On massless tree-level S-matrix in 2d sigma models

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scattering of on-shell 2d massless scalars (left/right):
analog of 4d gauge vector scattering

• motivation: search for integrable σ-models
exact solution for strings in curved backgrounds
integrable deformations of AdS/CFT
(recent examples: β, γ, η, λ,...)

• classical integrability: existence of Lax representation for e.o.m.
search for Lax pair is generally hard
(other attempts: stability under RG flow; reduction to 1d
and check for absence of chaotic behaviour – inconclusive)

• standard lore in massive 2d models: [Arefieva, Korepin 74; Parke 80]
integrability $\leftrightarrow$ no particle production
and factorization of S-matrix ($\rightarrow$ Yang-Baxter equation)

• S-matrix as guide to integrability for massless 2d theories?

• σ-model near trivial vacuum: massless scalar excitations

$$L = (G_{mn} + B_{mn})\partial_+ x^m \partial_- x^n$$
$$= (\delta_{mn} + h_{mnk} x^k + c_{mnkl} x^k x^l + \dots)\, \partial_+ x^m \partial_- x^n$$

constraints on coeffs $\to$ geometry of integrable models from no particle production/factorization of massless S-matrix?

• No: well-known integrable σ-models have massless particle production

• link to integrability is preserved if expand near non-trivial vacua with only massive excitations

• example: start with massive integrable 2d model

$$\mathcal{L} = \partial_+ x^n \partial_- x^n - V(x), \quad V = \tfrac{1}{2} m^2 x^2 + g x^3 + \dots$$

associated "pp-wave" σ-model : add "light-cone" directions

$$\hat{\mathcal{L}} = \partial_+ u\, \partial_- v + \partial_+ x^n \partial_- x^n - V(x)\, \partial_+ u \partial_- u, \qquad u, v = y \pm t$$

• near trivial vac $u = v = x = 0$: amplitudes for massless x-excitations and even no. of u-excitations may not factorize
• but in "light-cone" vac $u = \tau$, $(v, x = 0)$: x are massive resulting S-matrix factorizes for an integrable potential V
• similar: expand near BMN geodesic in $AdS_n \times S^n$ but breaks 2d Lorentz symmetry [Klose, McLoughlin, Roiban, Zarembo 06]

• massless S-matrix in 2d usually considered as suspect
standard interpretation may not apply – particles in 1d direction do not separate asymptotically
also: IR divergences at quantum level
"no Goldstone bosons in 2d" [Coleman 73]
[still: massless S-matrices were formally discussed in context of finite-density TBA [Zamolodchikov, Zamolodchikov 92]
S-matrix: relative phase if one particle is moved past another]

• one can certainly define massless S-matrix at tree level
e.g., from classical action on solution with scattering b.c.:
should resulting massless S-matrix reflect
classical (non-)integrability? if yes then how?
[seems likely: standard defn of classical integrability via
Lax pair makes no distinction between massless and
massive cases, does not depend on expansion point]
• early indications that connection between integrability
and no particle production / factorization does not apply
in massless scale-inv case: non-zero 5-point tree amplitude
in Zakharov-Mikhailov model (classical dual of PCM) [Nappi 80]
tree particle production claimed in PCM and S^N [Figueirido 89]
• if true, demanding factorization of tree-level S-matrix
may not be necessary for integrability of classical
scale-invariant 2d models with massless excitations

• aim: check and clarify why massless S-matrix exhibits this tree-level “anomaly of integrability” (role of IR ambiguities)

• massless case remains little known and controversial; may be in massless integrable case one may relax/modify condition of no particle production? (cf. [Wulff 18])

• recent work [Gabai,Mazac, Shieber,Vieira, Zhou 18]:
hermitian matrix massless fields with 2-derivative interactions alternative definition of no particle production for partial colour-ordered amplitudes as opposed to full amplitudes was claimed to lead directly to action of integrable $U(n)$ PCM; but unclear how to generalize this to other integrable σ-models that do not have notion of colour-ordered amplitudes

• massless 2d S-matrices were discussed in non scale invariant flat string Nambu-like models $L(\partial\phi)$ [Duvovsky,Flauger,Gorbenko 12]
no IR divergences if scalars appear in action only with ∂_μ
standard relation between factorization of S-matrix and integrability was taken for granted but indeed appears to hold (Nambu model is quantum-integrable only in critical dim)
corresponding 2d massless S-matrix was suggested as useful tool in search for effective action of confining QCD string

