Stability of chaos in a generalised SYK model

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Many-Body Quantum Chaos, Bad Metals and Holography
Stockholm, Oct. 2017

Based on arXiv: 1707.02197
With Antonio M. García-García, Bruno Loureiro and Masaki Tezuka

SYK and NAdS$_2$

$$H = \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \chi_i\, \chi_j\, \chi_k\, \chi_l\,,$$

$$\{\chi_i, \chi_j\} = \delta_{ij}\,, \qquad \mathbb{E}[J_{ijkl}^2] = \frac{6J^2}{N^3}$$

Large $\tau J \gg 1$ emergent conformal symmetry $\chi_i(\tau) \to \chi_i(f(\tau))$

[Kitaev 2016, Maldacena,... 2016, Sachdev 2016,...]

- Nearly-AdS$_2$

$$S = -\frac{\phi_0}{16\pi G}\left[\int \sqrt{g}R + 2\int_{bdy} K\right] - \frac{1}{16\pi G}\left[\int d^2x\sqrt{g}\phi(R+2) + 2\int_{bdy} \phi_b K\right] + \cdots$$

[Almheiri,... 2014, Maldacena,... 2016,...]

Random interactions and many-body physics

Long tradition in infinite-range random interaction models for many-body physics

[Mon,French, Wong,... 1970's, Sachdev, Ye, Georges, Parcollet,... 1990's]

Nuclear Physics: *k-body embedded ensembles*
Condensed Matter: spin models

- Spectral correlations close to RMT
- Ground state entropy
- Solvable at large-N

SYK

Replica trick:

$$\mathcal{Z} = \int D[\chi_i] e^{-S[\chi_i]} \to \mathcal{Z}^n = \int D[\chi_i] e^{-S_n[\chi_{i,a}]}$$

Disorder average: $\mathbb{E}[\mathcal{Z}^n] \implies$ get effective action:

$$S_{\text{eff}} = -\frac{1}{2} \operatorname{Tr} \log(\partial_\tau - \Sigma) + \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}\tau' \left[G(\tau, \tau') \Sigma(\tau, \tau') - \frac{J^2}{4} G(\tau, \tau')^4 \right]$$

$$G(\omega)^{-1} = -i\omega - \Sigma(\omega) \,, \; \Sigma(\tau) = -J^2 G(\tau)^2 G(-\tau)$$

IR limit $J\tau \gg 1$: $G(\tau) \propto \mathrm{sgn}(\tau)/|\tau|^{1/2}$

[Georges, Parcollet 1998, Polchinski, Rosenhaus, Maldacena, Stanford 2016,...]

Spectral properties, form factor and their relation to RMT

[García 2016, Cotler,... 2016,...]

SYK - generalizations

- Complex fermions
  [Sachdev, Davison, Fu, Georges, Gu, Jensen,...]
- Additional flavours
  [Gross, Rosenhaus, Banerjee, Altman,...]
- Supersymmetry
  [Fu, Gaiotto, Maldacena, Sachdev]
- Higher dimensional and transport
  [Davison, Fu, Gu, Lucas, Balents,...]
- No disorder
  [Witten, Klebanov, Tarnopolsky, Gurau...]

Model for today

$\mathrm{SYK}_4 + \mathrm{SYK}_2$

$$H = \frac{\kappa}{4!} \sum_{i,j,k,l}^{N} J_{ijkl}\, \chi_i\, \chi_j\, \chi_k\, \chi_l + \frac{i}{2!} \sum_{i,j}^{N} K_{ij}\, \chi_i\, \chi_j\,,$$

$$\mathbb{E}[J_{ijkl}^2] = \frac{6J^2}{N^3}\,, \quad \mathbb{E}[K_{ijkl}^2] = \frac{K^2}{N}$$

$$J = K = 1$$

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$$J = K = 1$$

- Replica trick
- Disorder average: $\mathbb{E}[\mathcal{Z}^n] \implies$ get effective action

$$S_{\mathrm{eff}} = S_{\mathrm{eff}}(\mathrm{SYK}_4) - \int \mathrm{d}\tau \mathrm{d}\tau' \frac{K^2}{4} G(\tau,\tau')^2$$

$$G(\omega)^{-1} = -i\omega - \Sigma(\omega)\,,\ \Sigma(\tau) = -\kappa^2 J^2 G(\tau)^2 G(-\tau) + K^2 G(\tau)$$

IR limit $J\tau \gg 1$ determined by SYK_2

Spectrum

Spectrum

Idea of ED

Choose a representation for $\mathcal{C}\ell_{0,2^{N/2}}$

$$\begin{aligned} \gamma_k^{(d+2)} &= \sigma_1 \otimes \gamma_k^d, \qquad \text{for} \quad k = 1, \cdots, d+1, \\ \gamma_{d+2}^{(d+2)} &= \sigma_2 \otimes 1_{2^{d/2}}, \\ \gamma_{d+3}^{(d+2)} &= \sigma_3 \otimes 1_{2^{d/2}}, \qquad \text{iterate till } d = N-2 \end{aligned}$$

Diagonalize $H_{2^{N/2} \times 2^{N/2}}$ and order spectrum: $\{E_k\}$ $(\mathcal{O}(10^6))$

Observables

Level spacing distribution: $P(s) = \sum_i \left\langle \delta(s - \frac{E_i - E_{i+1}}{\Delta}) \right\rangle$. Spectrum must be *unfolded* $(\Delta = 1)$!

