The melonic universality class

Răzvan Gurău

Nordita, 2017

Holography

Holography: in a theory with gravity one must have a correspondence between a volume of space and the boundary enclosing it (black hole entropy).

Gravity in $AdS_{d+1} \times \mathrm{Compact} \Leftrightarrow CFT_d$.

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Find interesting examples of (near) CFT_1!

An "interesting" CFT_1 should...

Non trivial scaling dimension (conformal weight Δ):

$$G(\tau_1 - \tau_2) \sim \frac{1}{|\tau_1 - \tau_2|^{2\Delta}} , \qquad \tilde{G}(\omega) = \int_{-\infty}^{\infty} d\tau \, \frac{1}{|\tau|^{2\Delta}} e^{\imath\omega\tau} \sim |\omega|^{2\Delta-1}$$

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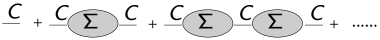

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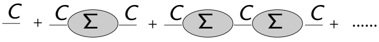

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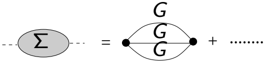

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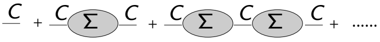

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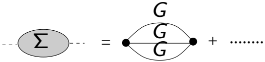

Simple equation if the rest is suppressed: $\Sigma=J^2\ G^{q-1}$

Melonic theories

Are there theories such that $\Sigma = J^2 \, G^{q-1}$?

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Yes: "Melonic" theories built on random tensors!

Random tensors

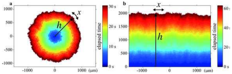
Random Tensors generalize Random Matrices to higher dimension.

Random matrices

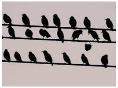
J. Wishart (1928) statistical analysis of large samples

E. Wigner (1955) spectroscopy of heavy nuclei

- Growing interfaces fluctuations



- Spacing between perched birds (parked cars)



- Distinguish "signal" from "noise"

A. Chakraborty (2011) identify DNA sectors in HIV that rarely undergo multiple mutations

The $1/N$ expansion in random matrices

Random $N \times N$ matrices:

- Feynman expansion in embedded graphs (combinatorial maps) $\leftrightarrow$ discretized surfaces.
- Have built in scales: the size of the matrix, N (number of degrees of freedom).

Canonical framework for the study of random surfaces (1980 onward): string theory, quantum gravity in $D = 2$, integrability, conformal field theory, invariants of algebraic curves, etc.

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$1/N$ expansion indexed by the genus

Higher dimensions

Generalize matrix models to higher dimensions

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Proposals in the 90s: tensor models and group field theories (Ambjørn, Durhuus, Jonsson, Boulatov, Ooguri).

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$1/N$ in 2010/2011.

A Baratin, D. Benedetti, J. Ben Geloun, V. Bonzom, S Carrozza, S. Dartois, T. Delepouve, A. Eichhorn, L. Freidel, T. Koslowski, T. Krajewski, V. Lahoche, L. Lioni, D. Oriti, V. Rivasseau, A. Riello, J.P. Ryan, D.O. Samary, M. Smerlak, L. Sindoni, A. Tanasa, F. Vignes-Tourneret, etc.

0–dimensional gauge theories

Random matrices ⇔ 0–dimensional gauge theories for at most two copies of the unitary (orthogonal, etc) group:

$$Z = \int [dA] \; e^{N\mathrm{Tr}[S(A)]} \, , \qquad A \to UAU^\dagger \quad U \in \mathcal{U}(N)$$

$$Z = \int [dM dM^\dagger] \; e^{N\mathrm{Tr}[S(MM^\dagger)]} \, , \qquad M \to UMV^\dagger \, , \; M^\dagger \to VM^\dagger U^\dagger \quad U \in \mathcal{U}(N) \, , \; V \in \mathcal{U}(N)$$

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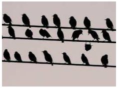
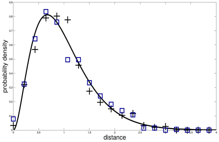
So where are the birds?

$$S(A) = -A^2/2 \xrightarrow{\text{eigenvalues}} Z = \int \Big(\prod_i d\lambda_i \Big) \underbrace{\Big(\prod_{i<j} |\lambda_i - \lambda_j|^2 \Big)}_{\text{eigenvalue repulsion}} \; \underbrace{e^{-\frac{N}{2} \sum_i \lambda_i^2}}_{\text{confining potential}}$$





Gap distribution at large N fits the distribution of the spacing between perched birds (squares) or parked cars (crosses)! (Šeba 2013)

Tensor models

Tensor Models are 0–dimensional gauge theories with gauge group the tensor product of $D \geq 3$ copies of the unitary (orthogonal, etc.) group.

