Microscopic model of quantum butterfly effect: out-of-time-correlators (OTOC) and traveling combustion waves

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THE BUTTERFLY EFFECT

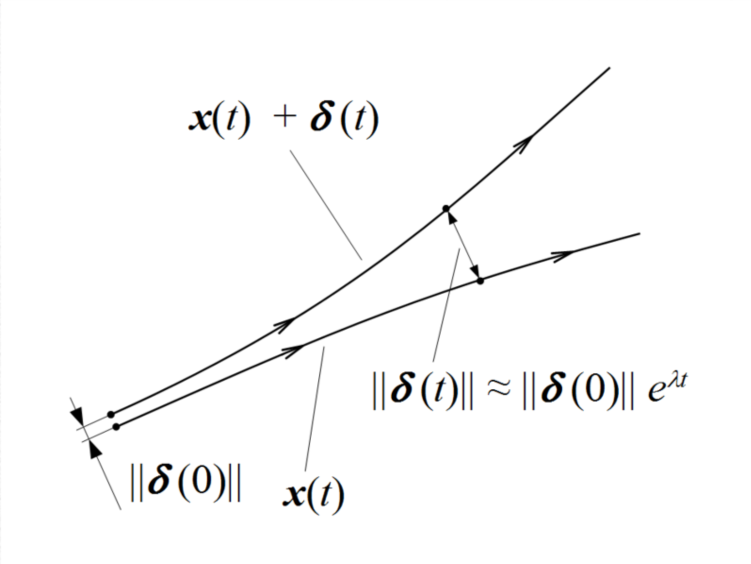
A sound of thunder (1952)

Ray Bradbury

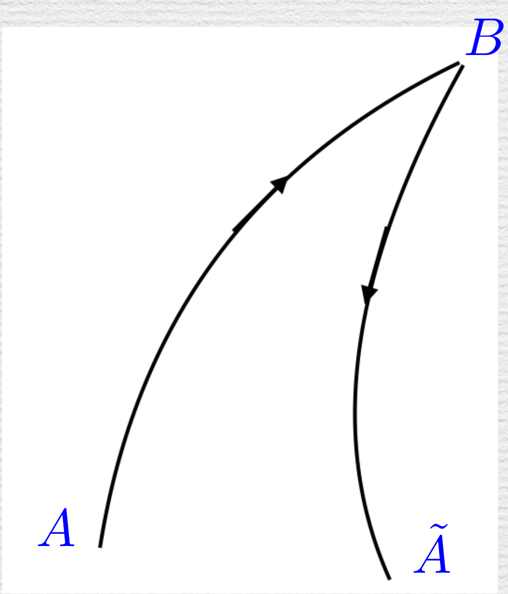
Tiny perturbation in the past has dramatic consequences at present in chaotic systems for local observables

THE BUTTERFLY EFFECT: classical vs. quantum physics

In a chaotic classical system a small perturbation leads to the exponential divergence of trajectories characterized by $1/\lambda$ Lyapanov time



Generalization to a closed chaotic quantum system via Loschmidt echo



go -> small kick -> back

$$\tilde{A} = \{e^{i\frac{t}{\hbar}H} e^{i\delta B} e^{-i\frac{t}{\hbar}H}\}^\dagger A \{e^{i\frac{t}{\hbar}H} e^{i\delta B} e^{-i\frac{t}{\hbar}H}\}$$

$$F_1 \propto \mathrm{Tr}\left[A^2\right] - \mathrm{Tr}\left[A\tilde{A}\right] \propto e^{\lambda t}$$

If $\delta = 0 \rightarrow F_1 = 0$

developing in powers of δ (linear response)

$$F_1 = -\frac{\delta^2}{2\hbar^2}\mathrm{Tr}\{[B(t), A]^2\} \propto -\mathrm{Tr}\{\left[e^{i\frac{t}{\hbar}H} B e^{-i\frac{t}{\hbar}H}, A\right]^2\}$$

DIAGNOSTIC OF QUANTUM CHAOS

D. Roberts and D. Stanford, PRL (2015)

Pair of rather general Hermitian operators W, V in a quantum mechanical system. System is chaotic.

$$-\langle [V, W(t)]^2 \rangle_\beta = \langle VW(t)W(t)V \rangle_\beta + \langle W(t)VVW(t) \rangle_\beta$$

$$-\langle VW(t)VW(t) \rangle_\beta - \langle W(t)VW(t)V \rangle_\beta$$

$$\langle \bullet \rangle_\beta \equiv Z^{-1} \mathrm{tr}\{e^{-\beta H} \bullet\} \quad Z^{-1} = \mathrm{tr} e^{-\beta H}$$

For large times:

approaches $\langle WW \rangle_\beta \langle VV \rangle_\beta$

These correlation functions will become small

overlap

$|W(t)V(0)\rangle \quad |V(0)W(t)\rangle$

$$-\langle [\mathbf{V}, \mathbf{W}(\mathbf{t})]^{\mathbf{2}} \rangle_\beta \Rightarrow$$ **becomes large**

quantum butterfly effect

FIRST APPEARANCE OF OTOC

SOVIET PHYSICS JETP VOLUME 28, NUMBER 6 JUNE, 1969

QUASICLASSICAL METHOD IN THE THEORY OF SUPERCONDUCTIVITY

A. I. LARKIN and Yu. N. OVCHINNIKOV

Institute of Theoretical Physics, USSR Academy of Sciences

Submitted June 6, 1968

Zh. Eksp. Teor. Fiz. 55, 2262–2272 (December, 1968)

It is shown that replacement of quantum-mechanical averages by the average values of the corresponding classical quantities over all trajectories with a prescribed energy is not valid in the general case. The dependence of the penetration depth on the field is found without making any assumptions about the weakness of the interaction between the electrons and the field of the impurities; the case of very dirty films is also considered.

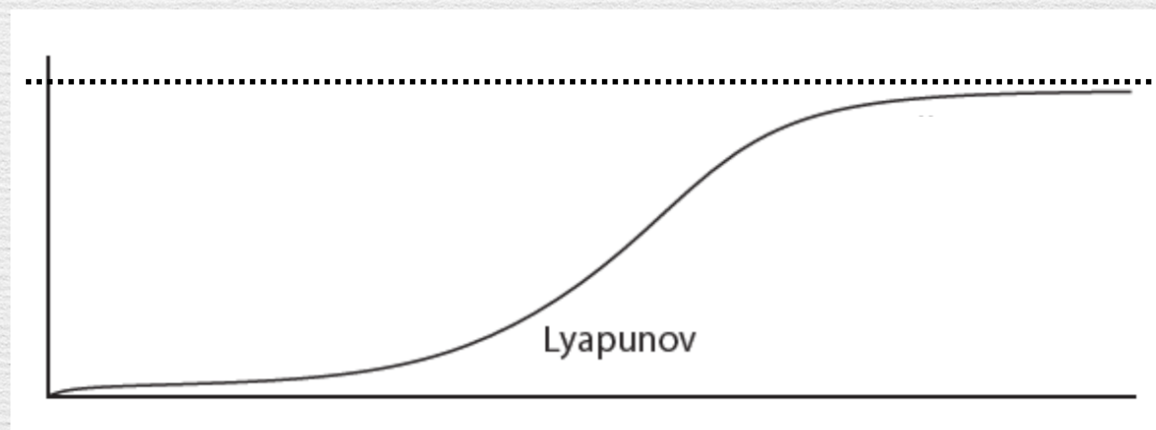
Used $\langle p(t)p(0)p(t)p(0)\rangle$ to get non linear response to electromagnetic field

In the semiclassical approximation
[single free particle in a smooth potential]

$$-\langle [p_z(t), p_z(0)]^2\rangle = \hbar^2 \langle \left(\frac{\partial p_z(t)}{\partial z(0)}\right)^2 \rangle$$

if the motion is chaotic $\frac{\partial p_z(t)}{\partial z(0)} \sim e^{\lambda t}$

