Kinematic Algebra Beyond the MHV Sector

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QCD Meets Gravity IV

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13.12.2018

Based on work in collaboration with G. Chen, H. Johansson and F. Teng

Color-kinematics duality

Cubic decomposition of tree-level scattering amplitudes in pure Yang-Mills theory

$$\mathcal{A}_n = g^{n-2} \sum_{i \in \text{cubic graphs}} \frac{c_i n_i}{D_i}$$

- Color factor c_i , kinematic numerator n_i , propagator denominator D_i
- Color Jacobi identities

$$c_i + c_j + c_k = 0 \sim f^{abe} f^{ecd} + f^{ace} f^{edb} + f^{ade} f^{ebc} = 0$$

- Same algebraic identities for kinematic numerators

$$c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0$$

  - Suggesting the existence of a kinematic algebra
- Generalized gauge freedom $n_i \to n_i + \Delta_i$, where $\sum_i \frac{c_i \Delta_i}{D_i} = 0$
- Double copy construction

Kinematic algebra

- State of the Arts
  - Monteiro-O'Connell approach (2011): self-dual YM
  - Cheung-Shen approach (2016): NLSM
- New method for characterizing kinematic algebra
  - Dimension-agnostic notations
  - Introduction of tensor currents and their fusion products
  - Special subsector of tree-level Yang-Mills amplitudes
  - Uniqueness of the algebra
  - Control over generalized gauge freedom

Tensor current

$$J^{(w)}_{\mathfrak{a}_1\otimes\mathfrak{a}_2\otimes\cdots\mathfrak{a}_m}(p)$$

- Tensor label $\mathfrak{a}_1\otimes\mathfrak{a}_2\otimes\cdots\mathfrak{a}_m$
  - Components: polarization vectors of external particles, momenta
- Momentum, dimension, w-index
- Rank: number of components in the tensor label
  - Rank-1: vector current
- Generalization of the standard tensor currents $\bar{v}(0)\not{\mathfrak{a}}_1\not{\mathfrak{a}}_2\cdots\not{\mathfrak{a}}_m u(q)$
- Linearity & "Clifford algebra"

$$J^{(w)}_{\cdots\otimes(x\mathfrak{a}_i+y\mathfrak{a}'_i)\otimes\cdots}(p) = xJ^{(w)}_{\cdots\otimes\mathfrak{a}_i\otimes\cdots}(p) + yJ^{(w)}_{\cdots\otimes\mathfrak{a}'_i\otimes\cdots}(p)$$

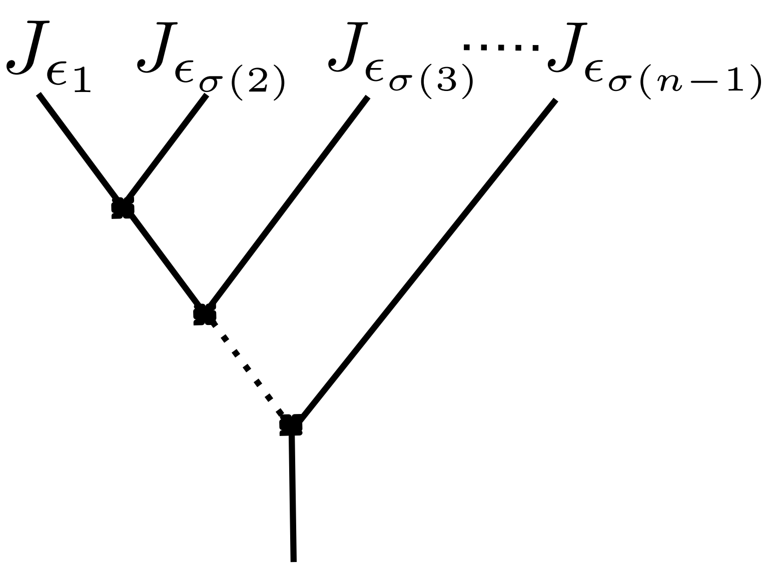
$$J^{(w)}_{\cdots\mathfrak{a}_i\otimes\mathfrak{a}_j\otimes\mathfrak{a}_k\otimes\mathfrak{a}_l\otimes\cdots}(p) + J^{(w)}_{\cdots\mathfrak{a}_i\otimes\mathfrak{a}_k\otimes\mathfrak{a}_j\otimes\mathfrak{a}_l\otimes\cdots}(p) = (2\mathfrak{a}_j\cdot\mathfrak{a}_k)J^{(w)}_{\cdots\otimes\mathfrak{a}_i\otimes\mathfrak{a}_l\otimes\cdots}(p)$$

Fusion Product

$$J \star J \sim J$$

- 2 to 1 fusions
- Properties of fusion products
  - Momentum conservation
  - Dimension, polarization vectors
- Goal: finding the fusion products that determine the BCJ numerators
  - Ordered fusion products (non-associativity)

    $$N(1\sigma(2)\cdots\sigma(n-1)n) = J_{\epsilon_1} \star J_{\epsilon_{\sigma(2)}} \star \cdots \star J_{\epsilon_{\sigma(n-1)}}$$
  - DDM basis $(1\sigma(2)\cdots\sigma(n-1)n)$
  - External legs: vector currents $J_{\epsilon_i}(p_i)$
  - Fusion products as "cubic interactions"