The field → rank D complex tensor (no symmetry) transforming in the external tensor product of D fundamental representations of $\mathcal{U}(N)^{\otimes D}$:

$$T'_{b^1\dots b^D} = \sum_a U^{(1)}_{b^1 a^1} \dots U^{(D)}_{b^D a^D} T_{a^1\dots a^D} , \quad \bar{T}'_{p^1\dots p^D} = \sum_q \bar{U}^{(1)}_{p^1 q^1} \dots \bar{U}^{(D)}_{p^D q^D} \bar{T}_{q^1\dots q^D}$$

The action and the gauge invariant observables → invariants built out of the tensor and its dual.

Tensor invariants as Colored Graphs

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Invariants ("traces") $\sum_{a^1,q^1} \delta_{a^1q^1} \dots T_{a^1\dots a^D}\, \bar T_{q^1\dots q^D} \dots$

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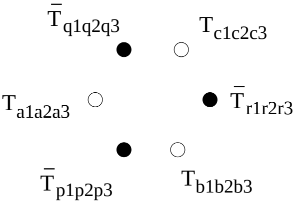

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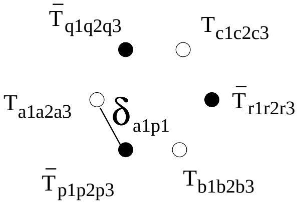

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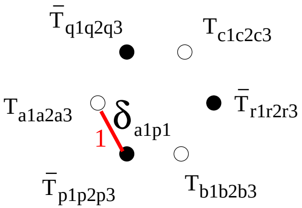

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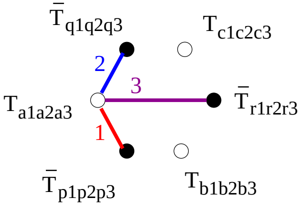

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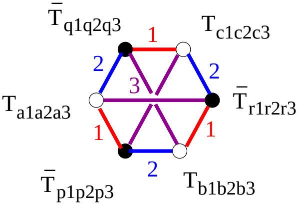

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$$\mathrm{Tr}_{\mathcal{B}}(T,\bar{T}) = \sum \prod_v T_{a^1_v \dots a^D_v} \prod_{\bar{v}} \bar{T}_{q^1_{\bar{v}} \dots q^D_{\bar{v}}} \prod_{c=1}^{D} \prod_{e^c=(w,\bar{w})} \delta_{a^c_w q^c_{\bar{w}}}$$

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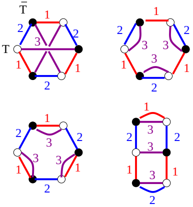

Single trace models

The field → tensor:

$$T'_{b^1 \dots b^D} = \sum_a U^{(1)}_{b^1 a^1} \dots U^{(D)}_{b^D a^D} T_{a^1 \dots a^D}$$

The action → "single trace" invariant:

$$S(T, \bar{T}) = \sum T_{b^1 \dots b^D} \bar{T}_{q^1 \dots q^D} \prod_{c=1}^{D} \delta_{b^c q^c} + \sum_{\substack{\text{connected graphs } \mathcal{B} \\ \text{with D colors}}} t_{\mathcal{B}} \mathrm{Tr}_{\mathcal{B}}(T, \bar{T})$$

The partition function:

$$Z(t_{\mathcal{B}}) = \int [d\bar{T} dT] \, e^{-N^{D-1} S(T, \bar{T})}$$

The gauge invariant observables:

$$\mathrm{Tr}_{\mathcal{B}}(T, \bar{T})$$

Objective: $\ln Z$, $\langle \mathrm{Tr}_{\mathcal{B}_1}(T, \bar{T}) \dots \mathrm{Tr}_{\mathcal{B}_1}(T, \bar{T}) \rangle$