λ Lyupanov exponent



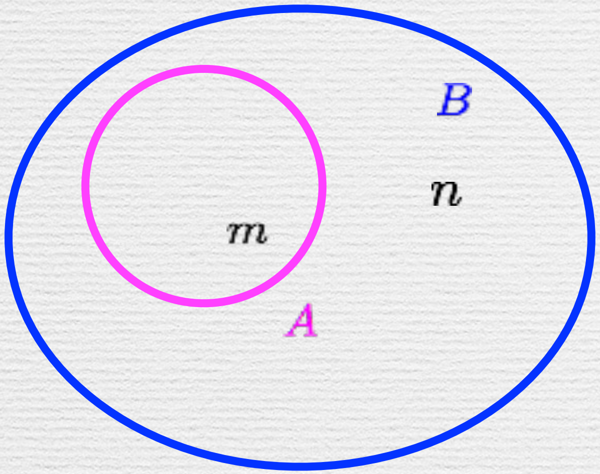
revived by Kitaev (2014)

OTOC AND SCRAMBLING

Scrambling

Page, gr-qc/9305007

and it is usually considered a property of a quantum state



Consider a random pure state of the system $|\psi\rangle$

$$\rho = |\psi\rangle\langle\psi| \qquad \rho_A = \mathrm{tr}_B \rho$$

Entropy of the subsystem A $\quad S_A = -\rho_A \ln \rho_A$

$$S_A^{\max} = \ln m \quad \text{when} \quad \rho_A = \frac{I_A}{|A|}$$

Examples:

$|\psi\rangle = |0\rangle_A |0\rangle_B \quad \text{then} \quad \rho_A = |0\rangle_{AA}\langle 0|$ No scrambled state

$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right] \quad \text{then} \quad \rho_A = \frac{I_A}{|A|}$ Scrambled state

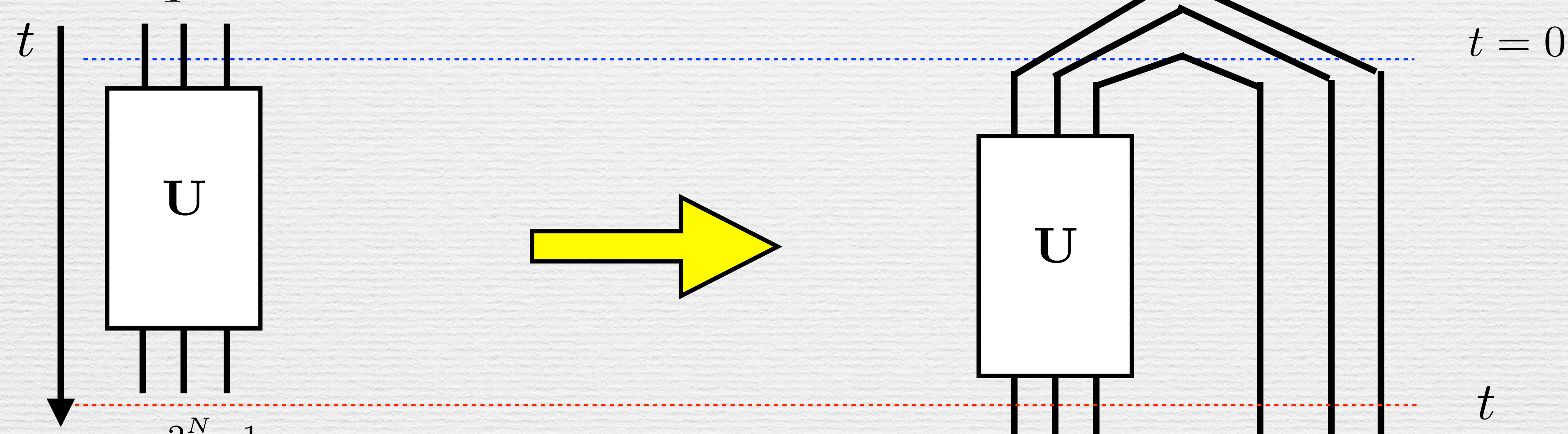
what has to do with $\langle W(t)V(0)W(t)V(0)\rangle$??

TRIPARTITE INFORMATION
(as a measure of scrambling)

Hosur et al. ,1511.04021

$$|I\rangle = \frac{1}{2^{N/2}} \sum_{i,j=0}^{2^N-1} |i\rangle_{in} \otimes |i\rangle_{out}$$

N qubits



$$U(t) = \sum_{i,j=0}^{2^N-1} u_{ij} |i\rangle\langle j|$$

$2^N \times 2^N$ matrix

$$|U(t)\rangle = \frac{1}{2^{N/2}} \sum_{i,j=0}^{2^N-1} |i\rangle_{in} \otimes U(t)|i\rangle_{out}$$

state

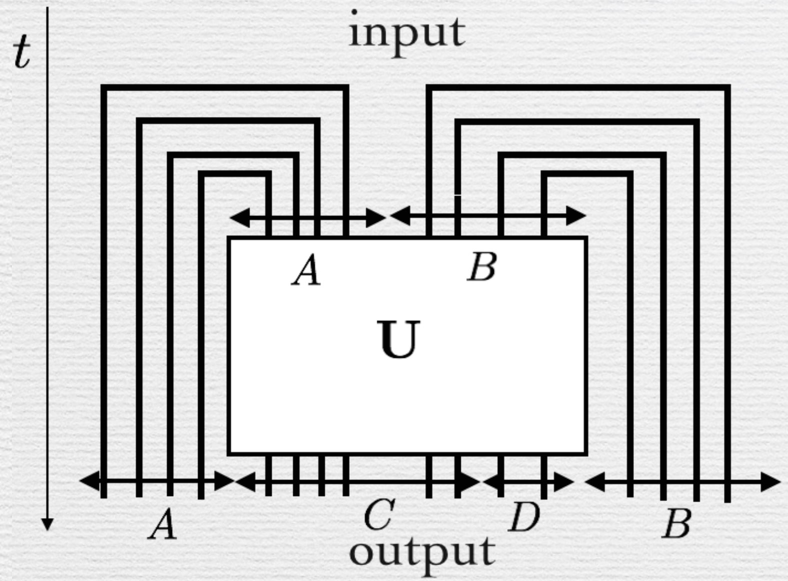
Generalized to initial statistical ensemble

$$\rho_{in} = \sum_j p_j |\psi_j\rangle\langle\psi_j| \longrightarrow |\Psi\rangle = I \otimes U(t) \sum_j \sqrt{p_j} |\psi_j\rangle_{in} \otimes |\psi_j\rangle_{out}$$

Ex:

$$|TDF(t)\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\beta E_i/2} e^{-iE_i t} |i\rangle \otimes |i\rangle \qquad \rho_{in} = \rho_T = \frac{1}{Z} e^{-\beta H}$$

Hosur et al. ,1511.04021



$$|U(t)\rangle = \frac{1}{2^{N/2}} \sum_{i,j=0}^{2^N-1} |i\rangle_{in} \otimes U(t)|i\rangle_{out}$$

$$\rho = |U(t)\rangle\langle U(t)|$$

Entanglement entropy $S_{AC} = -\mathrm{Tr}\rho_{AC} \ln \rho_{AC}$ $\rho_{AC} = \mathrm{Tr}_{BD}\rho$

Mutual information $I(A:C) = S_A + S_C - S_{AC}$

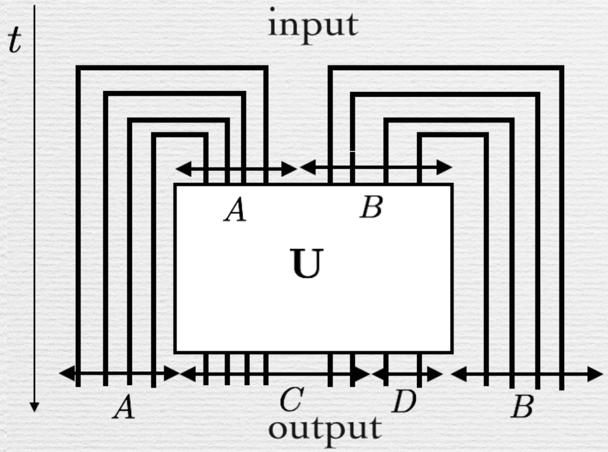
$I(A:CD) - I(A:C) - I(A:D)$ **is a natural measure of scrambling**