Adjacent gaps: (no unfolded needed)

$$\langle r \rangle = \left\langle \frac{1}{2^{N/2}} \sum_{i=1}^{2^{N/2}-1} \frac{\min(\delta_i, \delta_{i+1})}{\max(\delta_i, \delta_{i+1})} \right\rangle, \quad \delta_i \equiv E_i - E_{i-1}$$

Spectrum

Level spacing distribution

Probes system at *Heisenberg* time $\sim \hbar/\Delta$

$$\underbrace{P_{\text{Poisson}}(s) \propto e^{-s}}_{\text{SYK}_2} \qquad \underbrace{P_{WD}(s) \propto s^2 e^{-4s^2/\pi}}_{\text{SYK}_4}$$

Spectrum

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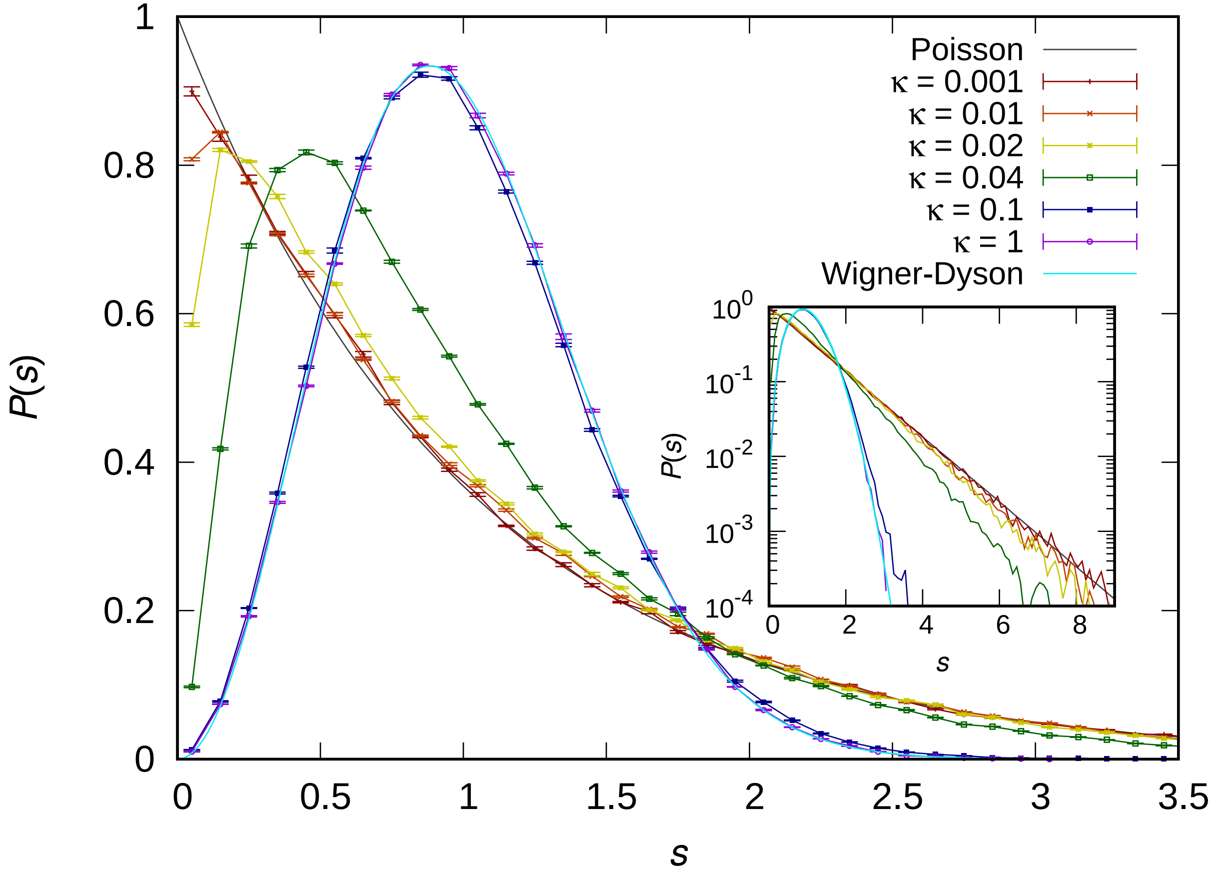
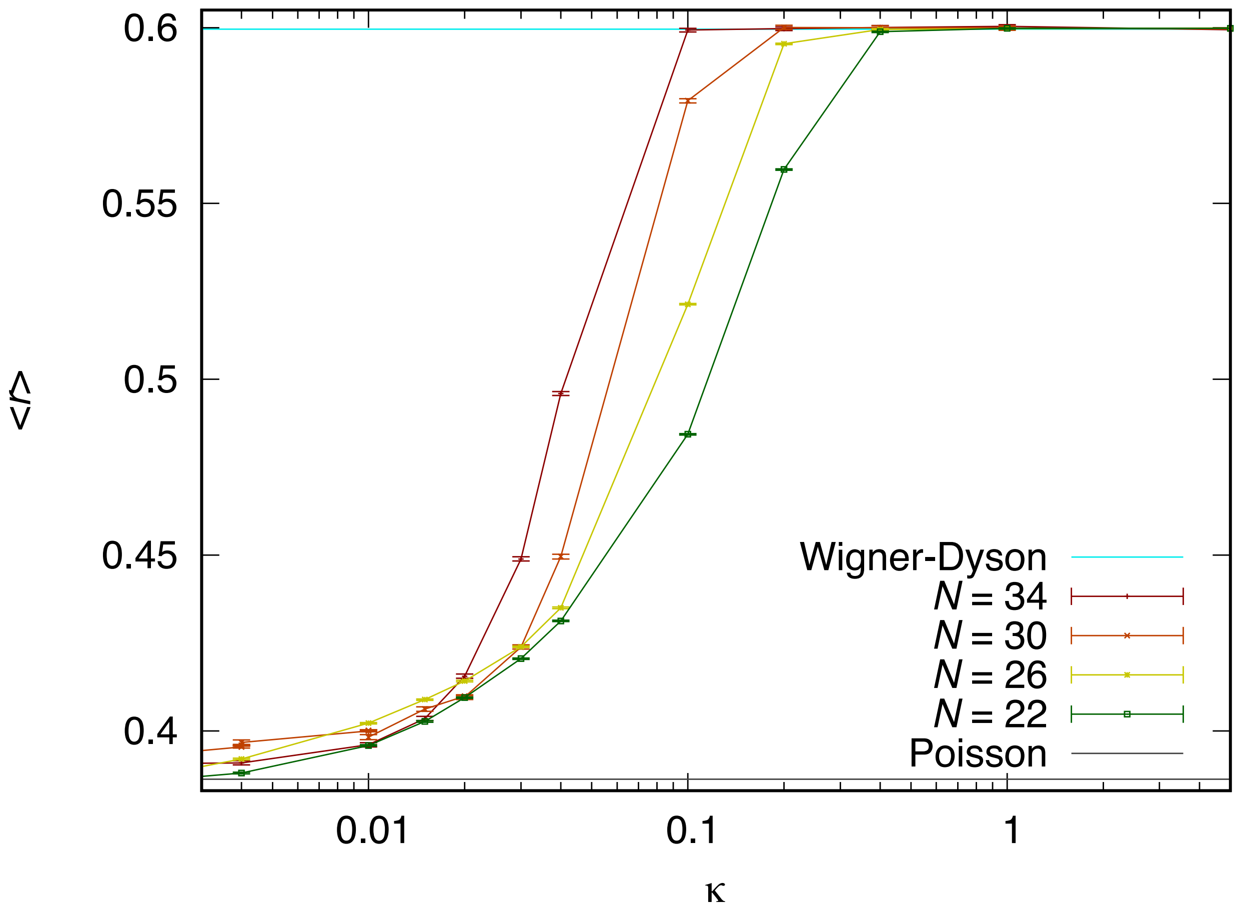


