Plane wave backgrounds and colour-kinematics duality

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Squaring relations: $(YM)^2 \simeq GR$

- Tree-level amplitudes (proof)
- Several examples @ loop-level
- Gravitational radiation
- Non-perturbative solutions

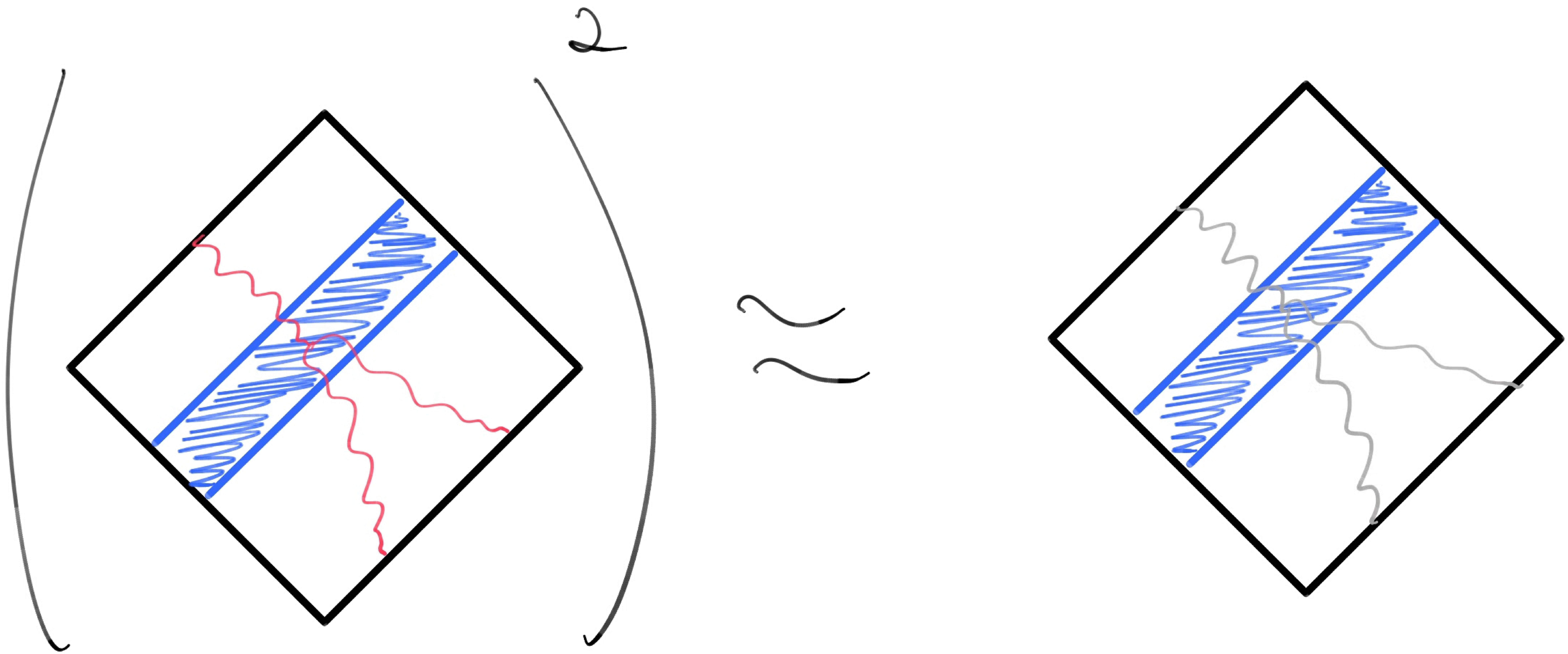
$\vdots$

- Perturbation around non-trivial background,

Plane wave

3-point gluons

3-point gravitons



Reflects the flat space statement

$$\left(\mathcal{A}_3^{YM}\right)^2 = \mathcal{A}_3^{GR}$$

Too trivial, no need for

Colour-Kinematics duality

$\longrightarrow$ Must look at higher points!

Flat space review (tree-level)

$$\mathcal{A}^{YM} = \sum_{\Gamma} \frac{\overbrace{n_i}^{\text{kinematical numerators}} \underbrace{c_i}_{\text{colour factors}}}{\underbrace{p_i^2}_{\text{propagators}}}$$

Γ cubic graphs

$$c_i - c_j + c_k = 0 \rightsquigarrow \text{Jacobi Id}$$

Flat space review (tree-level)

$\exists$ representation s.t. $n_i - n_j + n_k = 0$

and

$$\mathcal{A}^{GR} = \sum_{r} \frac{n_i \, n_i}{P_i}$$

gravitational amplitude

Question:

- Is there an analogue of $h_i - h_j + h_k = 0$ for amplitudes around plane wave backgrounds?