Feynman expansion

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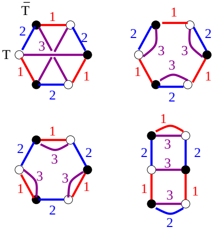
Feynman expansion:

- Taylor expand in $t_{\mathcal{B}}$ → graphs with D colors



$$Z(t_{\mathcal{B}}) = \sum \int_{T,\bar T} e^{-N^{D-1}\left(\sum T_{b^1\dots b^D}\bar T_{q^1\dots q^D}\prod_{c=1}^D \delta_{b^c q^c}\right)}$$
$$\mathrm{Tr}_{\mathcal{B}_1}(T,\bar T)\mathrm{Tr}_{\mathcal{B}_2}(T,\bar T)\dots$$

Feynman expansion

$$S(T,\bar T) = \sum T_{b^1\dots b^D}\bar T_{q^1\dots q^D}\prod_{c=1}^{D}\delta_{b^c q^c} + \sum_{\substack{\text{connected graphs } \mathcal{B} \\ \text{with D colors}}} t_{\mathcal{B}} \mathrm{Tr}_{\mathcal{B}}(T,\bar T)\,,$$

$$Z(t_{\mathcal{B}}) = \int [d\bar T dT]\; e^{-N^{D-1}S(T,\bar T)}$$

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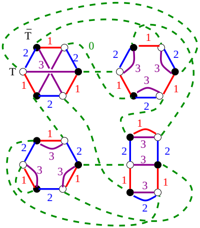
- Taylor expand in $t_{\mathcal{B}}$ → graphs with D colors
- compute the Gaussian integrals (Wick theorem) → graphs with $D+1$ colors



$$Z(t_{\mathcal{B}}) = \sum_{\text{graphs } \mathcal{G} \text{ with } D+1 \text{ colors}} A(\mathcal{G})$$

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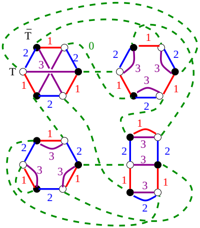
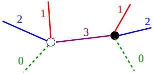


$$Z(t_{\mathcal{B}}) = \sum_{\text{graphs } \mathcal{G} \text{ with } D+1 \text{ colors}} A(\mathcal{G})$$

Each graph $\mathcal{G}$ is embedded in a D dimensional space
(Poincaré dual to a triangulation)

Colored graphs and vertex colored triangulations

White and black $D+1$ valent vertices connected by edges with colors $0, 1 \ldots D$.



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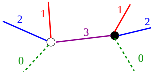
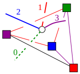
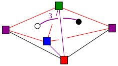


Vertex $\leftrightarrow$ D simplex with colored vertices .



Edges $\leftrightarrow$ gluings along $D-1$ simplices respecting all the colorings



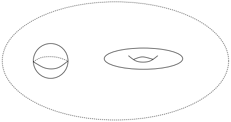
Observables and expectations

Observables = invariants $\mathrm{Tr}_{\mathcal{B}}$ encoding boundary triangulations.
Expectations =

$$\Big\langle \mathrm{Tr}_{\mathcal{B}_1}\mathrm{Tr}_{\mathcal{B}_2}\dots\mathrm{Tr}_{\mathcal{B}_q}\Big\rangle = \frac{1}{Z(t_{\mathcal{B}})}\int [d\bar{T}\,dT]\;\mathrm{Tr}_{\mathcal{B}_1}\mathrm{Tr}_{\mathcal{B}_2}\dots\mathrm{Tr}_{\mathcal{B}_q}\;e^{-N^{D-1}S(T,\bar{T})}$$

correlations between boundary states given by sums over all bulk triangulations compatible with the boundary states

- $\Big\langle \mathrm{Tr}_{\mathcal{B}_1}\mathrm{Tr}_{\mathcal{B}_2}\Big\rangle_c = \Big\langle \mathrm{Tr}_{\mathcal{B}_1}\mathrm{Tr}_{\mathcal{B}_2}\Big\rangle - \Big\langle \mathrm{Tr}_{\mathcal{B}_1}\Big\rangle\Big\langle \mathrm{Tr}_{\mathcal{B}_2}\Big\rangle$: transition amplitude between the boundary states $\mathcal{B}_1$ and $\mathcal{B}_2$