• Here: clarify properties of tree-level massless S-matrix in scale-inv 2d σ-models on standard integrable examples:
principal chiral model and models related by dualities
find non-zero particle production amplitudes and relate to IR ambiguities in tree-level scattering of chiral 2d scalars

PCM and related models

$$\mathcal{L}_{\rm PCM} = \tfrac{1}{\lambda^2}\,{\rm tr}\left(J^\mu J_\mu\right)\,, \qquad J_\mu = g^{-1}\partial_\mu g\,, \quad g = e^{\lambda X^a t_a}$$

$$\mathcal{L}_{\rm PCM} = -\tfrac{1}{2}\partial_\mu X^a \partial^\mu X^a - \tfrac{1}{12}\lambda^2 f_{ab}{}^e f_{cde} X^a X^c \partial_\mu X^b \partial^\mu X^d$$
$$+ \tfrac{1}{360}\lambda^4 f_{abl} f_{cm}{}^l f_{dn}{}^m f_{eg}{}^n X^b X^c X^d X^e \partial_\mu X^a \partial^\mu X^g + \ldots$$

add WZ term:

$$\mathcal{L}_{\rm PCM_q} = \mathcal{L}_{\rm PCM} + q\mathcal{L}_{\rm WZ}\,, \qquad (\eta^{\mu\nu} + q\,\epsilon^{\mu\nu})\partial_\mu J_\nu = 0$$

- PCM is classically dual to ZM model [Zakharov, Mikhailov 73]

$$\partial_\mu J^\mu = 0\,, \qquad F_{\mu\nu}(J) \equiv \partial_\mu J_\nu - \partial_\mu J_\nu + [J_\mu, J_\nu] = 0$$

$$J^\mu = \lambda\epsilon^{\mu\nu}\partial_\nu\phi\,, \qquad \Box\phi^a - \tfrac{1}{2}\lambda f_{bc}{}^a \epsilon^{\mu\nu}\partial_\mu\phi^b\partial_\nu\phi^c = 0$$

$$\mathcal{L}_{\rm ZM} = -\tfrac{1}{2}\partial^\mu\phi^a\partial_\mu\phi^a + \tfrac{1}{3}\lambda\epsilon^{\mu\nu} f_{abc}\,\phi^a\partial_\mu\phi^b\partial_\nu\phi^c$$

σ-model with flat metric and constant B-strength ($\phi^a \equiv X^a$)

$$\mathcal{L} = -\tfrac{1}{2}(G_{ab}\eta^{\mu\nu} + B_{ab}\epsilon^{\mu\nu})\partial_\mu X^a \partial_\nu X^b \ ,$$
$$G_{ab} = \delta_{ab} \ , \qquad B_{ab} = -\tfrac{2}{3}\lambda f_{abc} X^c \ , \qquad H_{abc} = -2\lambda f_{abc}$$

• similar "pseudodual" of PCM_q: ZM_q

$$J^\mu = \lambda(\epsilon^{\mu\nu}\partial_\nu\phi - q\,\partial^\mu\phi) \ , \qquad \Box\phi^a - \tfrac{1}{2}\lambda(1-q^2) f_{bc}{}^a \epsilon^{\mu\nu}\partial_\mu\phi^b\partial_\nu\phi^c = 0$$

$$\mathcal{L}_{\mathrm{ZM}_q} = -\tfrac{1}{2}\partial^\mu\phi^a\partial_\mu\phi^a + \tfrac{1}{3}(1-q^2)\lambda\epsilon^{\mu\nu} f_{abc}\,\phi^a\partial_\mu\phi^b\partial_\nu\phi^c$$

free at WZW points $q^2 = 1$

• PCM_q is classically equivalent to ZM_q model
but not equivalent at the quantum level:
e.g. 1-loop β-functions of PCM and ZM are opposite [Nappi 79]

• path integral dual of PCM: “non-abelian dual model” (NAD)
quantum equiv: same β-function [Fridling, Jevicki 84; Fradkin, AT 85]

$$\mathcal{L} = \tfrac{1}{\lambda^2}\mathrm{tr}\big[J^\mu J_\mu + \lambda\epsilon^{\mu\nu}\, Y F_{\mu\nu}(J)\big]$$

$$\mathcal{L}_{\rm NAD} = -\tfrac{1}{2}\big[\eta_{\mu\nu}\delta^{ab} - 2\lambda\epsilon_{\mu\nu}f^{ab}{}_c Y^c\big]^{-1}\partial_\mu Y^a\partial_\nu Y^b$$

$$= -\tfrac{1}{2}\partial_\mu Y^a\partial^\mu Y^a - \lambda\epsilon^{\mu\nu}f_{abc}Y^a\partial_\mu Y^b\partial_\nu Y^c - \lambda^2 f_{ac}{}^h f_{bdh}Y^c Y^d\partial_\mu Y^a\partial^\mu Y^b + \dots$$