Tripartite information

$$I_3 = -\left[I(A:CD) - I(A:C) - I(A:D)\right]$$

Have a large negative value for systems that scramble

BUTTERFLY EFFECT IMPLIES SCRAMBLING



Consider a complete basis of Hermitian operators

$\{A_j\}$ $\{D_i\}$

$\mathrm{tr}\{A_i A_j\} = 2^a \delta_{ij}$ $\mathrm{tr}\{D_i D_j\} = 2^d \delta_{ij}$

Define $|\langle \mathcal{O}_D(t)\mathcal{O}_A\mathcal{O}_D(t)\mathcal{O}_A\rangle_\beta| = \frac{1}{4^{a+d}}\sum_{ij}\langle D_i(t)A_jD_i(t)A_j\rangle_\beta$

If $|\langle \mathcal{O}_D(t)\mathcal{O}_A\mathcal{O}_D(t)\mathcal{O}_A\rangle_{\beta=0}| = \epsilon \quad \Rightarrow -\langle[\mathcal{O}_A, \mathcal{O}_D(t)]^2\rangle_{\beta=0}$ large

$$I_3 - I_{3,min} \leq \log_2 \frac{\epsilon}{\epsilon_{min}}$$

The butterfly effect $(\epsilon \ll 1)$ implies scrambling $I_3 \to I_{3,min}$

decay of OTOC gives information on scrambling time

THIS WORK

- To develop the analytic tools to study

$$\langle W(t)V(0)W(t)V(0)\rangle$$

choose as a quantum chaotic system in condensed matter physics where there is a microscopic theory and we can do calculations

main ingredient:

1. electrons interacting with localized bosonic degrees of freedom
2. electrons in disorder potential
3. electrons weakly interacting with each others

KELDYSH AND KINETIC EQS

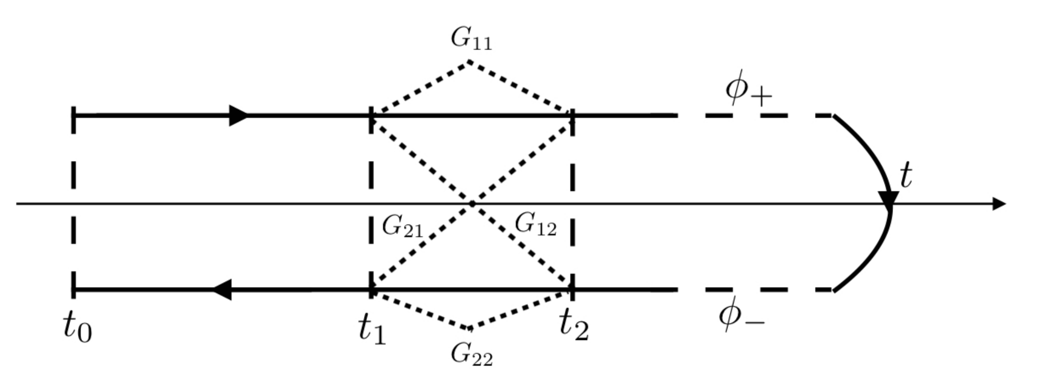
equilibrium

$$H = H_0 + H_i \qquad \rho(H) = \frac{e^{-\beta H}}{\mathrm{Tr}e^{-\beta H}}$$

out of equilibrium

$$\mathcal{H}(t) = H + H'(t) \qquad H'(t) = 0 \quad t > t_0$$

$$\langle O_{\mathcal{H}}(t)\rangle = \mathrm{Tr}[\rho(H) O_{\mathcal{H}}(t)]$$



correlation functions:

$$G^<(1,1') = \mp i \langle \psi^\dagger_{\mathscr{H}}(1')\psi_{\mathscr{H}}(1)\rangle ,$$

$$G^>(1,1') = -i \langle \psi_{\mathscr{H}}(1)\psi^\dagger_{\mathscr{H}}(1')\rangle ,$$

$$G_{c_K}(1,1') \mapsto \hat{G} \equiv \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{bmatrix}$$

$$\hat{G}_{11}(1,1') = -i\langle T(\psi_{\mathscr{H}}(1)\psi^\dagger_{\mathscr{H}}(1'))\rangle ,$$

$$\hat{G}_{12}(1,1') = G^<(1,1') ,$$

$$\hat{G}_{21}(1,1') = G^>(1,1') ,$$

$$\hat{G}_{22}(1,1') = -i\langle \widetilde{T}(\psi_{\mathscr{H}}(1)\psi^\dagger_{\mathscr{H}}(1'))\rangle .$$

contour-ordered Green function:

$$G(1,1') = -i\langle T_c(\psi_{\mathscr{H}}(1)\psi^\dagger_{\mathscr{H}}(1'))\rangle ,$$

$$T_c(\psi_{\mathscr{H}}(1)\psi^\dagger_{\mathscr{H}}(1')) \equiv \begin{cases} \psi_{\mathscr{H}}(1)\psi^\dagger_{\mathscr{H}}(1') & t_1 >_c t_{1'} , \\ \pm\psi^\dagger_{\mathscr{H}}(1')\psi_{\mathscr{H}}(1) & t_1 <_c t_{1'} \end{cases}$$

$$G(1,1') = \begin{cases} G^>(1,1') & t_1 >_c t_{1'} , \\ G^<(1,1') & t_1 <_c t_{1'} . \end{cases}$$

$$O_{\mathscr{H}}(t) = T_{c_t}\left[\exp\left[-i\int_{c_t} d\tau\, H'_H(\tau)\right] O_H(t)\right]$$

$$N_{\alpha\beta\gamma\delta}(t) = \left\langle \mathcal{T}_{\mathcal{C}_K} \hat{O}_\alpha(t_2)\hat{O}_\beta(t_1)\hat{O}_\gamma(t_2)\hat{O}_\delta(t_1) \exp\left(-i\int_{\mathcal{C}_K} \hat{H}_{int}(t)dt\right)\right\rangle$$

Larkin rotation

$$\check{G} \equiv \tau^3 \hat{G} \ ,$$

$$L = \frac{1}{\sqrt{2}}(\tau^0 - i\tau^2) \ .$$

$$\underline{G} \equiv L \check{G} L^\dagger \ ,$$

$$G^R(1,1') = \hat{G}_{11}(1,1') - \hat{G}_{12}(1,1') = \hat{G}_{21}(1,1') - \hat{G}_{22}(1,1') \ ,$$

$$G^A(1,1') = \hat{G}_{11}(1,1') - \hat{G}_{21}(1,1') = \hat{G}_{12}(1,1') - \hat{G}_{22}(1,1') \ ,$$

$$G^K(1,1') = \hat{G}_{21}(1,1') + \hat{G}_{12}(1,1') = \hat{G}_{11}(1,1') + \hat{G}_{22}(1,1') \ ,$$

$$\underline{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix} .$$

$$\underline{\Sigma} = \begin{pmatrix} \Sigma^R & \Sigma^K \\ 0 & \Sigma^A \end{pmatrix}$$

Dyson Equations

$$(\underline{G}_0^{-1} - \underline{\Sigma}) \otimes \underline{G} = \delta(1-1') \ ,$$

$$(A \otimes B)(1,1') = \int d\mathbf{x}_2 \int_{-\infty}^{\infty} dt_2 A(1,2) B(2,1') \ ,$$

$$\underline{G} \otimes (\underline{G}_0^{-1} - \underline{\Sigma}) = \delta(1-1') \ .$$

$$\underline{G}_0^{-1}(1,1') = [i\partial_{t_1} - \varepsilon(1)]\delta(1-1') \ ,$$

