# New results for 5 loop massless propagators integrals

A. Georgoudis

Department of Physics and Astronomy
Uppsala University

QCD Meets Gravity IV
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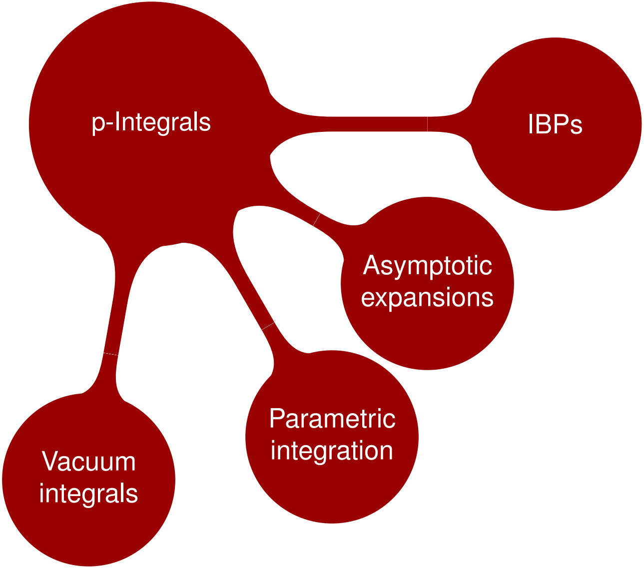
with Vasco Goncalves, Erik Panzer, Raul Pereira


p-Integrals
IBPs
Asymptotic expansions
Parametric integration
Vacuum integrals

# p-Integrals I

We want to compute massless 2-point integrals at 5-loops. We are interested in computing this two families of integrals:

- Momentum-space integrals
- Position-space integrals (i.e propagators of type $x_{ij} = x_i - x_j$).

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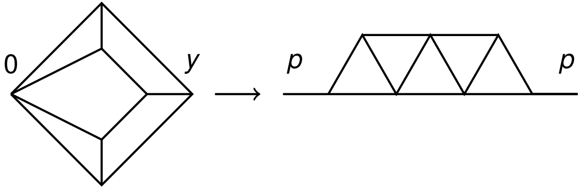
We can map the planar sector of this two types of integrals but we have to compute the non planar part separately.



# Integral-by-parts Identities

- IBPs are integration of a total derivative:

$$0 = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \cdots \frac{\mathrm{d}^D \ell_L}{\mathrm{i}\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu}{D_1^{\nu_1} \cdots D_m^{\nu_m}},$$

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- They can be used to express a general integral as a combination of Master Integrals:

$$\mathrm{I} = \sum_{i=1}^{N} c_i \; \mathrm{I}_i \,,$$

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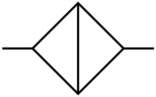
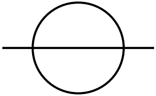
- Degree of divergence of a l-loop 2-point integral does not exceed l.
- IBP coefficients, in e.g. d=4-2 $\varepsilon$, might have poles in $\varepsilon$.
- Reducing convergent integrals!

# Integral-by-parts Identities: Example

Reduction of two-loop propagator integral



$$= \frac{2(3d-10)(8-3d)}{(d-4)^2}$$



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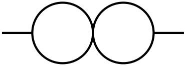

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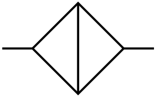

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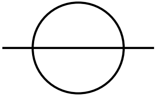

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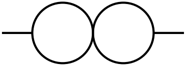

The spurious pole give a constrain on two epsilon orders. Convergence of the starting integral and the insertion of the value of the trivial integral $M_{\text{Dbubble}} = \frac{1}{\varepsilon^2}$ gives:

$$M_{\text{Sun}} = -\frac{1}{4\varepsilon} - \frac{5}{8} - \frac{27}{16}\varepsilon$$

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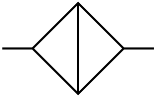

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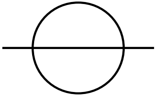

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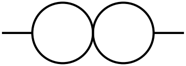

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**Simple algebra gave us a integral result!**

# Asymptotic expansions

- Start from a 4-point finite conformal integral.
- We can now take the limit $x_2 \to x_1 = 0$ which introduces two scales $x_2^2 \ll x_3^2$.
- $x_i$ is of the order of $x_2$ or order of $x_3$. For $x_2^2 < x_i^2$ we can write:

$$\frac{1}{(x_2 - x_i)^2} = \sum_{n=0}^{\infty} \frac{(2x_2 \cdot x_i - x_2^2)^n}{(x_i^2)^{n+1}}$$

- $2^l$ possible such regions.
- For finite integrals the expansion is still finite.

# Example

$$C_1 = \int \frac{\mathrm{d}^4 x_5 \ldots \mathrm{d}^4 x_9 \quad x_{13}^4 x_{24}^6 x_{35}^2}{x_{15}^2 x_{16}^2 x_{26}^2 x_{27}^2 x_{29}^2 x_{34}^2 x_{36}^2 x_{37}^2 x_{45}^2 x_{48}^2 x_{56}^2 x_{58}^2 x_{59}^2 x_{78}^2 x_{79}^2 x_{89}^2} .$$

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$$c_1 = c_2 = c_3 = c_4 = c_5 = 0\,.$$

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Take the 4-point conformal integral with points $\{x_7, x_8, x_9\}$.

