Constraints on hydrodynamics from many-body quantum chaos

Andrew Lucas

Stanford Physics

Many-Body Quantum Chaos, Bad Metals and Holography; NORDITA

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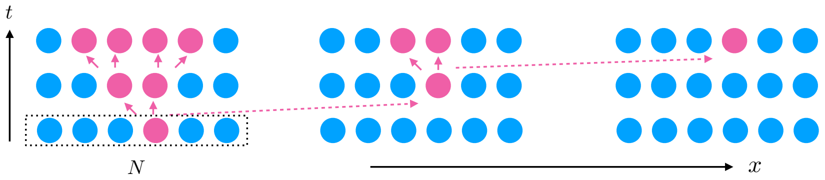
  - avatars of chaos if N degrees of freedom spread out in space

Spatial Butterfly Effect

- quantum information is scrambled among DOF:



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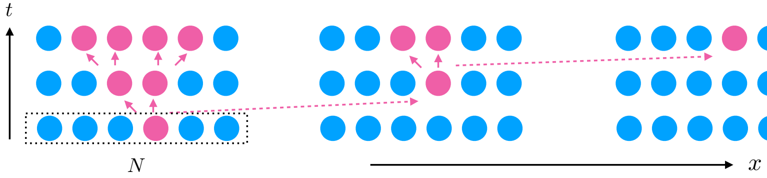

- on any site, we estimate the **scrambling time**

[Sekino, Susskind, 0808.2096]; [Shenker, Stanford, 1306.0622]

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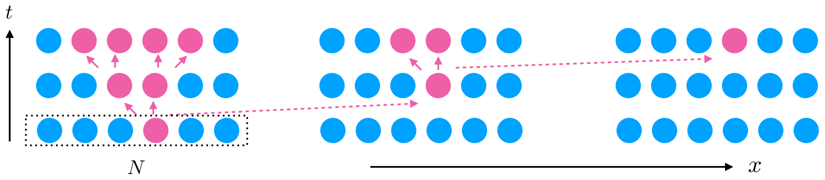

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- locality $\implies$ spatial scrambling much slower

let A, B, C, D be 'generic' (quasi)local operators: **define**

- **light cone velocity**: onset of chaos:

$$\langle A(x,t)[B(x,t),C(0)]D(0)\rangle_\beta \le \frac{\mathcal{P}}{N}\exp\left[\frac{v_{\mathrm{LC}}t-|x|}{v_{\mathrm{LC}}\tau_{\mathrm{LC}}}\right]+\mathrm{O}\left(\frac{1}{N^2}\right)$$

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Light Cone Velocity vs. Butterfly Velocity

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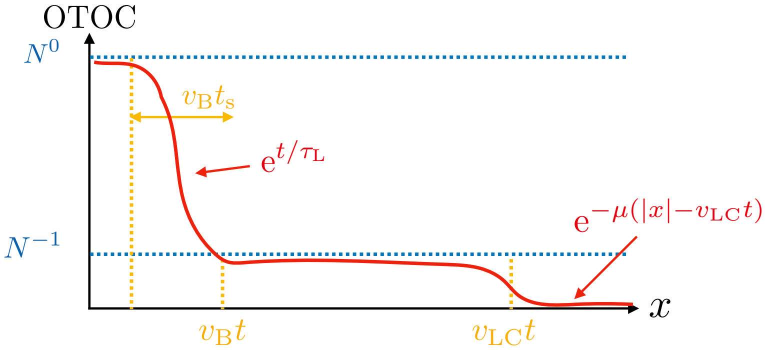
- conjecture: chaos insensitive to (lattice) regularization

Light Cone Velocity vs. Butterfly Velocity: Only at Large N



- v_B and v_LC only different at large N

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  chaos → (local) thermalization → hydrodynamics

Chaos is a Precursor to Thermalization

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- do these things really constrain each other? how? why?

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$$G^{\mathrm{R}}_{\rho\rho}(x,t) \equiv \mathrm{i}\Theta(t)\langle[\rho(x,t),\rho(0,0)]\rangle_\beta$$

then if there is hydrodynamic diffusion at late times: [Kadanoff, Martin (1963)]

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- if ρ instead couples to sound waves:

$$G^{\mathrm{R}}_{\rho\rho}(k,\omega) = \frac{w k^2}{v_{\mathrm{s}}^2 k^2 - \omega^2 - \mathrm{i}\omega k^2 \Gamma_{\mathrm{s}}} + \cdots$$

- using definition of v_{LC}, and $\rho = \sum \Phi_I^\dagger \Phi_I$:

$$\begin{aligned} |\langle [\rho(x,t), \rho(0,0)] \rangle_\beta| &= \sum_{IJ} \langle \Phi_I^\dagger(x,t) [\Phi_I(x,t), \Phi_J^\dagger(0,0)] \Phi_J(0,0) \rangle_\beta \\ &\quad + 3 \text{ similar terms} \\ &\leq 4\mathcal{P} N \exp\left[\frac{v_{\mathrm{LC}} t - |x|}{v_{\mathrm{LC}} \tau_{\mathrm{LC}}}\right] + \mathrm{O}\left(N^0\right) \end{aligned}$$