Off-Diagonal terms: contain information on the population of these states

Gradient Approximation

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{x}_1+\mathbf{x}_{1'}),\quad T=\tfrac{1}{2}(t_1+t_{1'}) ,$$

$$\mathbf{r}=\mathbf{x}_1-\mathbf{x}_{1'},\quad t=t_1-t_{1'} ,$$

$$\underline{G}(X,p)=\int dx\, e^{-ipx}\underline{G}(X+x/2,X-x/2) .$$

$$(A\otimes B)(X,p)=e^{i(\partial_X^A\partial_p^B-\partial_p^A\partial_X^B)/2}A(X,p)B(X,p) .$$

$$= \mathbf{A}(\mathbf{X},\mathbf{p})\mathbf{B}(\mathbf{X},\mathbf{p}) + \frac{\mathbf{i}}{\mathbf{2}}\{\mathbf{A},\mathbf{B}\}$$

$$[\underline{G}_0^{-1}-\underline{\Sigma}\otimes\underline{G}]_-=0 ,$$

Keldysh component:

$$i\{\omega-\xi_{\mathbf{p}},G^K\}=-G^K\left(\Sigma^A-\Sigma^R\right)+\Sigma^K\left(G^A-G^R\right)$$

FDT at equilibrium $\quad G^K(\omega,\mathbf{p})=f_0(\omega)\left(G^R(\omega,\mathbf{p})-G^A(\omega,\mathbf{p})\right),\quad f_0(\omega)=\tanh\frac{\omega}{2kT}$

quasiparticle approximation $\quad G^R(p,x)-G^A(p,x)=2\pi i\delta(\omega-\xi_{\mathbf{p}})$

$$G^K(\omega,\mathbf{p})=2\pi i\,(2n_{\mathbf{p}}-1)\,\delta(\omega-\xi_{\mathbf{p}})\qquad n_{\mathbf{p}}=[1+e^{\beta\xi_{\mathbf{p}}}]^{-1}$$

out of equilibrium search for solution of the form $\quad G^K(p,x)=f(p,x)\left(G^R(p,x)-G^A(p,x)\right)$

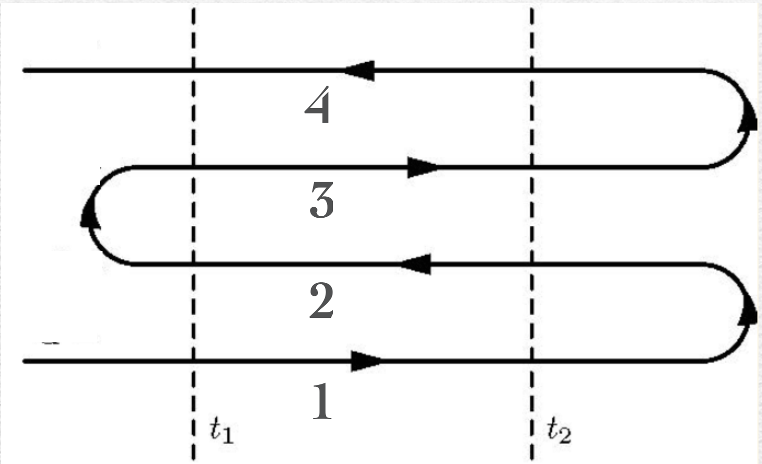
$$\left(\frac{\partial}{\partial t}+\nabla_{\mathbf{p}}\xi_{\mathbf{p}}\cdot\nabla_{\mathbf{x}}\right)f(p,x)=I[f(p,x)]$$

$$I[f]=i\left[\Sigma^K+f(\Sigma^A-\Sigma^R)\right]$$

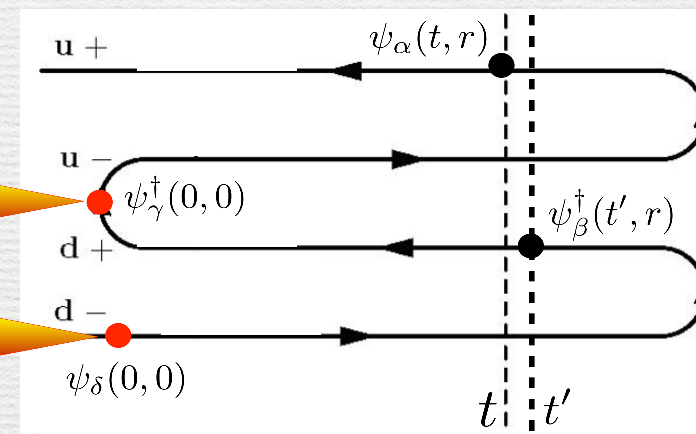
collision integral

AUGUMENTED KELDYSH

$$\langle B(t)A(0)B(t)A(0)\rangle = \langle e^{iHt}Be^{-iHt}Ae^{iHt}Be^{-iHt}A\rangle$$

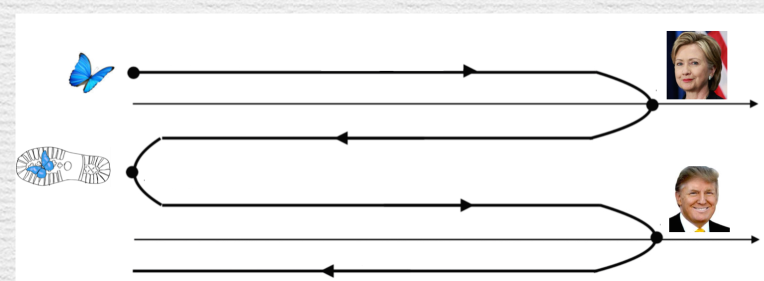


$$\hat{\mathcal{G}} = \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} & \hat{G}_{12} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} & \hat{G}_{12} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{21} & \hat{G}_{33} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{21} & \hat{G}_{21} & \hat{G}_{44} \end{bmatrix}$$



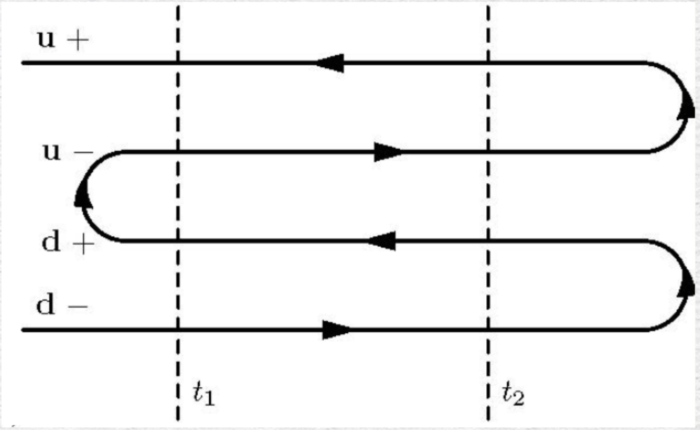
$$\mathcal{A}^{\gamma\delta}{}_{\alpha\beta}(t,r;t'r') = \left\langle T_C \left(\psi_\alpha(t,r)\psi^\dagger_\beta(t',r)\right)\left(\psi^\dagger_\gamma(0,0)\psi_\delta(0,0)\right)\right\rangle$$

$$\mathcal{A}^{\gamma\delta}_{\alpha\beta}(t,r:t',r') \sim G_{\alpha\beta}(t,r;t'r')$$



GREEN FUNCTIONS in UP/DOWN WORLDS

$$\hat{\mathcal{G}} = \begin{pmatrix} \hat{G}_{uu} & \hat{G}_{ud} \\ \hat{G}_{du} & \hat{G}_{dd} \end{pmatrix}_a$$



$$\hat{G}_{uu} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}_K \qquad \hat{G}_{ud} = \begin{pmatrix} 0 & \Gamma^K \\ 0 & 0 \end{pmatrix}_K$$

$$\hat{G}_{du} = \begin{pmatrix} 0 & \bar{\Gamma}^K \\ 0 & 0 \end{pmatrix}_K \qquad \hat{G}_{dd} = \begin{pmatrix} \tilde{G}^R & \tilde{G}^K \\ 0 & \tilde{G}^A \end{pmatrix}_K$$

Green Functions:

$$iG^R(1{,}2) = \left\langle \psi(1)\psi^\dagger(2) \pm \psi^\dagger(2)\psi(1) \right\rangle \theta(t_1 - t_2),$$
$$iG^A(1{,}2) = -\left\langle \psi(1)\psi^\dagger(2) \pm \psi^\dagger(2)\psi(1) \right\rangle \theta(t_2 - t_1),$$
$$iG^K(1{,}2) = \left\langle \psi(1)\psi^\dagger(2) \mp \psi^\dagger(2)\psi(1) \right\rangle$$