Spectrum

Adjacent gaps ratio

$$\langle r \rangle_{\text{Poisson}} = 2\log(2) - 1 \sim 0.386\,, \qquad \langle r \rangle_{\text{GUE}} = 0.599$$



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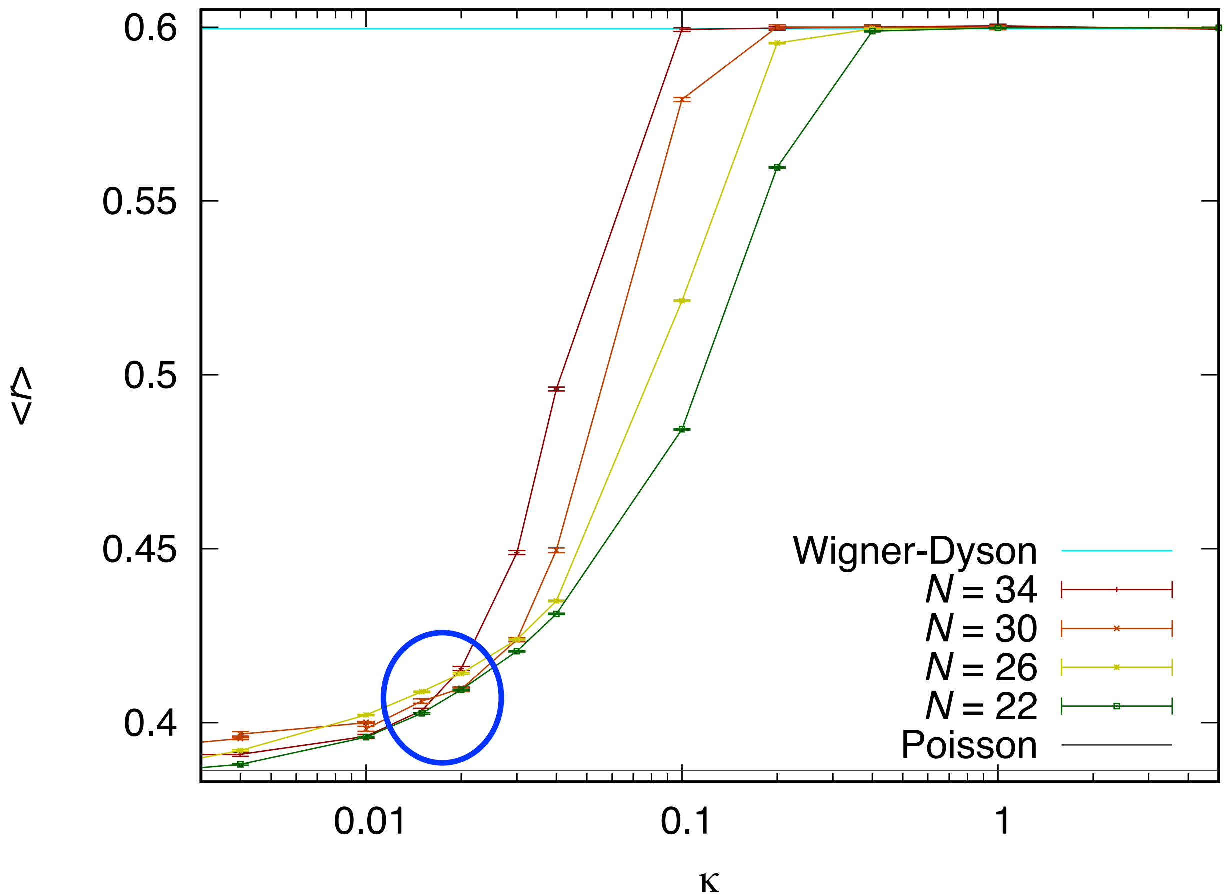


Thermodynamics

Thermodynamics

GS entropy

$$H = \frac{\kappa}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \chi_i\, \chi_j\, \chi_k\, \chi_l + \frac{i}{2!} \sum_{i,j=1}^{N} K_{ij}\, \chi_i\, \chi_j\,,$$

$$G(\omega)^2 + [i\omega + \kappa^2 \Sigma(\omega)] G(\omega) + 1 = 0\,, \quad \Sigma(\omega) = \int \mathrm{d}\tau e^{i\omega\tau} G(\tau)^3$$

Conformal limit: $\beta K \gg 1$, $\beta J \gg 1$:

- $\kappa = 0$, $G(\omega) \equiv G_2(\omega)$. Vanishing entropy at $T = 0$.

$$G_2(\tau) \sim \frac{1}{\beta K \sin(\pi\tau/\beta)} + \frac{\#}{[\beta K \sin(\pi\tau/\beta)]^3} + \dots$$

$$G_2(i\omega_n) \sim \frac{i\pi}{K} - \frac{\#}{(\beta K)^3}$$

- $\kappa \ll 1$, $G(\omega) = G_2(\omega) + \kappa^2 g(\omega) + \mathcal{O}(\kappa^4)$, $(K = 1 = J)$

$$G(i\omega_n) \sim \frac{i\pi}{K}\left(1 + \frac{\kappa^2}{4\beta^2}\right) - \frac{i\omega_n}{2}\left(1 + \frac{\kappa^2\pi}{16\beta^2}\right) + \mathcal{O}(\omega_n^2, \beta^{-4})$$

Same leading solution in the IR: GS entropy also vanishes!

