# Thermal diffusivity and chaos in holographic theories

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# Introduction

- In many-body systems, charge and energy typically diffuse over long distances.

- Diffusivities have units of $\text{velocity}^2 \times \text{time}$

- In a Fermi liquid, there is a natural velocity and timescale: Fermi velocity and quasiparticle lifetime.

- What about when there are no quasiparticles ? In various cases:

$$D \sim v_B^2 \tau_L$$

- I'm going to talk about when and why this works robustly in holographic theories.

# Main Results

- The main result is that the low temperature thermal diffusivity is related to the chaos parameters

$$D_T \equiv \frac{\kappa}{c_\rho} \sim v_B^2 \tau_L \sim v_B^2 \frac{\hbar}{k_B T}$$

- $D_T$ and $v_B^2 \tau_L$ depend in a complicated way on UV details of the theory.
- But the coefficient is a property of the IR fixed point only.
- This relation is robust to changing the theory in various ways.
- It works because $D_T$ and $v_B^2 \tau_L$ depend only on the metric near the horizon.

# Outline of the talk

- 1. Einstein relations and thermal diffusivity

- 2. Thermal diffusivity and chaos in holographic theories

- 3. Some special cases

# Diffusion of charge and energy I

- What is the thermal diffusivity?

- Perturbations of energy density $\delta\varepsilon$ & charge density $\delta\rho$ obey continuity equations
$$\partial_t \delta\varepsilon + \nabla \cdot j_\varepsilon = 0$$
$$\partial_t \delta\rho + \nabla \cdot j = 0$$

- It is easier to study heat rather than energy $T\partial_t \delta s + \nabla \cdot j_s = 0$
where $T\delta s = \delta\varepsilon - \mu\delta\rho \qquad j_s = j_\varepsilon - \mu j$

- Assume that at the longest scales, only excitations are long-wavelength perturbations of conserved quantities $\delta\varepsilon$ and $\delta\rho$

- Then
$$j = -C_1 \nabla\delta\varepsilon - C_2 \nabla\delta\rho + \ldots,$$
$$j_s = -C_3 \nabla\delta\varepsilon - C_4 \nabla\delta\rho + \ldots$$
assuming that momentum is not conserved.

# Diffusion of charge and energy II

- This produces coupled diffusion equations for charge & heat.
- Change variables using the susceptibility matrix $\chi$

$$\begin{pmatrix} \delta\rho \\ \delta s \end{pmatrix} = \begin{pmatrix} (\partial\rho/\partial\mu)_T & (\partial\rho/\partial T)_\mu \\ (\partial s/\partial\mu)_T & (\partial s/\partial T)_\mu \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta T \end{pmatrix}.$$

→

$$j = -\sigma\nabla\mu - \alpha\nabla T + \ldots$$

$$j_s = -\alpha T\nabla\mu - \bar{\kappa}\nabla T + \ldots$$

- In these variables, the coefficients are the elements of the conductivity matrix $\Sigma$

$$\begin{pmatrix} j \\ j_s/T \end{pmatrix} = \begin{pmatrix} \sigma & \alpha \\ \alpha & \bar{\kappa}/T \end{pmatrix} \begin{pmatrix} E \\ -\nabla T \end{pmatrix},$$

- So the diffusion equations are $\partial_t \begin{pmatrix} \delta\rho \\ \delta s \end{pmatrix} = \Sigma \cdot \chi^{-1} \nabla^2 \begin{pmatrix} \delta\rho \\ \delta s \end{pmatrix}$
- The diffusivity matrix $\Sigma \cdot \chi^{-1}$ is typically very complicated.

# Thermal diffusivity

- There are simpler diffusive processes.

- An external electric field will produce charge and heat currents

$$j = -\sigma \left( \nabla \mu - E \right) - \alpha \nabla T + \ldots$$

$$j_s = -\alpha T \left( \nabla \mu - E \right) - \bar{\kappa} \nabla T + \ldots$$

- Set up the electric field such that $\nabla \cdot j = 0$: $\quad E = \nabla \mu + \dfrac{\alpha}{\sigma} \nabla T$

- Then perturbations in the temperature $\delta T \equiv \left( \dfrac{\partial T}{\partial s} \right)_\rho \delta s + \left( \dfrac{\partial T}{\partial \rho} \right)_s \delta \rho$ obey

$$\partial_t \delta T = \frac{\kappa}{c_\rho} \nabla^2 \delta T$$

where $\kappa \equiv - \dfrac{j_s}{\nabla T} \Big|_{j=0} = \bar{\kappa} - \dfrac{T \alpha^2}{\sigma}$ and $c_\rho \equiv T \left( \dfrac{\partial s}{\partial T} \right)_\rho$