Inter-worlds Green Functions:

$$\Gamma^K(1,2) = -2i\left\langle \psi(1)\psi^\dagger(2) \right\rangle = G^K(1,2) + [G^R(1,2) - G^A(1,2)],$$
$$\bar{\Gamma}^K(1,2) = \pm 2i\left\langle \psi^\dagger(2)\psi(1) \right\rangle = G^K(1,2) - [G^R(1,2) - G^A(1,2)].$$

Meaning: distribution function of electrons (or holes) x times coherence between the Worlds

DYSON EQUATIONS

Dyson equations in the formalism 4x4

$$(\hat{H}_0 \tau_0^a - \hat{\Sigma}) \circ \hat{G} = \hat{1}, \qquad \hat{H}_0 G_0^R = 1 \qquad \hat{H}_0 G_0^A = 1$$

$$\hat{G} \circ (\hat{H}_0 \tau_0^a - \hat{\Sigma}) = \hat{1}, \qquad \hat{H}_0 = id/dt - H_0$$

Diagonal part of the Green function:

$$(\hat{H}_0 - \Sigma_\alpha^{R/A}) \circ G_\alpha^{R/A} = 1 \qquad \Sigma_{\alpha\beta}^{A,R} = \delta_{\alpha\beta} \Sigma_\alpha^{A,R}$$

$$G_\alpha^{R/A} \circ (\hat{H}_0 - \Sigma_\alpha^{R/A}) = 1, \ \alpha = u, d.$$

Non-Diagonal part of the Green function:

$$(\hat{H}_0 - \Sigma_\alpha^R) \circ G_{\alpha\beta}^K - \Sigma_{\alpha\beta}^K \circ G_\beta^A = 0,$$

$$G_{\alpha\beta}^K \circ (\hat{H}_0 - \Sigma_\beta^A) - G_\alpha^R \circ \Sigma_{\alpha\beta}^K = 0.$$

The Keldysh component of the Green function can be conveniently parametrized by:

$$G_{\alpha\beta}^K = G_\alpha^R \circ \mathcal{F}_{\alpha\beta} - \mathcal{F}_{\alpha\beta} \circ G_\beta^A.$$

→ $\alpha = \beta$ $\quad \mathcal{F}_{uu}$ and $\mathcal{F}_{dd}$

→ $\alpha \neq \beta$ $\quad \mathcal{F}_{ud}$ and $\mathcal{F}_{du}$

STABILITY AND INSTABILITY

quantum kinetic equation

$$H_0 \circ \mathcal{F}_{\alpha\beta} - \mathcal{F}_{\alpha\beta} \circ H_0 = \left[\Sigma_\alpha^R \circ \mathcal{F}_{\alpha\beta} - \mathcal{F}_{\alpha\beta} \circ \Sigma_\alpha^A\right] - \Sigma_{\alpha\beta}^K$$

quasiclassical approximation

outgoing scattering processes → dissipation

incoming scattering processes → instability

<u>Diagonal:</u>

$$\Sigma_{\alpha\alpha}^K(\epsilon) = \left[\Sigma_\alpha^R(\epsilon) - \Sigma_\alpha^A(\epsilon)\right] n_0(\epsilon), \quad n_0(\epsilon) = \tanh\left(\frac{\epsilon - \mu}{2T}\right)$$

$$\mathcal{F}_{uu} = \mathcal{F}_{dd} = n_0(\epsilon)$$

$\left[1 - \mathcal{F}_{uu}(\epsilon)\right]/2$ Fermi distribution function

<u>Off-diagonal:</u>

$\mathcal{F}_{ud} = 1 + n_0, \ \mathcal{F}_{du} = -1 + n_0$ correlated worlds solution (unstable)

$\mathcal{F}_{ud} = \mathcal{F}_{ud} = 0$ uncorrelated worlds solution (stable)

MODELS

In all these models the main ingredient are mobile electrons that form a Fermi sea. They are described by the quadratic Hamiltonian

$$H_{el} = \sum_p \xi_p \psi_p^\dagger \psi_p$$

1. Electrons interact with dispersionless phonons with frequency ω_0 (Einstein phonons) with Hamiltonians:

$$H_{ph} = \sum_r \hbar\omega_0 b_r^\dagger b_r \qquad H_{el-ph} = \sum_r u_r \psi^\dagger(r)\psi(r) \qquad u_r = \lambda(b_r + b_r^\dagger)$$

2. Electrons

$$H_{el-imp} = \sum_r U(r)\psi_r^\dagger \psi_r$$

3. Electron

$$H_{el-el} = \frac{1}{2}\int \psi_r^\dagger \psi_{r'}^\dagger \psi_{r'} \psi_r U(r-r') dr dr'$$

KINETIC EQUATIONS (semiclassical approximation)

$$H_0 \circ \mathcal{F}_{\alpha\beta} - \mathcal{F}_{\alpha\beta} \circ H_0 = \left[\Sigma^R_\alpha \circ \mathcal{F}_{\alpha\beta} - \mathcal{F}_{\alpha\beta} \circ \Sigma^A_\alpha\right] - \Sigma^K_{\alpha\beta}$$

In the semiclassical limit kinetic equation becomes local in time and phase space. This simplification is possible if the rate of the electron scattering is much smaller than T (for e-e or e-ph interactions) or Fermi energy (for electrons in disorder potential)

LHS

$$H_0 \circ \mathcal{F}_{\alpha\beta} - \mathcal{F}_{\alpha\beta} \circ H_0 = i\hbar \left\{ \frac{\partial}{\partial t} + \frac{\partial}{\partial r}\frac{d\xi}{dp} \right\} \mathcal{F}_{\alpha\beta}$$

(electrons)

$$H_0 \circ \mathcal{P}_{\alpha\beta} - \mathcal{P}_{\alpha\beta} \circ H_0 = i\hbar\omega \frac{\partial}{\partial t} \mathcal{P}_{\alpha\beta}(\omega, p; r, t)$$

(phonons)

RHS

$$\left\{ \frac{\partial}{\partial t} + \frac{\partial}{\partial r}\frac{d\xi}{dp} \right\} \mathcal{F}_{\alpha\beta}(\epsilon, p; r, t) = [\mathrm{St}^{\cdots}_{\mathrm{el}}]_{\alpha\beta}$$

collision integrals

$$\omega \frac{\partial}{\partial t} \mathcal{P}_{\alpha\beta}(\omega, p; r, t) = \omega \left[\mathrm{St}^{\cdots}_{\mathrm{ph}}\right]_{\alpha\beta}$$

COMPARE DIAGONAL AND NON DIAGONAL COLLISION INTEGRALS

el-ph

$$\left[\mathrm{St}_{\mathrm{el}}^{\mathrm{ph}}\right]_{\alpha\alpha}=n_{ph}\int\frac{d\omega dP_1 M(P;P_1,\omega)}{(2\pi)(2\pi\hbar)^{(d+1)}}\times\left\{-\left[\mathcal{L}_{el}^{ph}\right]_{\alpha}(P_1,\omega)\mathcal{F}_{\alpha\alpha}(P)+\left[\mathcal{P}_{\alpha\alpha}(\omega)\mathcal{F}_{\alpha\alpha}(P_1)+1\right]\right\}$$

$$\left[\mathrm{St}_{\mathrm{ph}}^{\mathrm{el}}\right]_{\alpha\alpha}=\frac{1}{2}\int\frac{dPdP_1 M(P;P_1,\omega)}{(2\pi\omega)((2\pi\hbar)^{(d+1)}}\times\left\{-\left[\mathcal{L}_{el}^{ph}\right]_{\alpha}(P_1,\omega)\mathcal{P}_{\alpha\alpha}(\omega)+\left[1-\mathcal{F}_{\alpha\alpha}(P)\mathcal{F}_{\alpha\alpha}(P_1)\right]\right\}$$

$$\left[\mathcal{L}_{el}^{ph}\right]_{\alpha}(P_1,\omega)=\mathcal{P}_{\alpha\alpha}(\omega)+\mathcal{F}_{\alpha\alpha}(P_1)$$

$$\left[\mathcal{L}_{ph}^{el}\right]_{\alpha}(P_1,\omega)=\mathcal{F}_{\alpha\alpha}(P)-\mathcal{F}_{\alpha\alpha}(P_1)$$