The $1/N$ expansion for matrices (G. 't Hooft, 1974)

$$Z = \int [dMdM^\dagger] e^{-N\left\{ \text{Tr}[MM^\dagger] - \sum_{p\geq 1} \frac{z^p}{p} \text{Tr}[(MM^\dagger)^p] \right\}}$$

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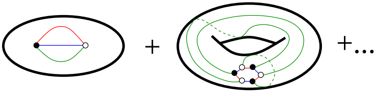
- perturbative expansion indexed by graphs embedded in surfaces

$$\ln Z = \sum_{\substack{\text{connected} \\ 2+1 \text{ colored graphs } \mathcal{G}}} A(\mathcal{G})$$



- the scaling with N senses the topology (genus $g(\mathcal{G}) \geq 0$):

$$A(\mathcal{G}) \sim N^{2-2g(\mathcal{G})}$$

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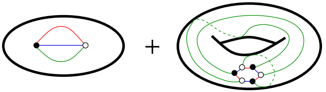

$+ \ldots$

- the scaling with N senses the topology (genus $g(\mathcal{G}) \geq 0$):

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- reorganize the perturbative series in powers of $1/N$:

$$\frac{1}{N^2} \ln Z = \underbrace{\sum_{g\geq 0} \left(\frac{1}{N}\right)^{2g}}_{\text{expansion in topologies}} \underbrace{\sum_{\substack{\text{connected} \\ \text{2+1 colored graphs } \mathcal{G}}}^{g(\mathcal{G})=g} \frac{1}{p(\mathcal{G})} z^{p(\mathcal{G})}}_{\text{convergent sum at fixed topology}}$$

number of white vertices

What replaces the genus?

The key in $D = 2$ is the Euler relation relating the number of vertices and faces (bi–colored cycles) of a graph

$$F(\mathcal{G}) = p(\mathcal{G}) + 2 - 2g(\mathcal{G})$$

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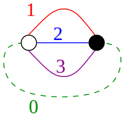
$$F(\mathcal{G}) = p(\mathcal{G}) + 2 - 2g(\mathcal{G})$$

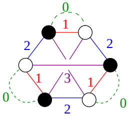
Theorem

For any D and any connected $\mathcal{G}$, with $2p(\mathcal{G})$ vertices and $F(\mathcal{G})$ faces there exists a non negative integer $\omega(\mathcal{G})$ such that

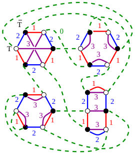
$$\boxed{F(\mathcal{G}) = \frac{1}{2}D(D-1)p(\mathcal{G}) + D - \frac{2}{(D-1)!}\omega(\mathcal{G}) \ , \qquad \omega(\mathcal{G}) \geq 0 \ .}$$



$\omega(\mathcal{G}) = 0$



$\omega(\mathcal{G}) = 4$



$\omega(\mathcal{G}) = 14$

The 1/N expansion in $D \geq 3$

$$S(T,\bar{T}) = \sum T_{a^1\dots a^D}\bar{T}_{q^1\dots q^D}\prod_{i=1}^{D}\delta_{a^i q^i} - \sum_{\mathcal{B}} N^{-\frac{2}{(D-2)!}\omega(\mathcal{B})}\frac{z^{p(\mathcal{B})}}{p(\mathcal{B})}\mathrm{Tr}_{\mathcal{B}}(T,\bar{T})$$

$$Z = \int [d\bar{T}dT]\, e^{-N^{D-1}S(T,\bar{T})}$$

Theorem

The free energy of this model admits the 1/N *expansion:*

$$\frac{1}{N^D}\ln Z = \sum_{\omega\geq 0} N^{-\frac{2}{(D-1)!}\omega} \sum_{\substack{\mathcal{G}\text{ connected bipartite}\\ D+1\text{ colored graphs}}}^{\omega(\mathcal{G})=\omega} \frac{1}{p(\mathcal{G})} z^{p(\mathcal{G})}\,.$$

Similar 1/N expansions exist for arbitrary observables.

Topology and the $1/N$ expansion

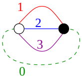
For $D \geq 3$ there does not exist a unique topological invariant that discriminates the topologies.