BCJ numerators with tensor currents

- $N(1\sigma(2)\cdots\sigma(n-1)n)$ consists of vector and tensor currents
- Irreducible basis b
  - Maximal set of tensor currents that can not be combined into vector currents

$$N(\rho) = N_{\mathsf{b}}^{\mathsf{V}}(\rho) + N_{\mathsf{b}}^{\mathsf{T}}(\rho)$$

  - Multiple choices for the irreducible basis
- Constraints
  - *null-space condition* $\qquad \sum_\rho m(\gamma|\rho) N_{\mathsf{b}}^{\mathsf{T}}(\rho) = 0\,,$
  - *gauge invariance* $\qquad \sum_\rho m(\gamma|\rho) N_{\mathsf{b}}^{\mathsf{V}}(\rho)\Big|_{\epsilon_i\to p_i} = 0\,.$
- Standard BCJ numerators

$$\mathsf{n}_{\mathsf{b}}(\rho) = N_{\mathsf{b}}^{\mathsf{V}}(\rho)|_{J_{\mathfrak{a}}\to\mathfrak{a}\cdot\epsilon_n}$$

Set-up: special sector

We restrict our discussions to the terms in the BCJ numerators that

- are up to the order $\mathcal{O}((\epsilon \cdot \epsilon)^2)$
  - In 4 dimensions: MHV & NMHV
- contain the common factor $\epsilon_1 \cdot \epsilon_n$
  - dimensional reduction of the first and the last legs
  - crossing symmetries involving these two legs broken

$$\epsilon_1 \cdot \epsilon_n \left(\prod \epsilon_m \cdot p_s \right) , \qquad s_{ij}\, \epsilon_1 \cdot \epsilon_n \, \epsilon_k \cdot \epsilon_l \left(\prod \epsilon_m \cdot p_s \right)$$

- Gauge invariance condition
  - mixes terms of the two types
  - is expected to hold up to the linear order in the Mandelstam variables

Determination of fusion products

- Simplest choice for irreducible basis: ascending order

$$\begin{array}{l} J^{(w)}_{p_{j_1}\otimes p_{j_2}\otimes\cdots\otimes p_{j_{m'}}\otimes\epsilon_{i_1}\otimes\epsilon_{i_2}\otimes\cdots\otimes\epsilon_{i_m}\otimes q}(q) \\ J^{(w)}_{p_{j_1}\otimes p_{j_2}\otimes\cdots\otimes p_{j_{m'}}\otimes\epsilon_{i_1}\otimes\epsilon_{i_2}\otimes\cdots\otimes\epsilon_{i_m}}(q) \end{array} , \qquad \begin{array}{l} 1 \leqslant i_1 < \ldots < i_m \leqslant n-1 \\ 2 \leqslant j_1 < \ldots < j_{m'} \leqslant n-1 \end{array}$$

- Toy model I: 3-point numerator

$$N(123) = J_{\epsilon_1}(p_1) \star J_{\epsilon_2}(p_2)$$

must reproduce $n(123) = \epsilon_2 \cdot p_1 \, \epsilon_1 \cdot \epsilon_3$

$$J_{\epsilon_1}(p_a) \star J_{\epsilon_i}(p_i) = \epsilon_i{\cdot}p_a \, J_{\epsilon_1}(p_a + p_i) - \frac{1}{2} J_{\epsilon_1\otimes\epsilon_i\otimes(p_a+p_i)}(p_a + p_i)$$

  - First term gives the standard numerator
  - Second term vanishes “on-shell”

Determination of fusion products

- Toy model II: 4-point numerator

$$N(1234) = J_{\epsilon_1}(p_1) \star J_{\epsilon_2}(p_2) \star J_{\epsilon_3}(p_3)$$
$$N(1324) = N(1234)|_{2\to 3}$$

  - New structure: $J_{\epsilon_1 \otimes \epsilon_i \otimes q} \star J_{\epsilon_k}$
  - We find $J_{\epsilon_1 \otimes \epsilon_i \otimes q} \star J_{\epsilon_k} = q^2 J^{(1)}_{\epsilon_1 \otimes \epsilon_i \otimes \epsilon_k}$