Except for outgoing terms the off diagonal part is very similar:

$$\left[\mathrm{St}_{\mathrm{el}}^{\mathrm{ph}}\right]_{\alpha\beta}=n_{ph}\int\frac{dP_1 dQ_1 M(P;P_1,\omega)}{(2\pi)(2\pi\hbar)^{(d+1)}}\left\{-\mathcal{L}_{el}^{ph}(P_1,\omega)\mathcal{F}_{\alpha\beta}(P)+\mathcal{P}_{\alpha\beta}(\omega)\mathcal{F}_{\alpha\beta}(P_1)\right\}$$

$$\left[St_{ph}^{el}\right]_{\alpha\beta}=\frac{1}{2}\int\frac{dPdP_1 M(Q;P_1,\omega)}{(2\pi\omega)(2\pi\hbar)^{(d+1)}}\left\{-\mathcal{L}_{ph}^{el}(P_1,\omega)\mathcal{P}_{\alpha\beta}(\omega)-\mathcal{F}_{\alpha\beta}(P)\mathcal{F}_{\beta\alpha}(P_1)\right\}$$

$$2\mathcal{L}_{\cdots}^{\cdots}(P_1,Q_1)\equiv\left[\mathcal{L}_{\cdots}^{\cdots}\right]_{u}(P_1,Q_1)+\left[\mathcal{L}_{\cdots}^{\cdots}\right]_{d}(P_1,Q_1)$$

COMPARE DIAGONAL AND NON DIAGONAL COLLISION INTEGRALS

el-el

$$\left[\mathrm{St}^{\mathrm{el}}_{\mathrm{el}}\right]_{\alpha\alpha} = \int \frac{dP_1 dP_2 dP_3 M(P, P_1; P_2 P_3)}{(2\pi\hbar)^{3(d+1)}} \Big\{ - \left[\mathcal{L}^{el}_{el}\right]_{\alpha} (P_1, P_2, P_3) \mathcal{F}_{\alpha\alpha}(P)$$
$$+ \left[\mathcal{F}_{\alpha\alpha}(P_3) + \mathcal{F}_{\alpha\alpha}(P_2) - \mathcal{F}_{\alpha\alpha}(P_1) - \mathcal{F}_{\alpha\alpha}(P_1)\mathcal{F}_{\alpha\alpha}(P_2)\mathcal{F}_{\alpha\alpha}(P_3)\right]\Big\} ,$$

$$\left[\mathcal{L}^{el}_{el}\right]_{\alpha} = \mathcal{F}_{\alpha\alpha}(P_2)\mathcal{F}_{\alpha\alpha}(P_3) - \mathcal{F}_{\alpha\alpha}(P_1)\mathcal{F}_{\alpha\alpha}(P_3) - \mathcal{F}_{\alpha\alpha}(P_1)\mathcal{F}_{\alpha\alpha}(P_2) + 1.$$

$$M = \frac{\left|U_{p_2-p} - U_{p_3-p}\right|^2}{8\hbar} \left[(2\pi\hbar)^{d+1}\, \delta(P + P_1 - P_2 - P_3)\right] \prod_{i=1}^{3} \left[2\pi\hbar\delta(\epsilon_i - \xi(p_i))\right]$$

For off-diagonal ($\alpha \neq \beta$) we obtain

$$\left[\mathrm{St}^{\mathrm{el}}_{\mathrm{el}}\right]_{\alpha\beta} = \int \frac{dP_1 dP_2 dP_3 M(P, P_1; P_2 P_3)}{(2\pi\hbar)^{3(d+1)}} \times$$
$$\Big\{ -\mathcal{L}^{el}_{el}(P_1, P_2, P_3)\mathcal{F}(P)_{\alpha\beta} + \mathcal{F}_{\alpha\beta}(P_2)\mathcal{F}_{\alpha\beta}(P_3)\mathcal{F}_{\beta\alpha}(P_1)\Big\} ,$$

where once again

$$2\mathcal{L}^{el}_{el}(P_1, P_2, P_3) \equiv \left[\mathcal{L}^{el}_{el}\right]_{u}(P_1, P_2, P_3) + \left[\mathcal{L}^{el}_{el}\right]_{d}(P_1, P_2, P_3).$$

COMPARE DIAGONAL AND NON DIAGONAL COLLISION INTEGRALS

electrons disorder potential

$$[\mathrm{St}^{\mathrm{im}}_{\mathrm{el}}]_{\alpha\beta} = \int \frac{dp_1 M(p,p_1)}{(2\pi\hbar)^d} \{-\mathcal{F}_{\alpha\beta}(p) + \mathcal{F}_{\alpha\beta}(p_1)\},$$

where

$$M(p,p_1) = \frac{2\pi}{\hbar} |V_{p-p_1}|^2 \, \delta(\xi_p - \xi_{p_1}).$$

Exactly the same of the diagonal Keldysh function

$$\frac{\partial}{\partial t} f_{\mathbf{x}}(\mathbf{p},t) = -\int (d\mathbf{p}') w(\mathbf{p},\mathbf{p}') \left(f_{\mathbf{x}}(\mathbf{p},t) - f_{\mathbf{x}}(\mathbf{p}',t) \right)$$

Born approximation: $w(\mathbf{p},\mathbf{p}') = 2\pi |V_{imp}(\mathbf{p}-\mathbf{p}')|^2 \delta(\xi_p - \xi_{p'})$

$$\frac{\partial f}{\partial t} = -\nu \int d\theta_{\mathbf{p}'} |V_{\mathbf{p}-\mathbf{p}'}|^2 (f(\mathbf{p}) - f(\mathbf{p}'))$$

ν density of states

Relaxation time $\tau_{\mathbf{p}}^{-1} = \nu \int (d\theta_{\mathbf{p}'}) |V_{\mathbf{p}-\mathbf{p}'}|^2$

KINETIC EQUATION FOR ELECTRON-PHONON PROBLEM

$$\mathcal{F}_{ud}(\epsilon, p; t) = f(\epsilon, t) \qquad \mathcal{P}_{ud}(\omega_0) = \theta$$
$$\mathcal{F}_{du}(\epsilon, p; t) = -\overline{f}(\epsilon, t) \qquad \mathcal{P}_{du}(\omega_0) = \overline{\theta}.$$

phonon scattering does not depend on electron momentum

$$\tau \frac{\partial f}{\partial t} = -L\left(\frac{\epsilon}{2T}, \frac{\omega_0}{2T}\right) f + \left[\theta f(\epsilon - \omega_0) + \overline{\theta} f(\epsilon + \omega_0)\right]$$

$$\tau \frac{\partial \overline{f}}{\partial t} = -L\left(\frac{\epsilon}{2T}, \frac{\omega_0}{2T}\right) \overline{f} + \left[\overline{\theta}\, \overline{f}(\epsilon - \omega_0) + \theta \overline{f}(\epsilon + \omega_0)\right]$$

$$\frac{1}{\tau} = \frac{2\pi \nu n_{ph} \lambda^2}{\hbar}$$

$$\eta \tau \frac{\partial \theta}{\partial t} = -\theta + I_-$$

$$\eta = \frac{n_{ph}}{\hbar \nu \omega_o}$$

$$\eta \tau \frac{\partial \overline{\theta}}{\partial t} = -\overline{\theta} + I_+$$

$$I_\pm = \frac{1}{2\omega_0} \int d\epsilon f(\epsilon) \overline{f}(\epsilon \pm \omega_0)$$

$$L(x, y) = 2\coth(y) - \tanh(x + y) + \tanh(x - y)$$

LIMIT $\eta \ll 1$

$$\eta = \frac{n_{ph}}{\hbar \nu \omega_o}.$$

$$\frac{1}{\tau} = \frac{2\pi \nu n_{ph} \lambda^2}{\hbar},$$

$$\tau \frac{\partial f}{\partial t} = -L\left(\frac{\epsilon}{2T}, \frac{\omega_0}{2T}\right) f + \left[I_- f(\epsilon - \omega_0) + I_+ f(\epsilon + \omega_0)\right],$$

$$\tau \frac{\partial \bar{f}}{\partial t} = -L\left(\frac{\epsilon}{2T}, \frac{\omega_0}{2T}\right) \bar{f} + \left[I_+ \bar{f}(\epsilon - \omega_0) + I_- \bar{f}(\epsilon + \omega_0)\right].$$

$$I_\pm = \frac{1}{2\omega_0} \int d\epsilon f(\epsilon) \bar{f}(\epsilon \pm \omega_0);$$

$$L(x, y) = 2\coth(y) - \tanh(x + y) + \tanh(x - y).$$

still non linear and non local in energy space

SIMPLEST CASE: LOW DENSITY PHONONS AND HIGH TEMPERATURE

$\eta \ll 1 \quad \omega_0 \ll T$

classical limit: large number of excitations are already present, perturbation results in the evolution that leads to uncorrelated fixed point $f = 0 \quad \bar{f} = 0$ with the characteristic time of the order of τ