- This is the thermal diffusivity $D_T \equiv \dfrac{\kappa}{c_\rho}$

# Holographic theories

- I will look at holographic theories where there is a conformally invariant fixed point at high energies.
- Deform this by turning on a density and a periodic potential.
- There is an RG flow to a different low energy (IR) fixed point.
- In general, it is not relativistic and does not obey hyperscaling.
- The class of gravitational actions are

$$S = \int d^4x\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{Z(\phi)}{4}F^2 - \frac{Y(\phi)}{2}\sum_{i=1,2}(\partial\varphi_i)^2\right)$$

Charmousis et al
Gouteraux
Donos & Gauntlett

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + h(r)d\vec{x}^2 \qquad A_t(r) \neq 0 \qquad \varphi_i = mx^i$$

breaks translational symmetry in a homogeneous & isotropic way

# IR fixed point solutions

- Near the horizon, assume exponential potentials

$$V(\phi) = V_0 e^{\delta\phi} \qquad W(\phi) = W_0 e^{\lambda\phi} \qquad Z(\phi) = Z_0^2 e^{\gamma\phi}$$

- These support solutions with logarithmically running dilaton and power law metric

Charmousis et. al.

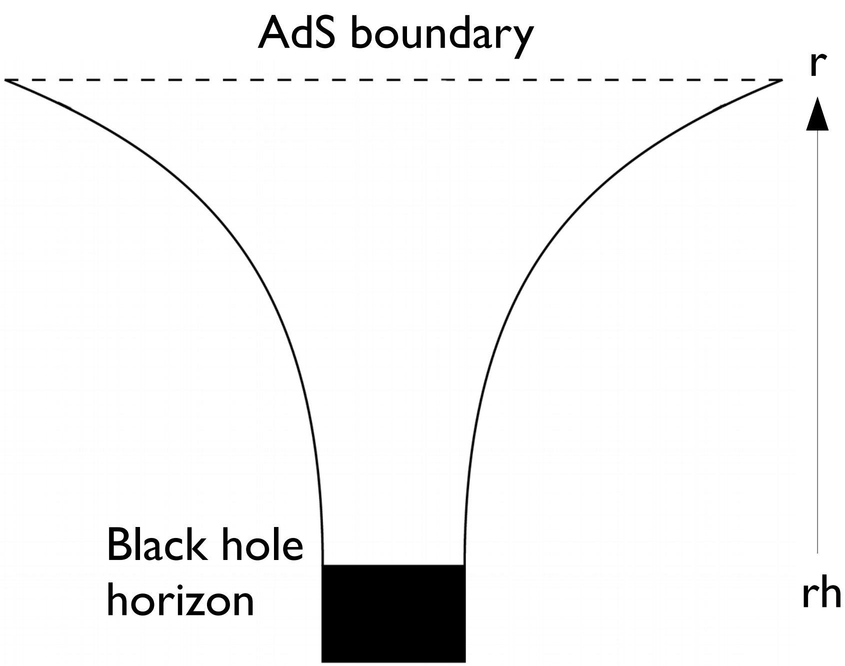
$$f(r) = L_t^{-2} r^{\frac{2z-\theta}{z-\theta}} \qquad h(r) = L_x^{-2} r^{\frac{2-\theta}{z-\theta}}$$

$$\phi(r) = \phi_1 \log r \qquad A_t(r) = A_0 r^{a_1}$$

- These are the IR fixed points.
- They are characterised by the dynamical critical exponent $z$ and the hyperscaling violating exponent $\theta$
- And also by the two length scales $L_t$ and $L_x$

# Small temperature solutions

- At small temperatures, there is a black hole in the IR.




$$f(r) = r^2 + \dots \qquad h(r) = r^2 + \dots$$

complicated intermediate metric

$$f(r) = L_t^{-2} r^{\frac{2z-\theta}{z-\theta}} \left(1 - \left(\frac{r_h}{r}\right)^{\frac{2+z-\theta}{z-\theta}}\right) + \dots$$

$$h(r) = L_x^{-2} r^{\frac{2-\theta}{z-\theta}} + \dots$$

- The IR exponents control scaling in T of some observables e.g.

$$s = 4\pi h(r_h) = 4\pi \left(\frac{4\pi(z-\theta)}{2+z-\theta}\right)^{\frac{2-\theta}{z}} \frac{L_t^{\frac{4-2\theta}{z}}}{L_x^2} T^{\frac{2-\theta}{z}}$$

- The prefactor depends in a complicated way on UV details.

# Chaos in holographic systems

- The chaos parameters are calculated by studying perturbations of the two-sided black hole.