  Yes, with caveats

- Is there a double copy for higher point amplitudes? Working on it

Curved space Challenges

- No momentum space, must generalize flat space representation
  ↳ definitions from 3-point calculation are not enough

- No momentum conservation
  ↳ work with plane waves, (d-1) conserved quantities

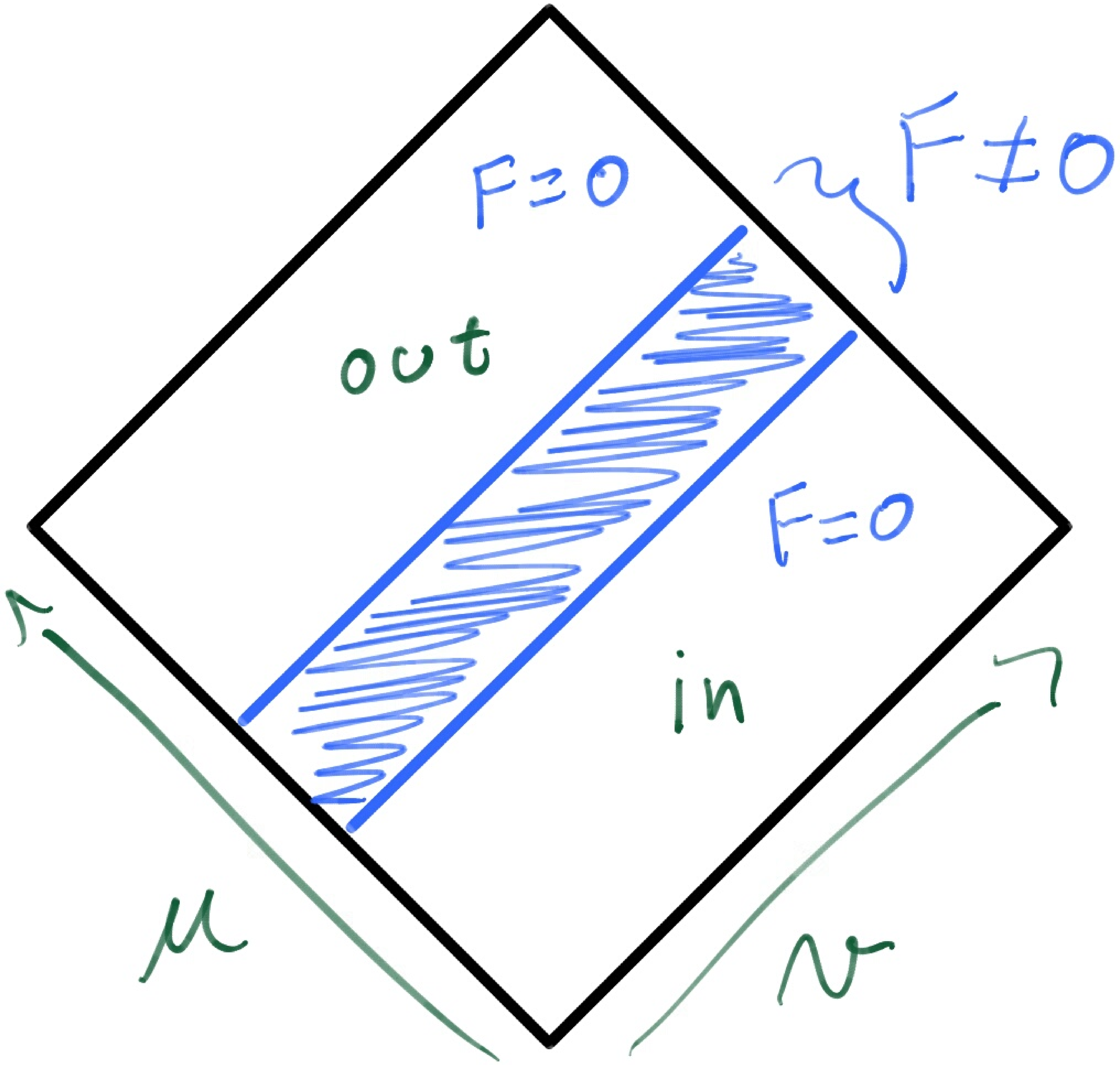
Today:

analogue of $n_i - n_j + n_k = 0$

for $\underline{A_4}$, 4-point gluon amplitude

on a gauge theory plane wave background

(Gauge) plane wave background



$$ds^2 = 2dx^+dx^- - dx_a dx^a$$

$$A = x_a \dot{A}^a(x^-)\,dx^-$$

$$F = \dot{A}_a\, dx^a \wedge dx^-$$

↳ in the Cartan

<u>Free fields:</u> $D_\nu D^\nu a_\mu + 2i\,[F_{\mu\nu}, a^\nu] = 0$

p.w.) $\left(2\partial_+\partial_- - \partial_a\partial^a - 2iex^a\dot{A}_a\partial_+\right)a_\mu$