- We can neglect ω_0 in $I_\pm$ and arguments of f. $\quad L = 2/y$
- We look for solutions of the form:

$$f(\epsilon) = \phi(t)\left[1 + \tanh(\epsilon / 2T)\right]$$
$$\bar{f}(\epsilon) = \phi(t)\left[1 - \tanh(\epsilon / 2T)\right]$$

Function φ(t) has a meaning of the correlation between two Worlds that multiplies distribution functions

$$\tau \frac{\partial \phi}{\partial t} = -\frac{4T}{\omega_0}\left(\phi - \phi^3\right)$$

$\phi = 0$ stable fixed point

$\phi = 1$ unstable fixed point

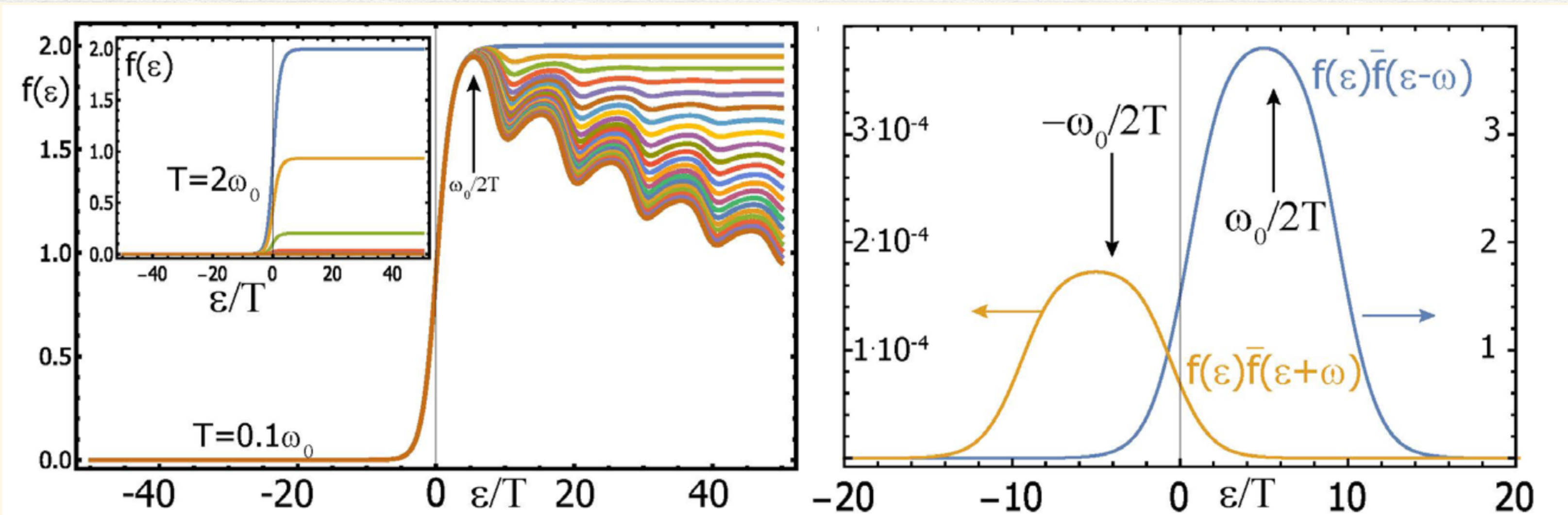
Correlated solution is unstable, fully correlated is stable

$$\phi(t) = \left(\frac{1}{1 + \exp\left[(t - t_d)/t_*^{cl}\right]}\right)^{1/2}, \qquad t_*^{cl} = \tau\omega_0 / 8T \qquad t_d = t_*^{cl} |\ln[1 - \phi(0)|$$

INSTABILITY AT LOW TEMPERATURE REGIME

$\omega_0 \gg T$ <u>quantum limit:</u> characteristic time dominated by exponentially small number of excitations

$I_- > I_+$



for $|\epsilon| > \omega_0$ L is large

$$\tau \frac{\partial f}{\partial t} = -L\left(\frac{\epsilon}{2T}, \frac{\omega_0}{2T}\right) f + \left[I_- f(\epsilon - \omega_0) + I_+ f(\epsilon + \omega_0)\right]$$

$$\tau \frac{\partial \bar{f}}{\partial t} = -L\left(\frac{\epsilon}{2T}, \frac{\omega_0}{2T}\right) \bar{f} + \left[I_+ \bar{f}(\epsilon - \omega_0) + I_- \bar{f}(\epsilon + \omega_0)\right]$$

$$I_\pm = \frac{1}{2\omega_0} \int d\epsilon f(\epsilon) \bar{f}(\epsilon \pm \omega_0)$$

$$L(x, y) = 2\coth(y) - \tanh(x + y) + \tanh(x - y)$$

Approximately:

$$f_\pm = f(\pm\omega_0 / 2) \quad I_+ \approx f_-^2 \quad I_- \approx f_+^2$$

$$\tau \frac{df_+}{dt} = -L_0 f_+ + f_+^2 f_-$$

$$\tau \frac{df_-}{dt} = -L_0 f_- + f_-^2 f_+$$

$$L_0 = 2\exp(-\omega_0 / 2T)$$

$$f_+ f_- = \frac{L_0}{1 + \exp\left[(t - t_d) / t_*^{qu}\right]}$$

$$f_+ + f_- = f_0 \left(\frac{1}{1 + \exp\left[(t - t_d)/t_*^{qu}\right]}\right)^{1/2}, \qquad t_*^{qu} = \tau/(2L_0)$$

INSTABILITY FOR ELECTRON_ELECTRON INTERACTION

$$\begin{aligned} \tau_{FL}\frac{\partial f}{\partial t} &= -L_F\left(\frac{\epsilon}{T}\right) f + K(\epsilon, f, \bar{f}), \\ \tau_{FL}\frac{\partial \bar{f}}{\partial t} &= -L_F\left(\frac{\epsilon}{T}\right) \bar{f} + \bar{K}(\epsilon, f, \bar{f}), \\ L_F(x) &= 1 + x^2/\pi^2, \end{aligned}$$

$$\tau_{FL}^{-1} \sim T^2/E_F^*$$

$$\begin{aligned} K(\epsilon, f, \bar{f}) &= \int_0^\infty I_-(\omega) f(\epsilon-\omega)\frac{d\omega}{2\pi T} + \int_0^\infty I_+(\omega) f(\epsilon+\omega)\frac{d\omega}{2\pi T}, \\ \bar{K}(\epsilon, f, \bar{f}) &= \int_0^\infty I_+(\omega) \bar{f}(\epsilon-\omega)\frac{d\omega}{2\pi T} + \int_0^\infty I_-(\omega) \bar{f}(\epsilon+\omega)\frac{d\omega}{2\pi T}, \\ I_\pm(\omega) &= 2\int \frac{d\epsilon}{2\pi T} f(\epsilon)\bar{f}(\epsilon\pm\omega). \end{aligned}$$