• three models are similar: global G symmetry: $X \to hXh^{-1}$
special cases of master Lagrangian

$$\mathcal{L}_{\rm p,q}(X) = -\tfrac{1}{2}\partial X^a\partial X^a - \tfrac{1}{12}{\rm p}\lambda^2 f_{ab}{}^e f_{cde}X^a X^c\partial_\mu X^b\partial^\mu X^d$$
$$+\tfrac{1}{3}{\rm q}\lambda\epsilon^{\mu\nu}f_{abc}X^a\partial_\mu X^b\partial_\nu X^c + \mathcal{O}(\lambda^3)$$

	PCM_q	ZM_q	NAD
p	1	0	12
q	q	$1-q^2$	-3

- all models classically integrable: admit flat Lax connection

$$L_+ = \tfrac{1}{2}(1-q+z\sqrt{1-q^2})\, J_+\,, \quad L_- = \tfrac{1}{2}(1+q+z^{-1}\sqrt{1-q^2})\, J_-$$

$$\partial_- J_+ - \partial_+ J_- + [J_-, J_+] = 0\,, \qquad (1-q)\partial_- J_+ + (1+q)\partial_+ J_- = 0$$

$$\text{PCM}_q : \quad J_\pm = g^{-1}\partial_\pm g$$

ZM_q : same Lax with $J_\mu \equiv \lambda(\epsilon_{\mu\nu}\partial^\nu\phi - q\partial_\mu\phi)$

NAD: same Lax with $J^\mu = -\lambda\epsilon^{\mu\nu}(\partial_\nu Y + [J_\nu, Y])$

IR ambiguties in massless 2d tree amplitudes

• 2d the mass-shell $k^2 \equiv -k_0^2 + k_1^2 = 0$ factorizes

$$k_+ k_- = 0\,, \qquad\qquad k_\pm \equiv \pm k_0 + k_1$$

$k_+ = 0$ and $k_- = 0$ (left- and right-moving)

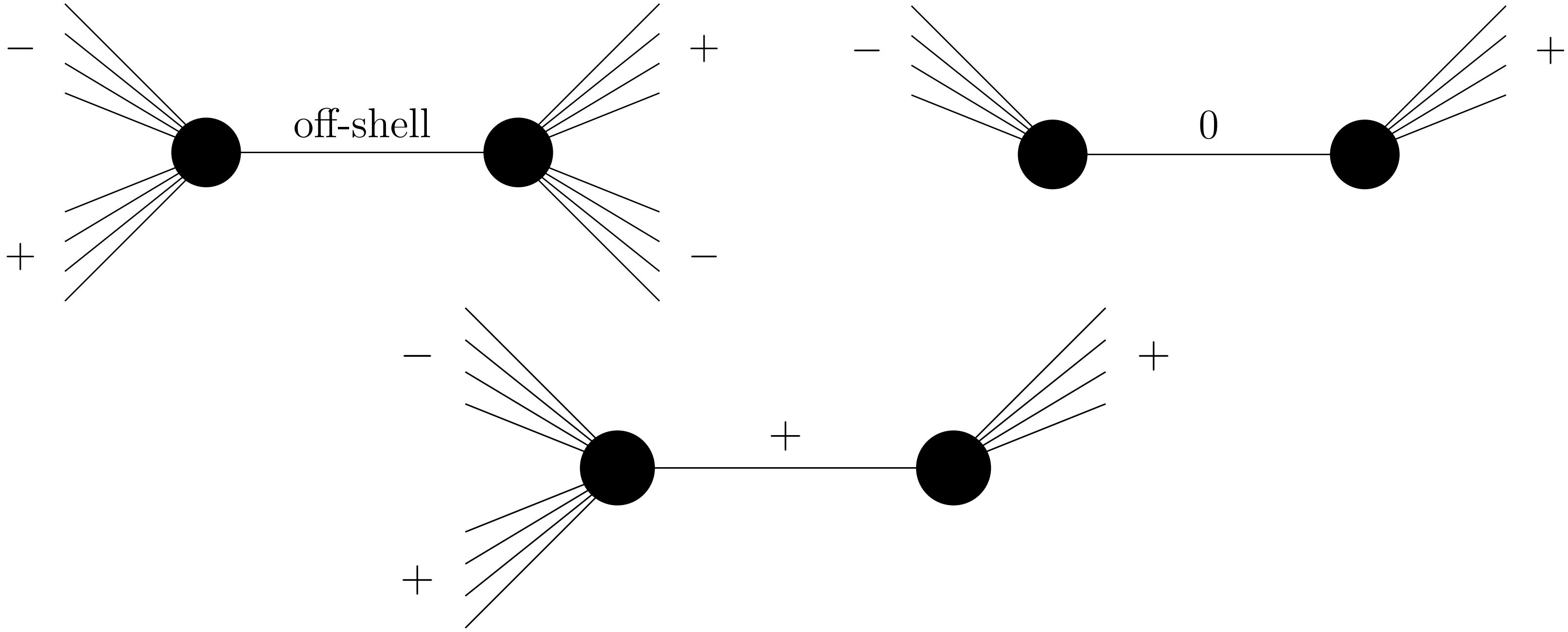
• conservation of momentum: separate for left- and right-movers

