right)\,, \quad w = e^{i\varepsilon k_i x^i}\,\left(-\delta\bar{\mu} + \varepsilon\,w_{[1]} + \dots\right)\,,$$

$$\delta g^{(0)}_{ij} = e^{i\varepsilon k_i x^i}\,\left(\frac{\partial g^{(0)}_{ij}}{\partial T}\,\delta T + \frac{\partial g^{(0)}_{ij}}{\partial\bar{\mu}}\,\delta\bar{\mu} + \dots\right)\,,$$

$$\delta\phi^{(0)} = e^{i\varepsilon k_i x^i}\,\left(\frac{\partial\phi^{(0)}}{\partial T}\,\delta T + \frac{\partial\phi^{(0)}}{\partial\bar{\mu}}\,\delta\bar{\mu} + \dots\right)$$

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[7] [Donos, Gauntlett, VZ '17]

Diffusive mode

We need to determine the dispersion relation $\omega(\varepsilon k)$ in order to show that this is indeed a diffusive mode.

At order $\mathcal{O}(\varepsilon)$ the horizon constraints imply that $\omega_{[1]} = 0$ and

$$\nabla_i v^i_{[1]} = 0\,, \qquad \nabla_j \left(Z^{(0)} \left(-i\, k^j \delta\bar{\mu} + \nabla^j w_{[1]} + v^j_{[1]} a_t^{(0)} \right) \right) = 0\,,$$

$$- 2\, \nabla^j \nabla_{(j} v_{[1]i)} + \nabla_i p_{[1]} - Z^{(0)} a_t^{(0)}\, \nabla_i w_{[1]} + v^j_{[1]} \nabla_j \phi^{(0)} \nabla_i \phi^{(0)}$$

$$+ i k_i 4\pi\, \delta T + Z^{(0)} a_t^{(0)}\, i k_i \delta\bar{\mu} = 0$$

This system has a unique solution for $v^i_{[1]}$, $p_{[1]}$ and $w_{[1]}$, so we can write

$$4\pi T i \oint \sqrt{g_{(0)}} v^i_{[1]} = T \alpha^{ij} k_j \delta\bar{\mu} + \bar{\kappa}^{ij} k_j \delta T\,,$$

$$i \oint \sqrt{g_{(0)}} Z^{(0)} \left(-i\, k^j \delta\bar{\mu} + \nabla^j w_{[1]} + v^j_{[1]} a_t^{(0)} \right) = \sigma^{ij} k_j \delta\bar{\mu} + \alpha^{ij} k_j \delta T$$

Diffusive mode

Finally, integrating the constraints at order $\mathcal{O}(\varepsilon^2)$ we get

$$
\begin{aligned}
i\omega_{[2]}\left(T^{-1}c_\mu\delta T+\xi\,\delta\bar{\mu}\right)-T^{-1}\bar{\kappa}^{ij}k_ik_j\,\delta T-\alpha^{ij}k_ik_j\,\delta\bar{\mu}=&0\,,\\
i\omega_{[2]}\left(\xi\,\delta T+\chi\,\delta\bar{\mu}\right)-\alpha^{ij}k_ik_j\,\delta T-\sigma^{ij}k_ik_j\,\delta\bar{\mu}=&0
\end{aligned}
$$

In the above we have written the thermodynamic susceptibilities

$$
\begin{aligned}
\delta s&\equiv T^{-1}c_\mu\,\delta T+\xi\,\delta\bar{\mu}\,,\\
\delta\rho&\equiv\xi\,\delta T+\chi\,\delta\bar{\mu}
\end{aligned}
$$

as horizon integrals given a family of background solutions, since we have

$$
s=4\pi\oint_H\sqrt{g^{(0)}}\,,\qquad \rho=\oint_H\sqrt{g_{(0)}}Z^{(0)}a_t^{(0)}
$$

Diffusive mode

So we can solve for $\omega_{[2]}$ and obtain the two eigenfrequencies $i\omega^{\pm}_{[2]}$ associated with the diffusion modes. Defining

$$\bar{\kappa}(k) = \bar{\kappa}^{ij} k_i k_j, \quad \alpha(k) = \alpha^{ij} k_i k_j, \quad \sigma(k) = \sigma^{ij} k_i k_j \tag{1}$$

we have[8]

$$\begin{aligned} i\omega^{+}_{[2]}\, i\omega^{-}_{[2]} &= \frac{\kappa(k)\,\sigma(k)}{c_\rho \chi}\,, \\ i\omega^{+}_{[2]} + i\omega^{-}_{[2]} &= \frac{\kappa(k)}{c_\rho} + \frac{\sigma(k)}{\chi} + \frac{T\,\left[\chi\,\alpha(k) - \xi\,\sigma(k)\right]^2}{c_\rho \chi^2 \sigma(k)} \end{aligned} \tag{2}$$

where

$$c_\rho \equiv c_\mu - \frac{T\xi^2}{\chi}\,, \qquad \kappa(k) \equiv \bar{\kappa}(k) - \frac{\alpha^2(k)\,T}{\sigma(k)} \tag{3}$$

The quantities $\bar{\kappa}$, α, σ and the susceptibilities are fixed by horizon data, given a family of solutions.

[8] [Hartnoll '15]

Einstein relations

For a holographic lattice, one can show that $\bar{\kappa}$, α, σ are the DC conductivities of the boundary theory and using the following facts:[9]

- Apply time-independent thermal and electric boundary sources $\delta g_{tj} \leftrightarrow \bar{\zeta}_j$, $\delta a_i \leftrightarrow \bar{E}_i$ and obtain linearised Stokes equations horizon:

$$
\begin{aligned}
&\partial_i Q^i_{(0)} = 0\,, \qquad \partial_i J^i_{(0)} = 0\,,\\
&-2\,\nabla^i \nabla_{(i}\, v_{j)} - Z^{(0)} a_t^{(0)} \nabla_j w + \nabla_j \phi^{(0)} \nabla_i \phi^{(0)} v^i + \nabla_j\, p\\
&= 4\pi T\, \bar{\zeta}_j + Z^{(0)} a_t^{(0)} \bar{E}_j
\end{aligned}
$$

- In general the *local* horizon current differs from the *local* boundary current, but their *fluxes* are equal

$$
\oint_H Q^i_{(0)} = \bar{Q}^i\,, \qquad \oint_H J^i_{(0)} = \bar{J}^i
$$

Thus we obtain a set of generalised Einstein relations.

[9] [Donos, Gauntlett '15], [Banks, Donos, Gauntlett '15]

Comments & outlook

- Thermodynamic instability $\Rightarrow$ dynamical instability
- Diffusion poles in retarded Green's functions for spatially inhomogeneous QFTs[10]
- Analogous construction of diffusive modes within hydrodynamics on curved manifolds[10]

- Sound and Goldstone modes in models with spontaneous symmetry breaking?
- Bounds on diffusion and relation to chaos?[11]

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[10] [Donos, Gauntlett, VZ '17]
[11] [Hartnoll '15, ...], [Blake '16, ...]

Thank you for your attention!