$$\phi(t) = \left(\frac{1}{1+\exp\left[(t-t_d)/t_*^{cl}\right]}\right)^{1/2}$$

$$t_* \sim \tau_{FL}$$

SPATIAL STRUCTURE OF INSTABILITY

$\eta \ll 1 \quad \omega_0 \ll T$

phonons remain local and relax fast

$$\theta(r) = \frac{1}{2\omega_0} \int d\epsilon f(\epsilon, r) \bar{f}(\epsilon - \omega_0, r),$$
$$\bar{\theta}(r) = \frac{1}{2\omega_0} \int d\epsilon f(\epsilon, r) \bar{f}(\epsilon + \omega_0, r),$$

$$\tau \frac{\partial f}{\partial t} = -L\left(\frac{\epsilon}{2T}, \frac{\omega_0}{2T}\right) f + \left[\theta f(\epsilon - \omega_0) + \bar{\theta} f(\epsilon + \omega_0)\right],$$
$$\tau \frac{\partial \bar{f}}{\partial t} = -L\left(\frac{\epsilon}{2T}, \frac{\omega_0}{2T}\right) \bar{f} + \left[\bar{\theta} \bar{f}(\epsilon - \omega_0) + \theta \bar{f}(\epsilon + \omega_0)\right],$$
$$\eta\tau \frac{\partial \theta}{\partial t} = -\theta + I_-,$$
$$\eta\tau \frac{\partial \bar{\theta}}{\partial t} = -\bar{\theta} + I_+,$$

$$\frac{\partial \phi}{\partial t} - D_* \nabla^2 \phi = -\frac{2\left(\phi - \phi^3\right)}{t_*}.$$

$$D_* = \frac{v_F^2 \tau_{tr}}{d} \qquad \frac{1}{\tau_{tr}} = \lambda^2 \nu \left(\frac{n_{ph} T}{\hbar \omega_0}\right)$$

$$t_*^{cl} = \tau \frac{\omega_0}{8T} \qquad \tau^{-1} = \frac{2\pi \nu n_{ph} \lambda^2}{\hbar}$$

Diffusion coefficient generally contains different time scale than instability time t_* that coincide only for simplistic models.

In the presence of elastic scattering D becomes smaller and propagation slows down

PROPAGATION OF INSTABILITY: COMBUSTION WAVES

$$\frac{\partial \phi}{\partial t} - D_* \nabla^2 \phi = -\frac{2\left(\phi - \phi^3\right)}{t_*}$$

$$\frac{dy}{dt} - \nabla^2 y = y(1-y)$$ COMBUSTION WAVES (FKPP eq)

two stationary solutions: y=0 (stable) and y=1 (unstable)

Our eq. displays the instability at $\phi(r) = 1$ according to this scenario:

after being seeded at time t=0 $\delta\phi(r) = 1 - \phi(r) \ll 1$ in a region around 0

$\delta\phi = 0$ for $r > R_c$

the instability remains localized in the area (r<Rc) for $t_d \sim \ln(1/\delta\phi)$

Then instability grows spatially forming a non-linear wave that moves with well defined velocity v_{cw}

In 1D the solution $\phi_f(x - v_{cw}t)$ of the front obeys

$$t_* \left(v_{cw} \frac{d\phi_f}{dx} + D_* \frac{d^2\phi_f}{dx^2} \right) = 2\phi_f(1 - \phi_f^2) \qquad v_{cw} = 4\sqrt{D_*/t_*}$$

For e-ph and e-e models in the absence of electron impurity and elastic scattering $D_* \sim t_* v_F^2$ $v_{cw} \sim v_F$

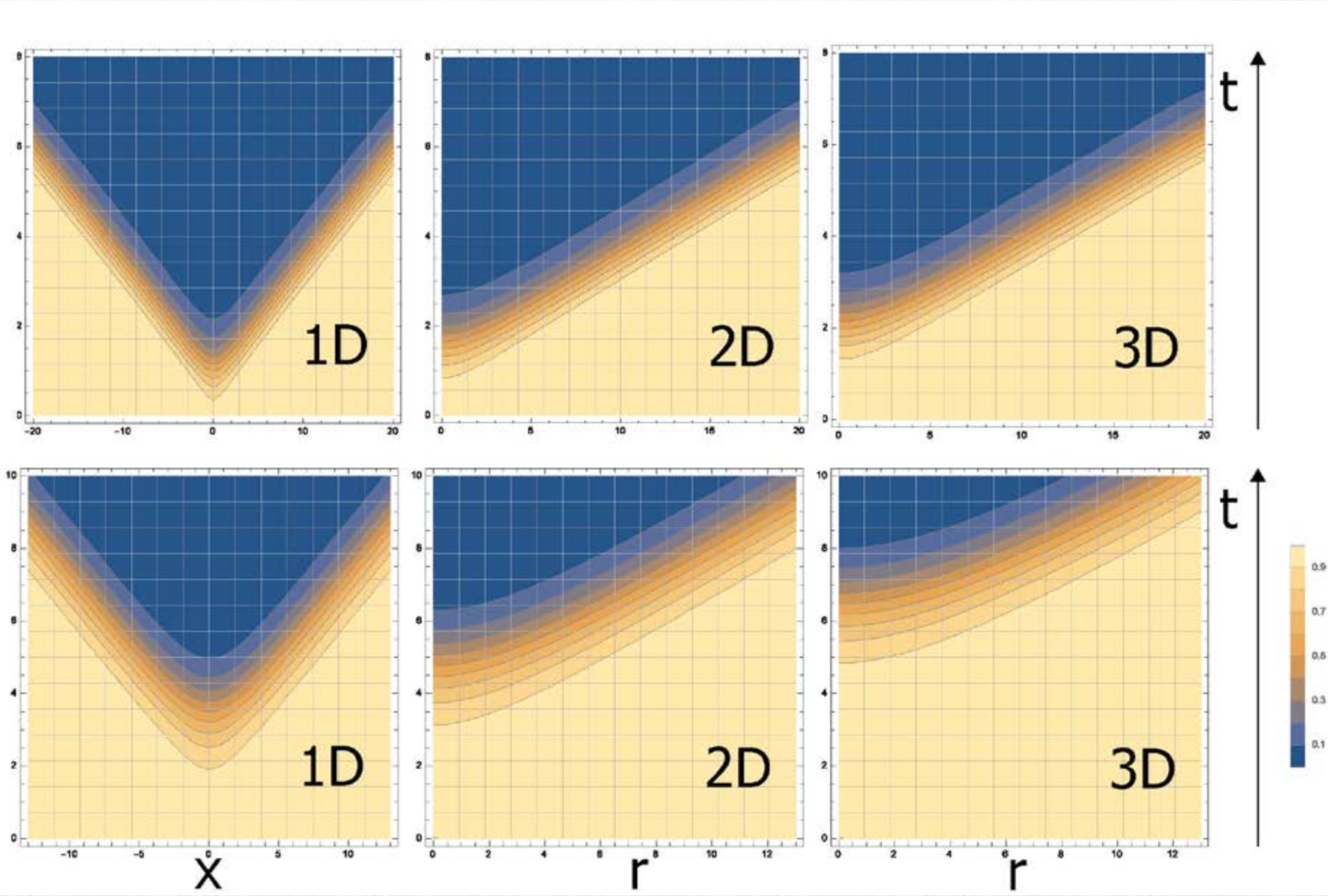
NUMERICAL SOLUTION FOR DIFFERENT MODELS

$$\frac{d\phi}{dt} = \nabla^2\phi + 2\phi(\phi^2 - 1)$$

$$\frac{d\phi}{dt} = \nabla^2\phi + 2(\Theta - 1)\phi,$$

$$\frac{d\Theta}{dt} = \phi^2 - \Theta$$

phonon dynamics same order of electron one (i.e. $\eta = 1$)



CONCLUSIONS

Propagation of decoherence in many body systems introduced by backward time evolution is described by the same equations as combustion wave.

- For el-ph and el-el interactions the mathematical description of OTOC is similar to the description of combustion waves:
  - The small initial perturbation first grows exponentially remaining local and then it starts to propagate with a constant velocity and a well defined front.
  - The velocity of the front is always slower than Fermi velocity
  - The constant velocity of the quantum butterfly propagation agrees with the result obtained in holographical theory of BH

We have seen that in the model of electrons in disorder potential there is no instability