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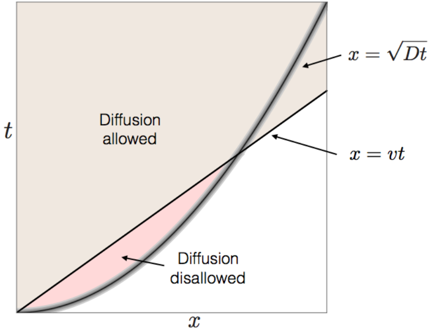
- these two equations are only consistent if

$$\boxed{v_{\mathrm{s}} \le v_{\mathrm{LC}}}$$

[Lucas, 1710.01005]

- diffusion exponentially suppressed when $x^2 \gtrsim Dt$:



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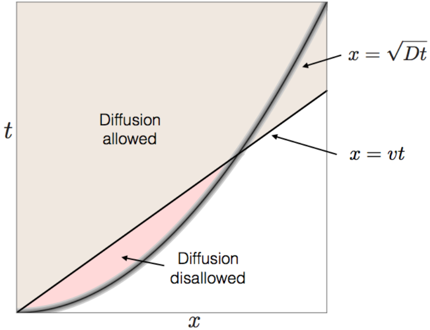

- if hydrodynamic diffusion is valid for $t \geq \tau$:
[Hartman, Hartnoll, Mahajan, 1706.00019]

$$\boxed{D \leq v^2 \tau}$$

with $v = v_{\mathrm{LC}}$ (as computed at $\mathrm{O}(1/N)$) [Lucas, 1710.01005]

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A Puzzle: Sound Waves in Holography

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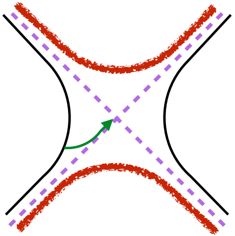

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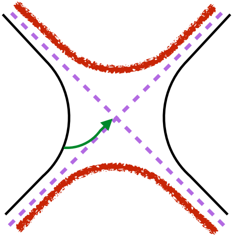
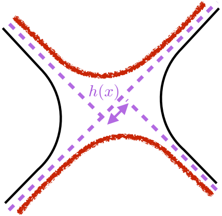
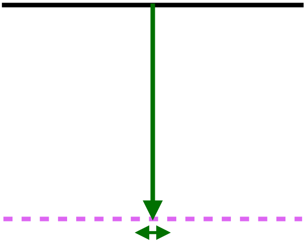




OTOC $\sim$ gravitational shock across horizon:
$h(x,t) \sim \frac{1}{N}\mathrm{e}^{2\pi T(t-|x|/v_{\mathrm{B}})}$

- if UV matter $\Phi_I^\dagger \Phi_I$ decouples from bulk fields (at low enough order):



$|x| \lesssim 1/T$?

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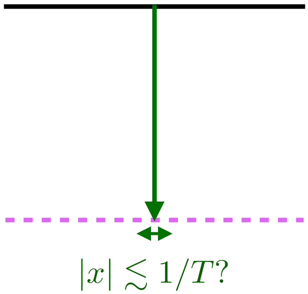
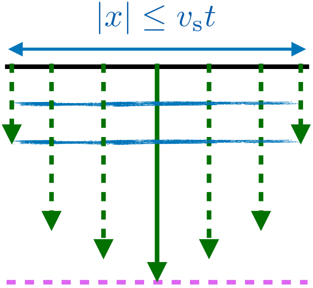


$$h(x,t) \sim \frac{1}{N} \mathrm{e}^{2\pi T(t-|x|/v_\mathrm{B})}$$

- if UV matter $\Phi_I^\dagger \Phi_I$ "overlaps" with sound:



$$h_\mathrm{sound}(x,t) \sim \frac{1}{N} f\left(t - \frac{|x|}{v_\mathrm{s}}\right)$$

[Lucas, 1710.01005]

A Puzzle: Holographic Charge Diffusion

- no sound waves (momentum relaxation) $\Longrightarrow$

$$D \le v_{\mathrm{B}}^2 \tau ?$$

- in "scaling" holographic theories: [Blake, 1603.08510]

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- but if $\Delta < 0$, hydrodynamics fails unexpectedly early?

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- what physics is responsible for τ when it is large?

[Lucas, 1710.01005]

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- $v_{\mathrm{LC}} \to v_{\mathrm{B}}$ with incoherent diffusion modes?
  - Fermi surface coupled to gauge field? [Patel *et al*, 1611.00003]
  - (charged) Sachdev-Ye-Kitaev chain? [Davison *et al*, 1612.00849]