$$\sum_i k_\mu^{(i)} = 0\,, \qquad \sum_j l_\mu^{(j)} = 0\,, \qquad\qquad k_+^{(i)} = 0\,,\ \ l_-^{(j)} = 0$$

• implies two types of possible on-shell IR divergences:
when internal propagator blows up (u=internal momentum)

$$\text{type 1:}\quad \mathrm{u}_\mu = 0\,; \qquad\qquad \text{type 2:}\quad \mathrm{u}^2 = 0\,,\ \mathrm{u}_\mu \neq 0$$

type 1: particles on sides of propagator of opposite chirality
type 2: particles on one side of propagator are of same chirality



• classically scale invariant σ-models : $X^n \partial X \partial X$ interactions

infinite propagator $\times$ vanishing vertex factor

possible $\frac{0}{0}$ ambiguities $\frac{V}{\mathrm{u}^2}$ with $V \to 0$ and $\mathrm{u}^2 \to 0$

when external legs go on-shell

• one way to resolve: $i\epsilon$-regularization $\frac{V}{\mathrm{u}^2} \to \frac{V}{\mathrm{u}^2 - i\epsilon}$

vanishing of V implies such $\frac{0}{0}$ terms simply set to zero

• "massive regularization": add $m \to 0$ for all fields
different from $i\epsilon$-regularization: mass shell conds modified

$$k_+^{(i)} = 0 \;\to\; k_+^{(i)} = -\frac{m^2}{k_-^{(i)}}\,, \qquad l_-^{(j)} = 0 \;\to\; l_-^{(j)} = -\frac{m^2}{l_+^{(j)}}$$

• type 1 terms vanish in $i\epsilon$-reg

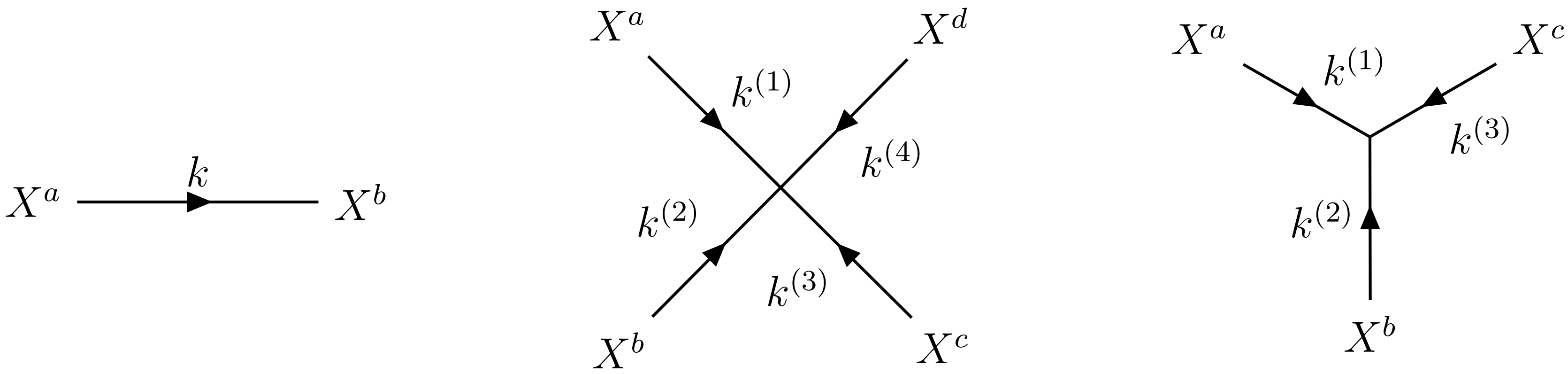
$\frac{V_1(\mathrm{u})V_2(\mathrm{u})}{\mathrm{u}^2 - i\epsilon} \to 0$ as $V_{1,2} \to 0$ as $\mathrm{u}_\mu \to 0$