$+\, 2i\left(\delta^a_\mu \dot{A}_a a_+ + \delta^-_\mu \dot{A}^a a_a\right) = 0$

$a_\mu = \vec{T}^A \varepsilon_\mu e^{i\phi}$ $\rightsquigarrow \varepsilon_\mu$ depends on (K, A)

$$\phi = \underbrace{K_+ x^+ + (K_a}_{} + eA_a)x^a + \frac{1}{2K_+}\int^{x^-}(K+eA)^2\,ds$$

↳ $d-1$ cons. momenta

Free fields: curved momentum

$$K_\mu = -i e^{-i\phi} D_\mu e^{i\phi} = \left(K_+, K_a + eA_a, \frac{(K_a + eA_a)^2}{2K_+} \right)$$

$$\underline{\epsilon_\mu K^\mu} = 0 \; ; \; \underline{K^\mu K_\mu} = 0$$

$\Rightarrow$ **not** constants

Propagator

$$\left(D_\lambda D^\lambda \delta^\mu_\sigma + 2ieF^\mu_\sigma\right) g_{\mu\nu}(X,Y) = \delta^{AB} \eta_{\sigma\nu}\, \delta^d(X-Y)$$

$$g_{\mu\nu}(X,Y) = \frac{N\delta^{AB}}{2\pi i} \int \frac{d^D K}{K^2 + i\varepsilon} D_{\mu\nu}(x^-, y^-)\, e^{(i\phi_K(X) - i\phi_K(Y))}$$

$$D_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\delta_{ab} & \frac{e\Delta A_a}{K_+} \\ 1 & \frac{e\Delta A_b}{K_+} & \frac{e^2 \Delta A^2}{2K_+^2} \end{pmatrix}$$