Exchange $x_2 \leftrightarrow x_3$, $x_4 \leftrightarrow x_5$ and expand:

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This also fixes the finite term of the five-loop p-integral to be

$$c_6 = 120\,\zeta(3)\zeta(5)\,.$$

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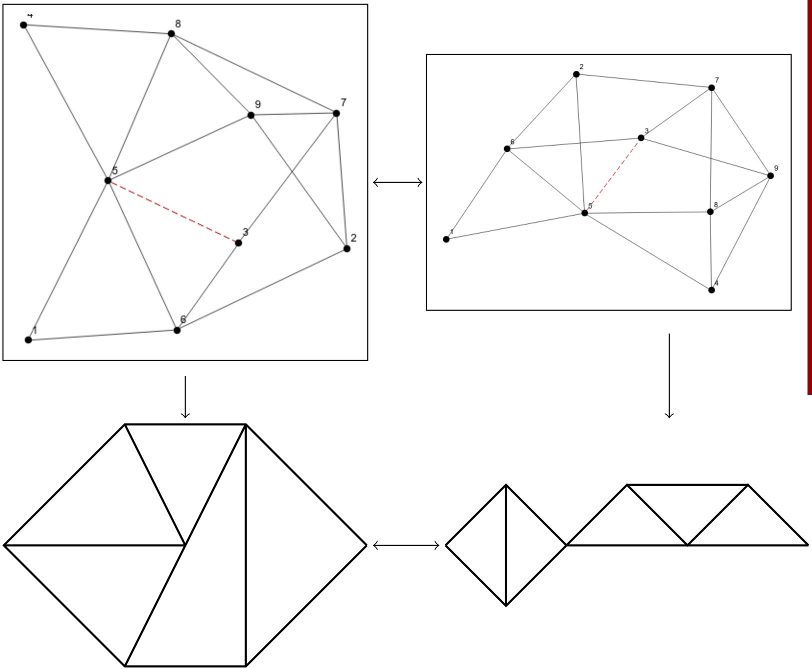
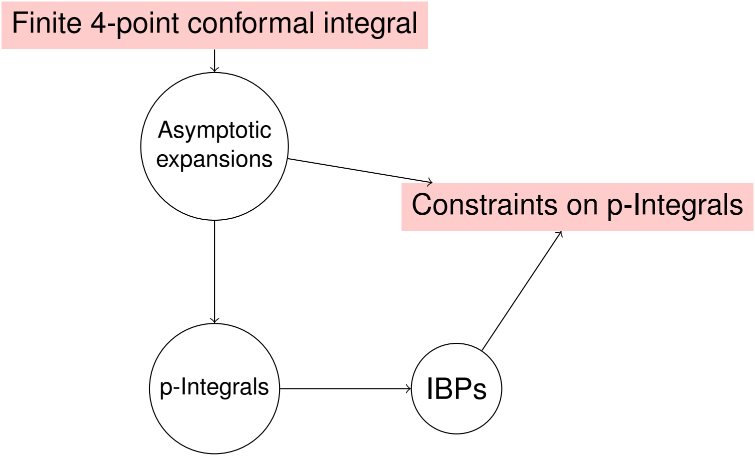


# How?


Finite 4-point conformal integral
Asymptotic expansions
Constraints on p-Integrals
p-Integrals
IBPs

# Results

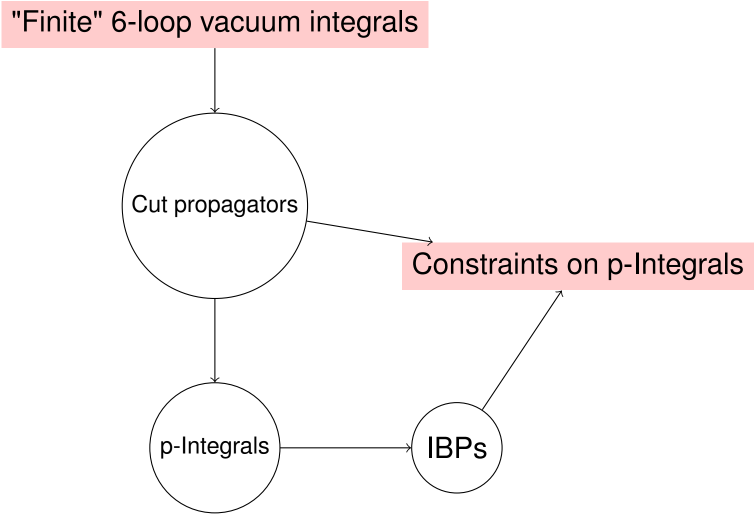
With IBPs and conformal symmetry constraints we were able to fix:

- Expansions of 169 out of the 187 genuine five-loop masters.
- All but two of the 95 planar masters were fixed to transcendental weight 9 in this way.
- In order to get the 169 integrals up to trascendentality 9 we used parametric integration (8 single orders of 7 integrals needed).

# Glue and Cut symmetry

If we want to compute non planar momentum space integrals we can use a different method.

# Glue and Cut symmetry



# Conclusions

In the $G$-scheme normalization:

- We find that $\zeta_2$ is absent!
- The coefficients of $\varepsilon^{-n}$ never have transcendental contributions of weight larger than $9-2n$ (where $n>0$).
- We can redefine odd zeta values to eliminate even zetas in the expansion of all master integrals.
- Any dependence on multiple zeta values comes in the combination $\varpi = \zeta_{2,6} - \zeta_{5,3}$.

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# Summary and future directions

- Presented recent result for 5-loop master integral.
- All the complication is encoded in the IBP reductions.
- Applying CaG symmetry to higher loops to check and obtain new results.
- Derive similar results for different dimensions (e.g $d = 3$).

Thank You