and in massive reg: u_μ, V_1 and V_2 are $\mathcal{O}(m^2)$

$\frac{V_1(\mathrm{u})V_2(\mathrm{u})}{\mathrm{u}^2 + m^2} = \frac{\mathcal{O}(m^2)\mathcal{O}(m^2)}{\mathcal{O}(m^4) + m^2} \to 0$

• ambiguities: potential issues with equivalence theorem, preservation of (hidden) symmetries, T-duality, etc.

4-point scattering amplitudes



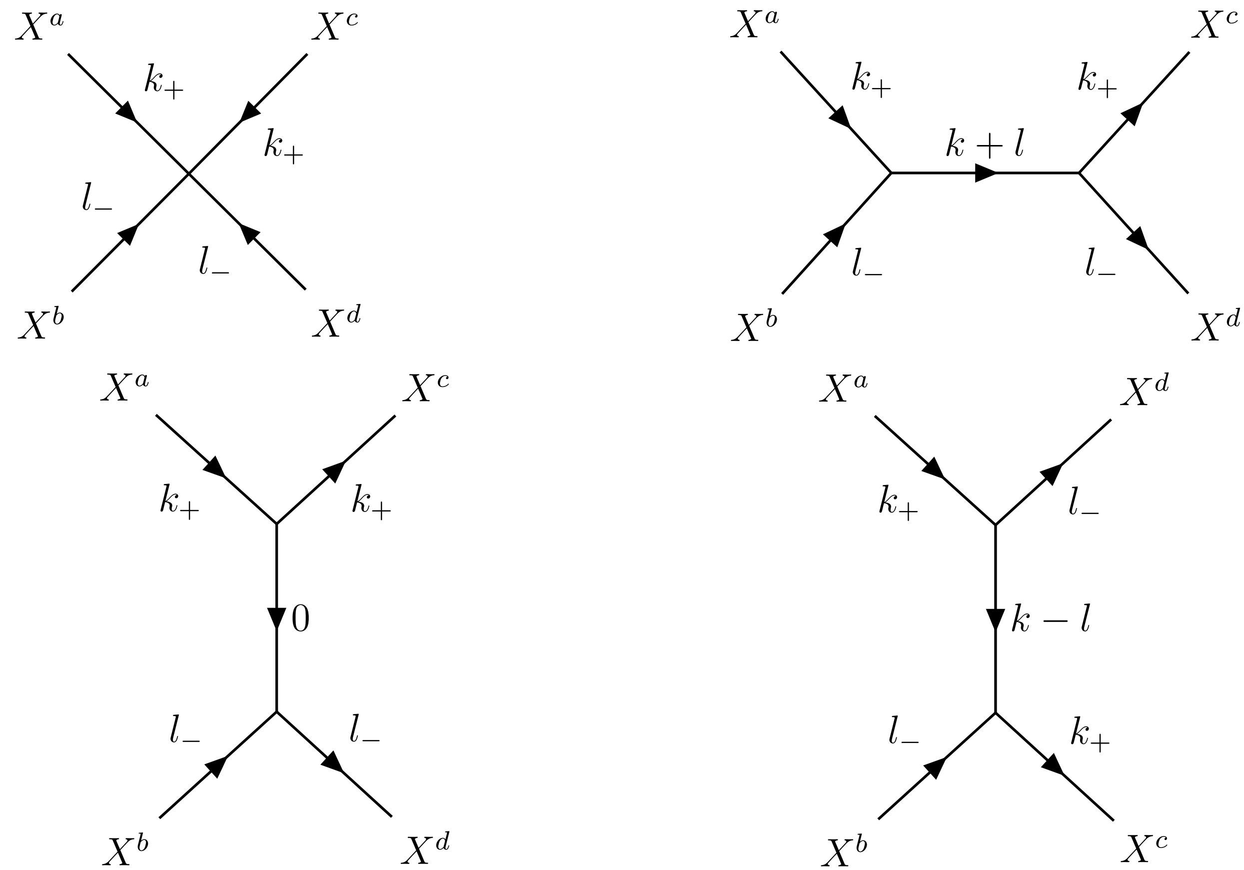
compute tree-level scattering of massless scalars $+- \to +-$