- A small amount of energy injected at early time $t_w$ from one boundary produces a gravitational shockwave solution

$$ds^2 = A(uv)dudv + h(uv)d\vec{x}^2 - A(uv)\delta(u)g(x)du^2 \qquad g(x) \sim \frac{1}{\sqrt{x}} e^{2\pi T(t_w - v_B/x)}$$

- This gives exponential growth in the correlator

$$C(x,t) = -\langle [W(x,t), V(0,0)]^2 \rangle_T$$

- The Lyapunov time and butterfly velocity are

$$v_B^2 = \frac{2\pi T}{h'(r_h)} \qquad \tau_L = \frac{1}{2\pi T}$$

- They depend on the metric near the horizon.

Roberts, Shenker & Stanford
Blake
Roberts & Swingle

# Holographic transport

- All of the conductivities depend only on the action and solution near the horizon  Blake & Tong ; Donos & Gauntlett

$$\sigma = Z(\phi(r_h)) + \frac{4\pi\rho^2}{Y(\phi(r_h))sm^2}, \qquad \alpha = \frac{4\pi\rho}{Y(\phi(r_h))m^2}, \qquad \bar{\kappa} = \frac{4\pi sT}{Y(\phi(r_h))m^2}$$

- The thermal conductivity we are interested in is

$$\kappa \equiv \bar{\kappa} - \frac{T\alpha^2}{\sigma} = \frac{4\pi sTZ(\phi(r_h))h(r_h)}{\rho^2 + m^2Y(\phi(r_h))Z(\phi(r_h))h(r_h)}$$

- In contrast, the elements of the susceptibility matrix typically depend on the details of the entire bulk solution.

- One exception is the heat capacity at low temperatures, due to the Bekenstein-Hawking entropy formula:

$$c_\rho = T\frac{\partial s}{\partial T} = \frac{2-\theta}{z}s = \frac{2-\theta}{z}4\pi h(r_h)$$

# Thermal diffusivity and chaos

- So thermal diffusivity and chaos parameters can be determined from the solution near the horizon (the IR fixed point)

$$\kappa = \frac{4\pi s T Z(\phi(r_h)) h(r_h)}{\rho^2 + m^2 Y(\phi(r_h)) Z(\phi(r_h)) h(r_h)} \qquad c_\rho = \frac{2-\theta}{z} 4\pi h(r_h) \qquad v_B^2 \tau_L = \frac{1}{h'(r_h)}$$

- Therefore they are power laws in temperature, with a prefactor that depends on microscopic details e.g.

$$D_T = \frac{z(z-\theta)}{(2-\theta)(2z-2)} \left( \frac{4\pi(z-\theta)}{2+z-\theta} \right)^{\frac{z-2}{z}} L_t^{\frac{2z-4}{z}} L_x^2 T^{1-\frac{2}{z}}$$

- But remarkably: $$D_T = \frac{z}{2(z-1)} v_B^2 \tau_L$$

- All dependence on the length scales is gone!

- The prefactor depends only on the IR fixed point exponent z.

# Why does this work?

- Why does all dependence on the IR length scales disappear ?

- The thermal conductivity appears to depend on the matter fields and couplings in a complicated way

$$\kappa = \frac{4\pi s T Z(\phi(r_h))h(r_h)}{\rho^2 + m^2 Y(\phi(r_h))Z(\phi(r_h))h(r_h)}$$

- One of the Einstein equations is

$$h(r)f''(r) = m^2 Y(\phi(r)) + h(r)Z(\phi(r))A_t'(r)^2 + f(r)h''(r)$$

- By evaluating this on the horizon, find that all matter dependence in $\kappa$ can be replaced by geometry:

$$\kappa = 4\pi \frac{f'(r_h)}{f''(r_h)}$$

- This is not true of any other of the conductivities.

# Cancellation of length scales

- So in general we have

$$\kappa = 4\pi \frac{f'(r_h)}{f''(r_h)} \qquad c_\rho = \frac{2-\theta}{z} 4\pi h(r_h) \qquad v_B^2 \tau_L = \frac{1}{h'(r_h)}$$

- And therefore

$$D_T = \frac{\kappa}{c_\rho} = \frac{z}{2-\theta} \frac{f'(r_h)h'(r_h)}{f''(r_h)h(r_h)} \times v_B^2 \tau_L$$

- The metric near the IR fixed point is $f(r) \sim L_t^{-2} r^{\#}$ $h(r) \sim L_x^{-2} r^{\#}$

- And so it is manifest that all length scales will drop out so that

$$D_T \sim v_B^2 \tau_L$$

- This is a consequence of the fact that $\kappa$ , $c_\rho$ and $v_B^2 \tau_L$ all depend only on the metric near the horizon.