Thermodynamics

Free energy

$$-\frac{\beta F}{N} = -\frac{1}{2}\,\mathrm{Tr}\log(G) + \text{subleading in } 1/\beta$$
$$= -\frac{1}{2}\beta\sum_{n\in\mathbb{Z}}\log G(i\omega_n)e^{i\omega_n 0^+} = -\beta\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon}{\pi}\frac{\arg G^R(\epsilon)}{1+e^{\beta\epsilon}} \qquad (1)$$

- SYK_2, $\kappa = 0$:

$$-\frac{\beta F}{N} \underset{\beta\gg 1}{\sim} \beta(\pi-1) + \log\left(1+e^{-2\beta}\right) + \frac{\pi}{12\beta} + O(\beta^{-3})$$

- $\mathrm{SYK}_2 + \kappa\mathrm{SYK}_4$, $\kappa > 0$:
  - $G^R(\epsilon) = G_2^R(\epsilon) + \kappa^2 g(\epsilon)$
  - Access only to IR region of $G^R(\epsilon)$. Introduce cutoff λ in Eq. (1) consistent with sum rule $\int_{-\lambda}^{\lambda}\mathrm{d}\epsilon\rho(\epsilon) = 2\pi$.
  - Cutoff affects only the GS energy

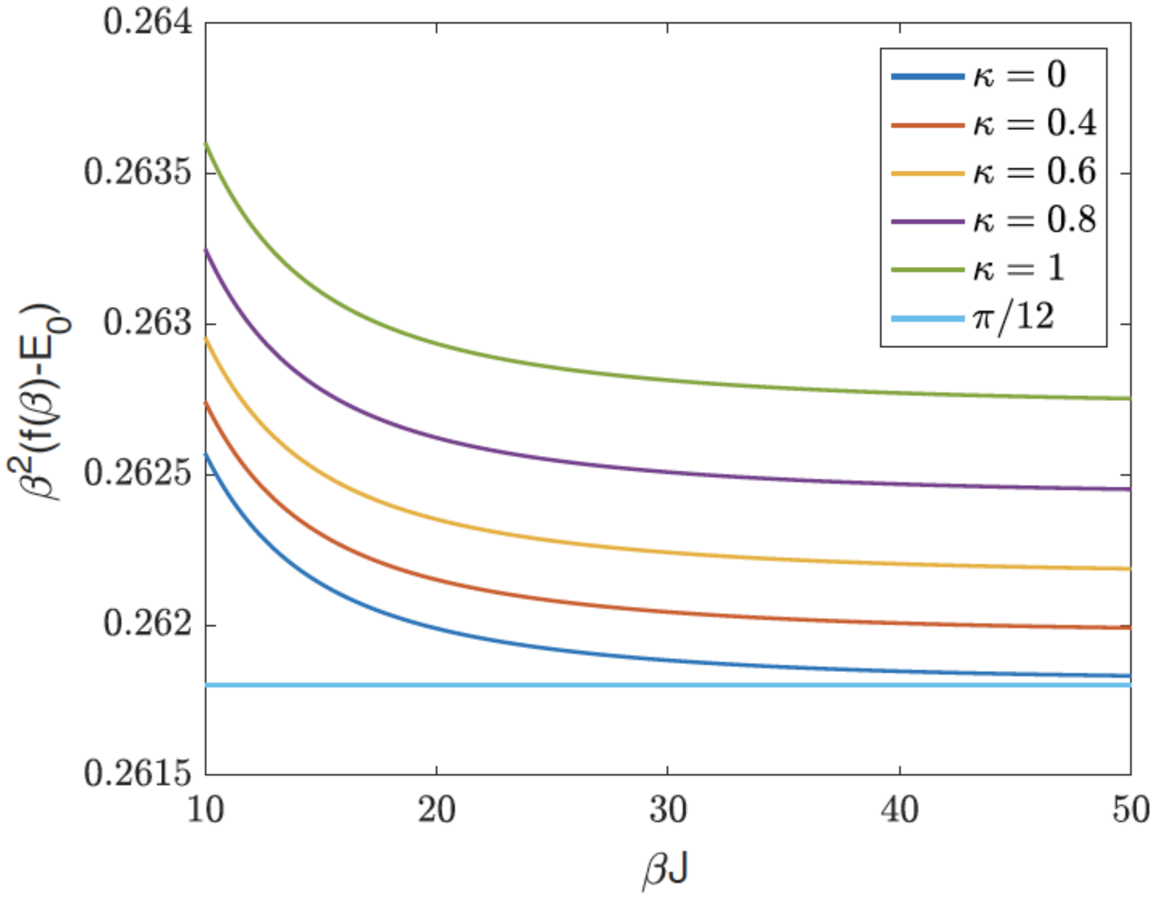
$$f(\beta) \equiv \frac{F}{N} = E_0 - \frac{s_0}{\beta} - \frac{c}{2\beta^2}\dots$$