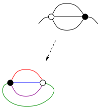
The $1/N$ expansion can not be indexed by a topological invariant: $\omega(\mathcal{G})$ mixes topological and triangulation dependent information.

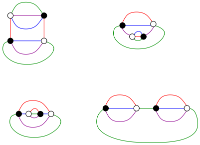
Leading order

Theorem

Graphs $\mathcal{G}$ with degree $\omega(\mathcal{G}) = 0$ are melonic.







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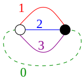

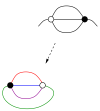

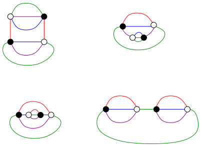

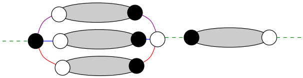

$$\Sigma = J^2 G^D$$

Melonic theories

Whenever a random tensor is present, melonic graphs dominate.

The self energy factors at leading order in terms of the two point function.

A first example: The SYK model

Vector Majorana fermions $\chi_a(\tau)$, q–body interaction with quenched random couplings:

$$S = \frac{1}{2}\sum_a \int \chi_a \partial_\tau \chi_a + \sum_{a_1 \dots a_q} J_{a_1 \dots a_q} \int_\tau \chi_{a_1} \cdots \chi_{a_q} \ ,$$

$$d\nu(J) = e^{-\frac{N^{(q-1)}}{J^2} \sum_a (J_{a_1 \dots a_q})^2}$$

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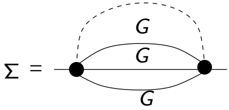
$G(\tau_1, \tau_2) \to$ the disorder average of the thermal two point function:

$$G(\tau_1, \tau_2) = \int d\nu(J) \left[\frac{\int [d\chi]\, e^{-S} \chi(\tau_1)\chi(\tau_2)}{\int [d\chi]\, e^{-S}} \right]$$

Disorder: Gaussian random tensor, melonic at leading order in N

$$\Sigma = J^2 G^{q-1}$$



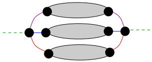
Tensor SYK models

Promote everybody to tensor fields (Witten, 2016):

$\chi^{0}{}_{a^{03}a^{02}a^{01}}(\tau),\ \ \chi^{1}{}_{a^{01}a^{13}a^{12}}(\tau),\ \ \chi^{2}{}_{a^{12}a^{02}a^{23}}(\tau),\ \ \chi^{3}{}_{a^{23}a^{13}a^{03}}(\tau)$:

$$S = \frac{1}{2}\sum\int \chi\partial_\tau\chi + \frac{J}{N^{3/2}}\sum\int_\tau \chi^0\chi^1\chi^2\chi^3 , \qquad G = \frac{\int[d\chi]\, e^{-S}\chi\chi}{\int[d\chi]e^{-S}}$$

Color tensor model (R.G. 2011) melonic at leading order in 1/N:



$$\Sigma = J^2 G^D$$

Why Tensor SYK?

- Eliminates the quenching without drowning the fermions (Witten)
- Number of fields does not proliferate for $N \to \infty$: as in QCD, it is the gauge group which grows (Witten)
- Solves the question of the singlets: only gauge invariant observables (Klebanov)
- Better controlled 1/N series (Klebanov)

Simpler tensor SYK

- Witten: 4 tensors, $O(N)^6$ gauge group (colored tensors)
- Klebanov and Tarnopolsky (2016): 1 tensor, $O(N)^3$ gauge group (Tanasa-Carrozza, non bipartite invariant tensor)

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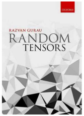
- symmetric tensors don't work (antisymmetric might...)
- Klebanov and Tarnopolsky (2017): numerical checks for symmetric traceless and antisymmetric: 1 tensor, $O(N)$
- R.G. (2017): proof for symmetric and antisymmetric but bipartite: 2 tensors, $O(N)$ gauge group

The bottom line

Random tensors:

- canonical path integral formulation.
- built in scales (tensors of size N^D).
- new universal $1/N$ expansion, melons dominate.
- "interesting" CFT_1s, random geometries in arbitrary dimension, etc.



- extend maximally the melonic universality class
- escape the melonic universality class