- They are otherwise insensitive to the matter fields & couplings.

# Special cases

- There are two special cases: $z = \infty$ and $z = 1$

- For $z = \infty$ ( $\mathrm{AdS}_2 \times R^2$ metric ), $h(r) = \text{constant}$

  $c_\rho$ of the IR fixed point vanishes. So it is determined by the leading irrelevant deformation of the fixed point.

- One still finds $D_T = C v_B^2 \tau_L$

  Blake & Donos
  Baggioli, Gouteraux, Kiritsis, Li

- For $z = 1$, $f''(r_h) = 0$ for the fixed point since $\kappa = 4\pi \frac{f'(r_h)}{f''(r_h)}$
  The leading deformation around the fixed point determines $\kappa$

- In this case, $D_T \gg v_B^2 \tau_L$

  RD, Gentle, Gouteraux

- The IR fixed point is neutral and translationally invariant.

# Generalisations

- The relation for $\kappa$ is robust to various generalizations.
- It generalizes to any dimension d $\quad \kappa = 4\pi \frac{f' h^{d-2}}{(f' h^{d/2-1})'}\Big|_{r_h}$
- It works in the presence of an external magnetic field B, where

$$\kappa_L \equiv \bar{\kappa}_L - \frac{T\alpha_L^2}{\sigma_L} = 4\pi \frac{f'(r_h)}{f''(r_h)}$$

- It applies when the Maxwell action is replaced by a DBI action.
- There is an analogue for anisotropic black hole solutions.
- So in all these cases, the relation $D_T \sim v_B^2 \tau_L$ will still hold.

# Conclusions

- For a wide class of holographic examples, the thermal diffusivity is related to the chaos parameters

$$D_T = \frac{z}{2(z-1)} v_B^2 \tau_L$$

- $D_T$ and $v_B^2 \tau_L$ depend only on the metric near the horizon.

- Two important generalisations to investigate more carefully are
  1). inhomogeneous solutions
  2). higher derivative theories of gravity

- Similar relations also been observed in various non-holographic systems.

- We would like to better understand why.

# Extra slides

# Translationally invariant theories

- Translational symmetry breaking is not necessary for the results

- When m=0, the elements of the conductivity matrix are infinite. But $\kappa$ is finite.

- At low temperatures, $D_T \equiv \frac{\kappa}{c_\rho} = \frac{z}{2(z-1)} v_B^2 \tau_L$

- Holographic theories with m=0 are described by relativistic hydrodynamics.

- Relativistic hydro has a diffusive excitation with diffusivity

$$D = \sigma_Q \left[ \chi_{\rho\rho} - \frac{2\rho}{\epsilon + P} \chi_{\varepsilon\rho} + \frac{\rho^2}{(\epsilon + P)^2} \chi_{\varepsilon\varepsilon} \right]^{-1}$$

- At low temperatures $D \to D_T$

RD, Gentle, Gouteraux

# Chaos in holography

- The chaos commutator can be written as

$$C(x,t) = -\langle [W(x,t), V(0,0)]^2 \rangle_T = \langle \mathrm{TFD} |\, [W(x,t), V(0,0)]^2 | \mathrm{TFD} \rangle$$

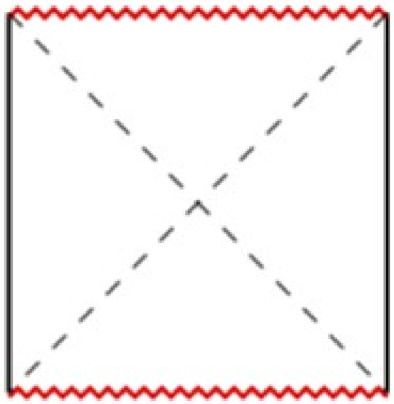
$$|\mathrm{TFD}\rangle = Z^{-1/2} \sum_n e^{-\beta E_n/2} |n\rangle_L |n\rangle_R$$

- Thermofield double state is the two-sided black hole

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + h(r)d\vec{x}^2$$

$$uv = -e^{f'(r_h)r_*(r)} \qquad u/v = -e^{-f'(r_h)t} \qquad r_* = \int_\infty^r \frac{dr}{f(r)}$$

$$ds^2 = A(uv)dudv + h(uv)d\vec{x}^2$$



- A small perturbation at an early time $t_w$ can backreact a lot

$$\delta T_{uu} \sim E e^{2\pi T t_w} \delta(u)\delta(x) \qquad t_s \sim T^{-1} \log N^2$$

Shenker & Stanford