Thermodynamics

Specific heat

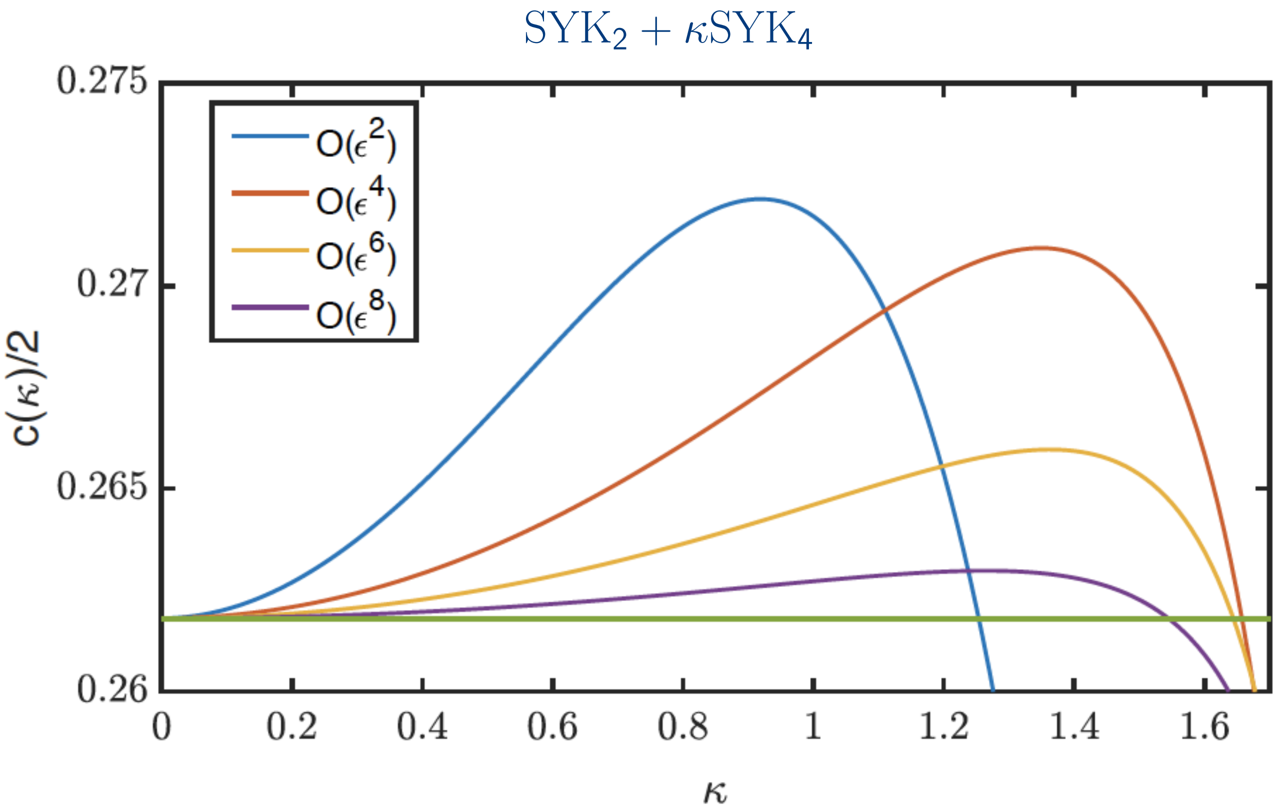
$G^R(\epsilon)$ to $\mathcal{O}(\epsilon^8)$

$\mathrm{SYK}_2 + \kappa \mathrm{SYK}_4$



Thermodynamics

Specific heat



We could have anticipated this:

$$-\frac{\beta F}{N} = -\beta \int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{\pi} \frac{\arg G^R(\epsilon)}{1+e^{\beta\epsilon}} \underset{\beta\gg 1}{=} -\beta \int_{-\lambda}^{0} \frac{\mathrm{d}\epsilon}{\pi} a(\epsilon,\kappa,\beta) + \frac{\pi}{6\beta} \underbrace{\frac{\mathrm{d}}{\mathrm{d}\epsilon} a(\epsilon,\kappa,\beta)|_{\epsilon=0}}_{=1/2} + O(\beta^{-3})$$

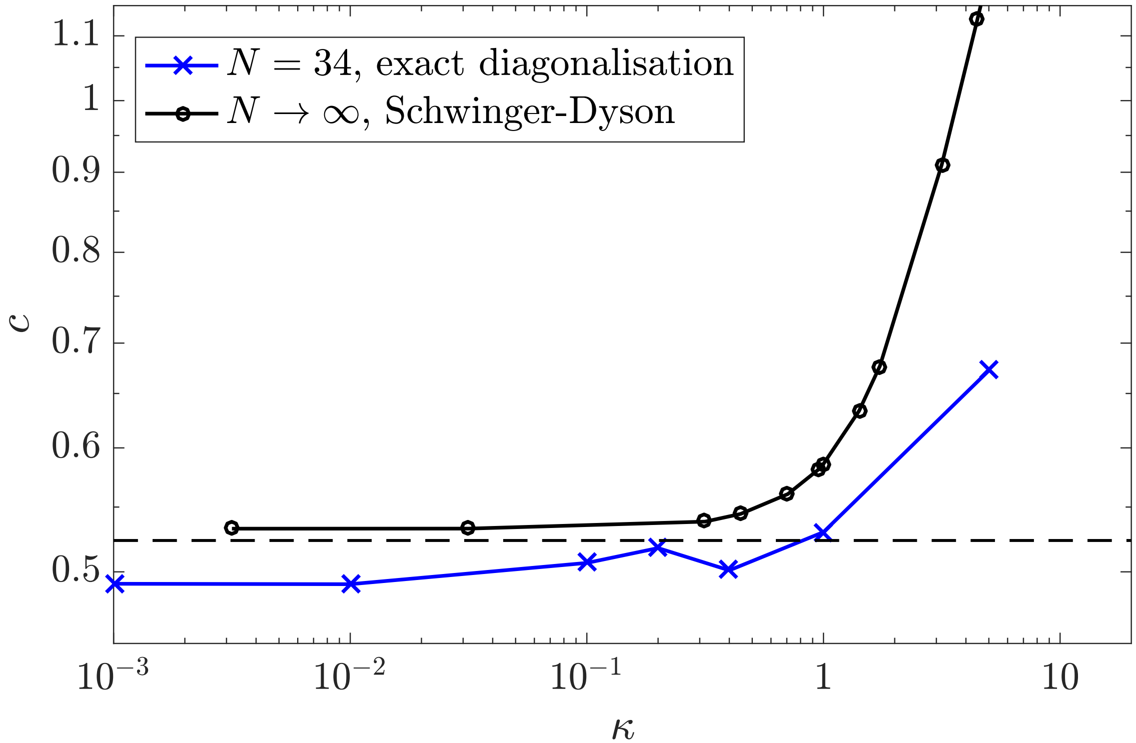
Thermodynamics

Specific heat - Numerics

Finite N (ED): $\mathrm{SYK}_2 + \kappa \mathrm{SYK}_4$

$$C(T) = \left\langle \frac{1}{NZ} \sum_k \frac{(E_k - \bar{E})^2}{T^2} e^{-\beta E_k} \right\rangle$$