$$\underline{\underline{D_{\mu\nu}(x^-, y^-)\, K^\nu(y^-) = K_\mu(x^-)}}$$

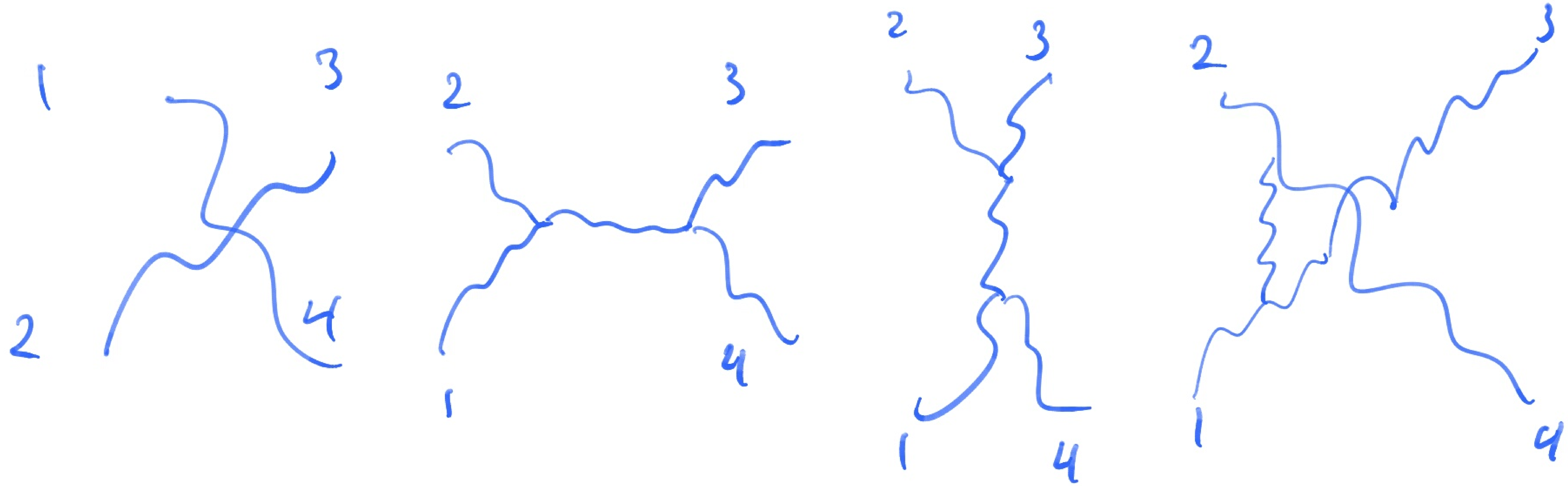
Interactions

$$S_{int}[a,A] = -\frac{1}{4g^2}\int d^dx\, \mathrm{tr}\Big(4[a_\mu,a_\nu]\underline{D}^{[\mu}a^{\nu]} + [a_\mu,a_\nu][a^\mu,a^\nu]\Big)$$

→ Only explicit dependence on background

4-point Amplitude

$$A_4 = A_4^{\text{cont}} + A_{4,s} + A_{4,t} + A_{4,u}$$



Contact vertex

$$A_4^{cont} = \frac{g^2}{N} \delta^{d-1}\left(\sum_r^4 k_r\right)\left[f^{A_1A_2B} f^{A_3A_4B}\left(\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4 - \epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4\right)\right.$$

$$\left. + \text{permutations}\right] \times \int d\tilde{x}\, \exp\left[i \sum_{r=1}^{4} \frac{f_r}{2k_{+r}} \right]$$

just constants as in flat space

$$f_r = \int^{\tilde{x}} \left(2k_+k_- - k_a k^a + (k_a + eA_a(x))^2\right) dx$$

Exchange vertex (s-channel)

$$A_{4,s} = g^2 \delta^{d-1}\left(\sum_r^4 k_r\right) f^{A_1 A_2 B} f^{A_3 A_4 B} \int d^2\mu[s] \Big[\epsilon_1 \cdot \epsilon_2 (k_1 - k_2)^\mu$$

$$+ 2\epsilon_1 \cdot k_2\, \epsilon_2^\mu - 2\epsilon_2 \cdot k_1\, \epsilon_1^\mu \;(x^-)\Big] D^s_{\mu\nu}(x^-, y^-)$$

$$\times \Big[\epsilon_3 \cdot \epsilon_4 (k_4 - k_3)^\nu - 2\epsilon_3 \cdot k_4\, \epsilon_4^\nu + 2\epsilon_4 \cdot k_3\, \epsilon_3^\nu\Big](y^-) + (x^- \leftrightarrow y^-)$$

Color-Kinematics duality

$$A_4\Big|_{\text{flat}} = g^2 \delta^d\left(\sum k_r\right)\left(\frac{N_s c_s}{s} + \frac{N_t c_t}{t} + \frac{N_u c_u}{u}\right)$$

$$c_s = f^{A_1 A_2 B} f^{B A_3 A_4} \quad , \quad c_t = f^{A_1 A_3 B} f^{B A_2 A_4}$$

$$c_u = f^{A_1 A_4 B} f^{B A_2 A_3}$$

$$c_s - c_t + c_u = 0$$

$$N_s - N_t + N_u = 0$$

Splitting the contact term

$$A_4^{\mathrm{cont}} \propto \Big(c_s \left(\epsilon_1 \cdot \epsilon_3 \, \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \, \epsilon_2 \cdot \epsilon_3 \right) + c_t \left(\epsilon_1 \cdot \epsilon_2 \, \epsilon_3 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \, \epsilon_2 \cdot \epsilon_3 \right) + c_u \left(\epsilon_1 \cdot \epsilon_2 \, \epsilon_3 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_3 \, \epsilon_2 \cdot \epsilon_4 \right) \Big) \times$$

$$\left. \times \int d^d X \exp\left(i \sum_{r=1}^{4} \phi_r(X) \right) \right\} \text{Have to split this integral}$$

Splitting the contact term

$$C_1 \left(\epsilon_1 \cdot \epsilon_3 \, \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \, \epsilon_2 \cdot \epsilon_3 \right) \times$$

$$\int d^d x \, d^d y \, \delta^d(x-y) \exp\left[i \sum_{r=1,2} \phi_r(x) + i \sum_{l=3,4} \phi_l(y) \right]$$

$$\underline{D_\mu D^\mu g}(x,y) = \delta^d(x-y)$$

$\hookrightarrow$ Use scalar propagator

Splitting the contact term

$$\int d^d x\, d^d y \left(D_\mu D^\mu G(x,y) + D'_\mu D'^\mu G(x,y) \right)$$

$$\times \exp\left[i \sum_{r=1,2} \phi_r(x) + i \sum_{l=3,4} \phi_l(y) \right]$$

$$\simeq \int \underline{d^2\mu[r]} \left(K_1 \cdot K_2(\bar{x}) + K_3 \cdot K_4(\bar{y}) + (\bar{x} \leftrightarrow \bar{y}) \right)$$