As a result

$$\begin{bmatrix} \frac{1}{s_{12}} + \frac{1}{s_{23}} & -\frac{1}{s_{23}} \\ -\frac{1}{s_{23}} & \frac{1}{s_{13}} + \frac{1}{s_{23}} \end{bmatrix} \begin{bmatrix} N_{\mathsf{b}}^{\mathsf{T}}(1234) \\ N_{\mathsf{b}}^{\mathsf{T}}(1324) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{s_{12}} + \frac{1}{s_{23}} & -\frac{1}{s_{23}} \\ -\frac{1}{s_{23}} & \frac{1}{s_{13}} + \frac{1}{s_{23}} \end{bmatrix} \begin{bmatrix} -s_{12} J^{(1)}_{\epsilon_1 \otimes \epsilon_2 \otimes \epsilon_3} \\ +s_{13} J^{(1)}_{\epsilon_1 \otimes \epsilon_2 \otimes \epsilon_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  - Tensor part satisfies the null-space condition automatically
  - Vector part satisfies the gauge invariance condition (= reproduces the correct BCJ numerators)

Determination of fusion products

- 5-point numerator
  - New structure: $J^{(1)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j} \star J_{\epsilon_k}$

Solution to the null-space & gauge inv. conditions

$$J^{(1)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j}(p_a) \star J_{\epsilon_k}(p_b) = (\epsilon_j \cdot p_a) J^{(w_1)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_k}(p_a + p_b) - (\epsilon_i \cdot p_a) J^{(w_2)}_{\epsilon_1\otimes\epsilon_j\otimes\epsilon_k}(p_a + p_b) + (\epsilon_k \cdot p_a) J^{(w_3)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j}(p_a + p_b)$$

  - w_i generic as fusion products involving these new tensor currents unknown at five points

- 6-point numerator
  - New structure $J^{(w_i)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j} \star J_{\epsilon_k}$
  - We find $w_1 = w_2 = 1 \,, w_3 = 2$

$$J^{(2)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j}(p_a) \star J_{\epsilon_k}(p_b) = (\epsilon_k \cdot p_a) J^{(w)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j}(p_a + p_b)$$

Determination of fusion products

- 7-point numerator
  - New structure $J^{(w)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j} \star J_{\epsilon_k}$
  - We find $w = 2$
- Closed algebra!

$$J_{\epsilon_1}(p_a) \star J_{\epsilon_i}(p_i) = \epsilon_i \cdot p_a \, J_{\epsilon_1}(p_a + p_i) - \frac{1}{2} J_{\epsilon_1\otimes\epsilon_i\otimes(p_a+p_i)}(p_a + p_i)$$

$$J_{\epsilon_1\otimes\epsilon_i\otimes p_a}(p_a) \star J_{\epsilon_k}(p_b) = p_a^2 J^{(1)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_k}$$

$$J^{(1)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j}(p_a) \star J_{\epsilon_k}(p_b) = (\epsilon_k \cdot p_a) J^{(2)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j}(p_a + p_b) + (\epsilon_j \cdot p_a) J^{(1)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_k}(p_a + p_b) - (\epsilon_i \cdot p_a) J^{(1)}_{\epsilon_1\otimes\epsilon_j\otimes\epsilon_k}(p_a + p_b) \,,$$

$$J^{(2)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j}(p_a) \star J_{\epsilon_k}(p_b) = (\epsilon_k \cdot p_a) J^{(2)}_{\epsilon_1\otimes\epsilon_i\otimes\epsilon_j}(p_a + p_b)$$

- Higher-point numerators
  - Null-space & gauge invariance conditions satisfied automatically
  - Closed formula for $N(1\sigma(2)\sigma(3)\cdots\sigma(n-1)n)$ in terms of tensor currents

Generalized gauge freedom

- Generate multiple versions of BCJ numerators by choosing different basis for tensor currents
- All versions are equivalent (differences vanish on BCJ relations)
- Degrees of freedom

	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
Degree of freedom	1	9	90	1080	15435

- Degrees of freedom (Crossing-symmetric numerators)

	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
degree of freedom	0	1	3	9	22

Outlook

- Factorization properties
  - Connection with the self-consistency of the algebra
  - Connection with the soft-limit behavior
- Prescription beyond the special sector
- Prescription beyond the DDM basis
  - Restoration of complete crossing symmetries
  - Kinematic Jacobi identities

Closed formula for numerators

$$N^{\mathsf{V}}(123\cdots n) = \left(\prod_{j=2}^{n-1} \epsilon_j \cdot p_{1\cdots j-1}\right) J_{\epsilon_1}$$

$$N^{\mathsf{T}}(123\cdots n) = \sum_{i=2}^{n-2} \sum_{\substack{\ell,m=i \\ m>\ell}}^{n-1} (-1)^{\ell-i-1} s_{1\cdots i} \left(\prod_{j\in \mathsf{S}_{i\ell m}} \epsilon_j \cdot p_{1\cdots j-1}\right) \det(\mathbf{P}_{[i,\ell-1]}) J_{\epsilon_1\otimes\epsilon_\ell\otimes\epsilon_m}$$

$$\mathsf{S}_{i\ell m} = \{2\cdots i-1\} \cup \{\ell+1\cdots \hat{m}\cdots n-1\}$$

$$\mathbf{P}_{r_1 r_2} = \begin{cases} \epsilon_{r_1} \cdot p_{1\cdots(r_2+1)} & r_1 \leqslant r_2+1 \\ 0 & \text{otherwise} \end{cases}$$