$N \to \infty$ (Schwinger-Dyson): fitting $\frac{\log Z}{N} = -E_0\beta + s_0 + \frac{c}{2\beta} + \frac{c_1}{\beta^2} + \frac{c_2}{\beta^3}$



Spectral Form Factor

Probes system at *Ehrenfest* time

$$g(t,\beta) \equiv \left\langle \frac{Z(t,\beta)Z^*(t,\beta)}{Z(0,\beta)^2} \right\rangle , Z(t,\beta) = \mathrm{Tr}\, e^{-\beta H - iHt}$$

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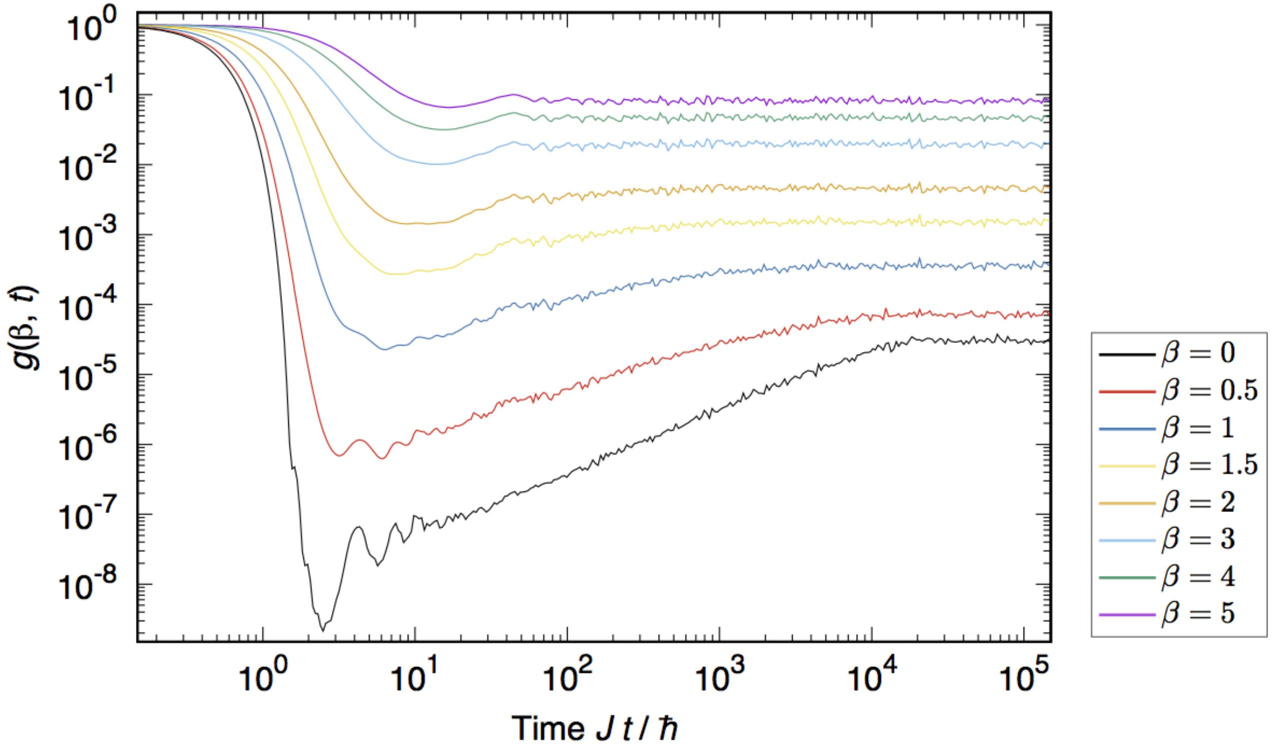
$K = J = \kappa = 1$



OTOC

$$H = \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\,\chi_i\,\chi_j\,\chi_k\,\chi_l + \kappa\frac{i}{2!}\sum_{i,j=1}^{N} K_{ij}\,\chi_i\,\chi_j\,, \qquad J = K = 1$$

$$\begin{aligned} F(t_1,t_2) &\equiv \frac{1}{N^2}\sum_{i,j}^{N}\mathbb{E}\left\{ Tr\left[\rho(\beta)^{\frac{1}{4}}\chi_i(t_1)\rho(\beta)^{\frac{1}{4}}\chi_j(0)\rho(\beta)^{\frac{1}{4}}\chi_i(t_2)\rho(\beta)^{\frac{1}{4}}\chi_j(0)\right]\right\} \\ &\simeq G_R(t_1)G_R(t_2) + \frac{1}{N}\mathcal{F}(t_1,t_2) + \mathcal{O}(N^{-2})\,, \end{aligned}$$