↳ Same measure as exchange channel

New representation for A_4

$$A_4 = g^2\, \delta^{d-1}\left(\sum_{r=1}^{4} K_r\right)\left(c_s \int d^2\mu[s]\, h_s + c_t \int d^2\mu[t]\, h_t \right.$$

$$\left. + c_u \int d^2\mu[u]\, h_u \right)$$

Check:

$$\int d^2\mu[s]\, h_s \Big|_{A=ct} = \delta\left(\sum_{r=1}^{4} K_{-r}\right) \frac{N_s}{s}$$

Explicit kinematical integrand

$$n_s = \left[\epsilon_1\cdot\epsilon_2 (K_1 - K_2)^\mu + 2\epsilon_1\cdot K_2\, \epsilon_2^\mu - 2\epsilon_2\cdot K_1\, \epsilon_1^\mu\right](x^-)$$

$$\times\ D^{\Lambda}_{\mu\nu}(x^-, y^-)\left[\epsilon_3\cdot\epsilon_4 (K_4 - K_3)^\nu - 2\epsilon_3\cdot K_4\, \epsilon_4^\nu + 2\epsilon_4\cdot K_3\, \epsilon_3^\nu\right](y^-)$$

$$-\left(\epsilon_1\cdot\epsilon_3\, \epsilon_2\cdot\epsilon_4 - \epsilon_1\cdot\epsilon_4\, \epsilon_2\cdot\epsilon_3\right)\left(K_1\cdot K_2(x^-) + K_3\cdot K_4(y^-)\right)$$

$$+\ (x^- \leftrightarrow y^-)$$

Kinematic Jacobi identity

$$A_4 = g^2 \delta^{d-1}\left(\sum_{r\geq 1}^{4} K_r\right)\left(c_s \int d^2\mu[s]\, n_s + c_t \int d^2\mu[t]\, n_t + c_u \int d^2\mu[u]\, n_u\right)$$

$n_s - n_t + n_u =$ nonsense $\rightsquigarrow$ Still contains propagators

$\hookrightarrow$ Must mod out by propagators.

Non-trivial, D is not algebraic as in flat space

Kinematic Jacobi identity

Set all prop. to the same channel

$$\sigma\left(D^{t}_{\mu\nu}\right) = D^{s}_{\mu\nu} = \sigma\left(D^{u}_{\mu\nu}\right)$$

In flat space this is equivalent to

$$\frac{n_t}{t} \to \frac{n_t}{s} \; ; \quad \frac{n_u}{u} \to \frac{n_u}{s}$$

S. t. $\quad \frac{1}{s}\left(n_s - n_t + n_u\right) = 0$

Kinematic Jacobi identity

After some algebra

$$\sigma(n_s - n_t + n_u) = \underline{K_3(\bar{x})\cdot D^{1}\cdot \epsilon_3(\bar{y})}\,(K_2 - K_1)(\bar{x})\cdot D^{1}\cdot \epsilon_4(\bar{y})$$

$$+\ \underline{\epsilon_3(\bar{x})\cdot D^{1}\cdot K_3(\bar{y})}\,\epsilon_4(\bar{x})\cdot D^{1}\cdot (K_2 - K_1)(\bar{y})$$

$$+\ (K_1 - K_2)(\bar{x})\cdot D^{1}\cdot \epsilon_3(\bar{y})\,\underline{K_4(\bar{x})\cdot D^{1}\cdot \epsilon_4(\bar{y})}$$

$$+\ \epsilon_3(\bar{x})\cdot D^{1}\cdot (K_1 - K_2)(\bar{y})\,\underline{\epsilon_4(\bar{x})\cdot D^{1}\cdot K_4(\bar{y})}$$

All prop to $\epsilon_i \cdot D \cdot K_i$

↳ like a "deformed gauge cond"

Remarks

- $\sigma(n_s - n_t + n_u) \neq 0$, but req. $= 0$ up to deformed gauge conditions is highly constraining

  → Constrains the representation of amp. at higher points. Interesting to check at 5-points.

- Can attempt to double-copy this rep. but need gravity calculation to check (in progress)

- Ambitwistor string calculation might give C-K representation directly, like in flat space

Thanks!