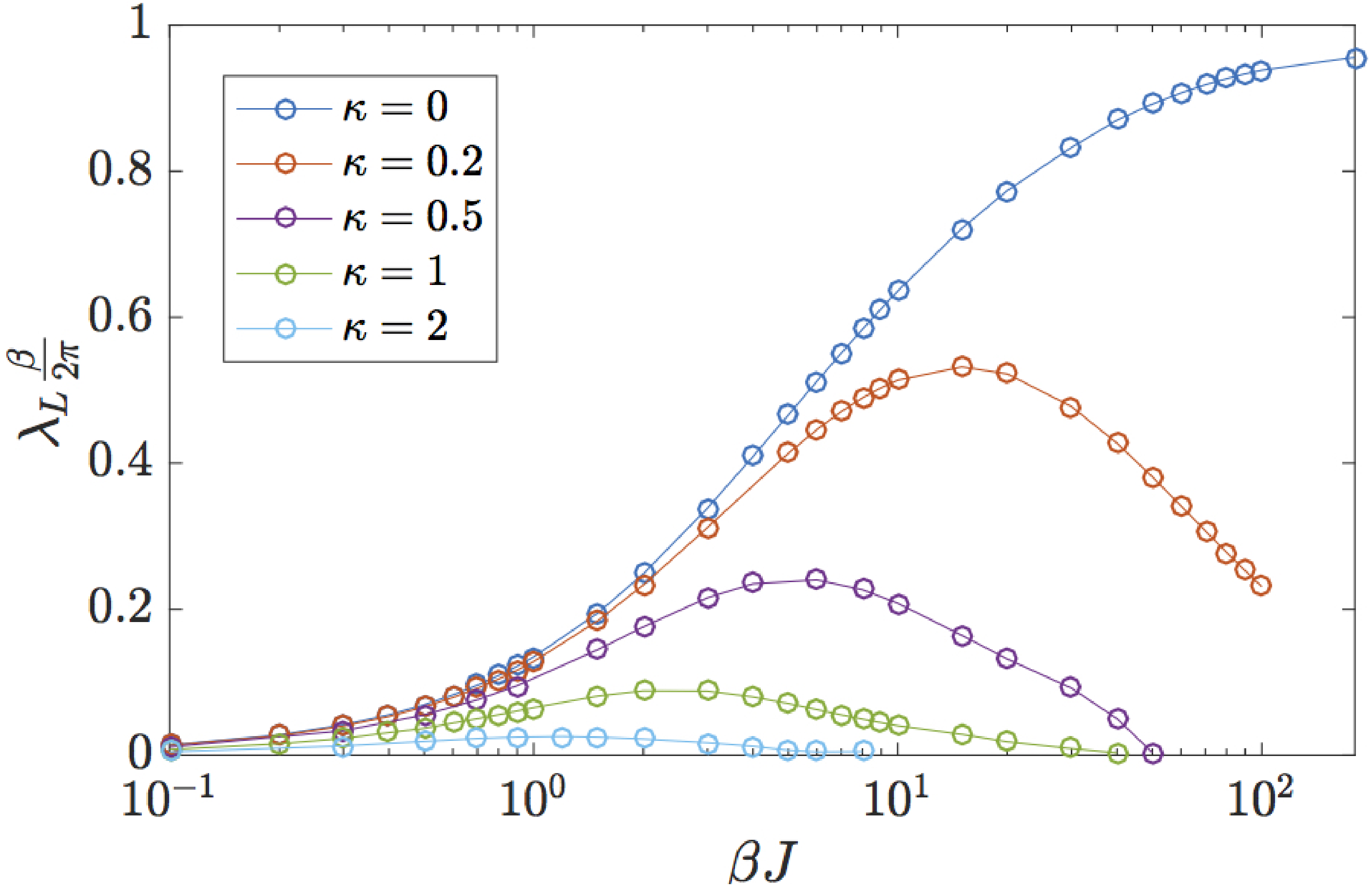
$$\begin{aligned} \mathcal{F}(t_1,t_2) &= \int \mathrm{d}t_3\mathrm{d}t_4 K_R(t_1,t_1,t_2,t_3,t_4)\mathcal{F}(t_3,t_4)\,, \\ K_R(t_1,t_1,t_2,t_3,t_4) &= G^R(t_{13})G^R(t_{24})\left[3J^2G_{lr}^2(t_{34}) + \kappa^2K^2\right] \end{aligned}$$

- Requires analytic continuation $i\omega_n \to \omega + i0^+$ of SD equations
- Ansatz: $\mathcal{F}(t_1,t_2) = e^{\lambda_L(t_1+t_2)/2}f(t_{12}) \implies$ solve eigenvalue equation for $f(t_{12})$

OTOC numerical

Lyapunov exponent

$$H = \mathrm{SYK}_4 + \kappa \mathrm{SYK}_2$$



OTOC analytical - large-q

$$H = i^{q/2} \sum_{i,\dots,i_q}^{N} J_{i\dots i_q} \chi_i \dots \chi_{i_q} + \kappa i \sum_{i,j=1}^{N} K_{ij} \chi_i \chi_j \,, \qquad \mathcal{J}^2 = 2^{1-q} q J^2 \,, \mathcal{K}^2 = q K^2$$

We need $G^R(t)$ for large-q:

$$G(\tau) \underset{\beta K \ll 1}{=} \frac{1}{2}\mathrm{sgn}(\tau)\left(1 + \frac{1}{q} g(\tau) + O(q^{-2})\right)$$

- Equation for $g(\theta = \tau/\beta)$: $\partial_\theta^2 g = 2(\beta\mathcal{J})^2 e^{g(\theta)} + \kappa^2(\beta\mathcal{K})^2$
- Perturbatively in κ: $g(\theta) = g_{(0)}(\theta) + (\beta\mathcal{K})^2\, g_{(1)}(\theta) + O(\beta\mathcal{K}^2)$

OTOC analytical - large-q

- Kernel at large-q: $K_R(t_1, t_1, t_2, t_3, t_4) = \theta(t_{13})\theta(t_{24}) \left[2\mathcal{J}^2 e^{g(\tau=it_{34}+\beta/2)} + \frac{1}{q}\kappa^2\mathcal{K}^2\right]$
- Ansatz: $\mathcal{F}(t_1, t_2) = e^{\lambda_L(t_1+t_2)/2} f(t_{12}) \implies$ Schödinger equation for $f(t_{12})$

$$\left[\partial_y^2 + V(y)\right] f(y) = \left(\frac{\beta\lambda_L}{2\pi\nu}\right)^2 f(y)$$

$$V(y) = 2\left(\frac{\beta\lambda_L}{2\pi\nu}\right)^2 e^{g(iy)} \underset{\beta\mathcal{K}\ll 1}{\simeq} 2\,\mathrm{sech}^2\, y + \kappa^2(\beta\mathcal{K})^2 g_{(1)}(iy) e^{g_{(0)}(iy)}$$

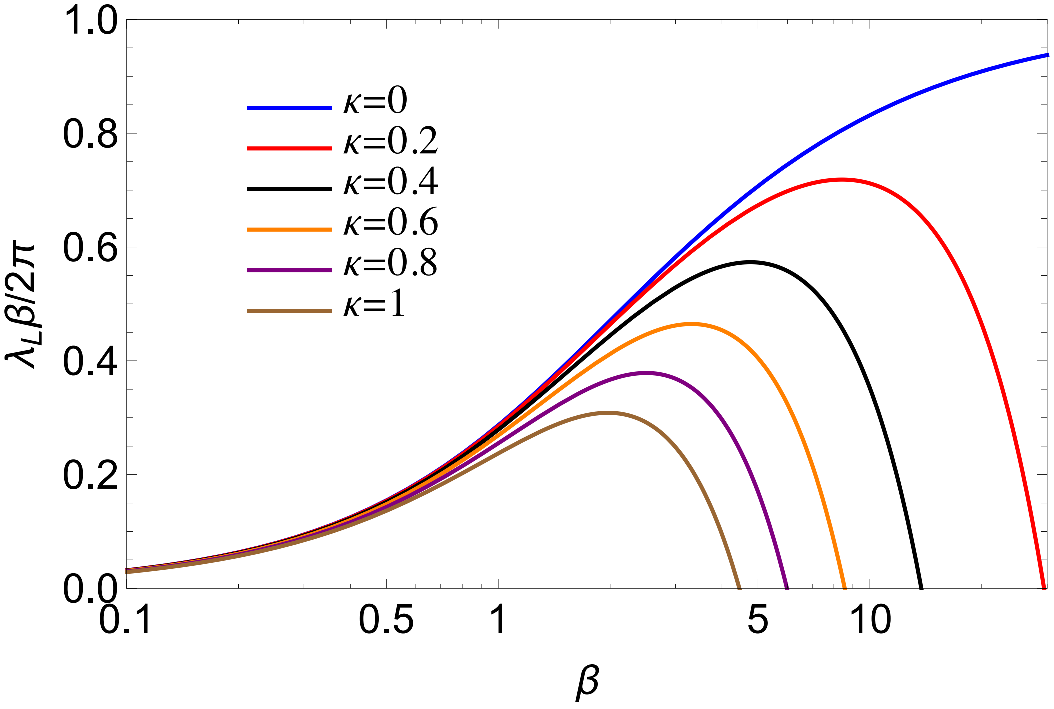
- Bound states: $E_\lambda = -1 + \kappa^2(\beta\mathcal{K})^2 E_\lambda^{(1)}(\nu) = -\left(\frac{\beta\lambda_L}{2\pi\nu}\right)^2$

$\nu = 0$ for $\beta\mathcal{J} = 0$
$\nu \to 1$ for $\beta\mathcal{J} \to \infty$

OTOC analytical - large-q

Lyapunov exponent

$$H = \mathrm{SYK}_q + \kappa \mathrm{SYK}_2$$



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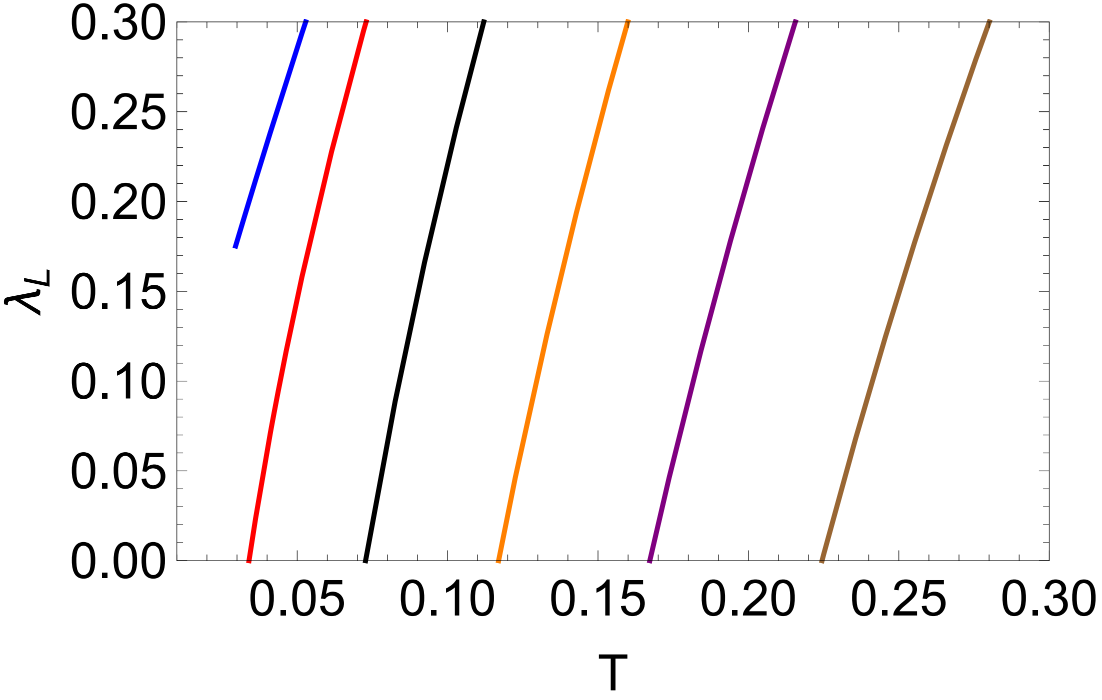

- SYK$_q$ in the presence of a relevant one-body deformation:
  - Vanishing GS entropy. Small corrections to specific heat.
  - Persistence of chaos (though weakened)

- For sufficiently low temperature there is a **chaotic-integrable transition** vanishing λ_L,

- This type of transition is common in low-dim. systems. Need to **identify gravity dual**
  - Centaur geometries? [Anninos, Hofman 2017]
  - Hawking-Page transition? thermal gas corresponding to non-chaotic phase