$$S[X^a(k_+)X^b(l_-) \to X^c(k_+)X^d(l_-)] = A_{\text{cont}} + A^{(\text{s})}_{\text{exch}} + A^{(\text{t})}_{\text{exch}} + A^{(\text{u})}_{\text{exch}}$$

$$A_{\text{cont}} = \tfrac{1}{2^4}\tfrac{1}{3} i\text{p}\lambda^2 \left(f^{abm} f^{cd}{}_m + 2 f^{acm} f^{bd}{}_m + f^{adm} f^{bc}{}_m \right) k_+ l_-$$

$$A^{(\text{s})}_{\text{exch}} = \tfrac{1}{2^4}\tfrac{i}{2}(2i\text{q}\lambda)^2 (f^{ab}{}_m \epsilon_{\mu\nu} k^\mu l^\nu) \frac{\gamma^{mn}}{(k+l)^2} (f^{cd}{}_n \epsilon_{\rho\sigma} k^\rho l^\sigma) = -\tfrac{i}{16}\text{q}^2\lambda^2 f^{abm} f^{cd}{}_m\, k_+ l_-$$

$$A^{(\text{u})}_{\text{exch}} = \tfrac{1}{2^4}\tfrac{i}{2}(2i\text{q}\lambda)^2 (f^{ad}{}_m \epsilon_{\mu\nu} k^\mu l^\nu) \frac{\gamma^{mn}}{(k-l)^2} (f^{bc}{}_n \epsilon_{\rho\sigma} l^\rho k^\sigma) = -\tfrac{i}{16}\text{q}^2\lambda^2 f^{adm} f^{bc}{}_m\, k_+ l_-$$



- t-channel: type 1 ambiguity = 0 in both $i\epsilon$- and massive reg

$$A^{(\mathrm{t})}_{\mathrm{exch}}\Big|_{\mathrm{u}_\mu\to 0} = \tfrac{i}{8}\mathrm{q}^2\lambda^2 f^{acm} f^{bd}{}_m \frac{(\epsilon_{\mu\nu}k^\mu \mathrm{u}^\nu)(\epsilon_{\rho\sigma}l^\rho \mathrm{u}^\sigma)}{\mathrm{u}^2} \quad\to\quad 0$$

$$S[X^a(k_+)X^b(l_-) \to X^c(k_+)X^d(l_-)] = \tfrac{i}{32}\,\kappa\,\lambda^2 f^{acm} f^{bd}{}_m\, k_+ l_-$$

	PCM_q	ZM_q	NAD
$\kappa \equiv \mathrm{p} - \mathrm{q}^2:$	$1-q^2$	$-(1-q^2)^2$	3

- $+- \to +-$ vanishes in WZW model ($q^2 = 1$)

(cf. decoupling of left and right modes in classical eqs)

- classically dual PCM_q and ZM_q : different tree amplitudes
- path integral dual PCM and NAD: different tree amplitudes
- classical solutions are in one-to-one correspondence

(relations between integrable structures or Lax pairs)
but need not have equivalent massless S-matrices
tree-level S-matrix = action on solution with asymptotic b.c.
but classical actions not same

• also elementary scattering fields are non-locally related:

PCM vs. ZM: $J_\mu = e^{-\lambda X}\partial_\mu e^{\lambda X} \to J^\mu = \lambda\epsilon^{\mu\nu}\partial_\nu\phi$

PCM vs. NAD: $J_\mu = e^{-\lambda X}\partial_\mu e^{\lambda X} \to J^\mu = -\lambda\epsilon^{\mu\nu}(\partial_\nu Y + [J_\nu, Y])$

• different discrete symmetries: PCM is parity-inv
while ZM and NAD contain parity-odd interactions
$\to$ different S-matrices

• still, relation between classical solutions suggests
some map between S-matrix elements
expected from off-shell duality of quantum correlators
in PCM and NAD: $J \to f(Y)\partial Y$
should also imply relations between
certain on-shell amplitudes

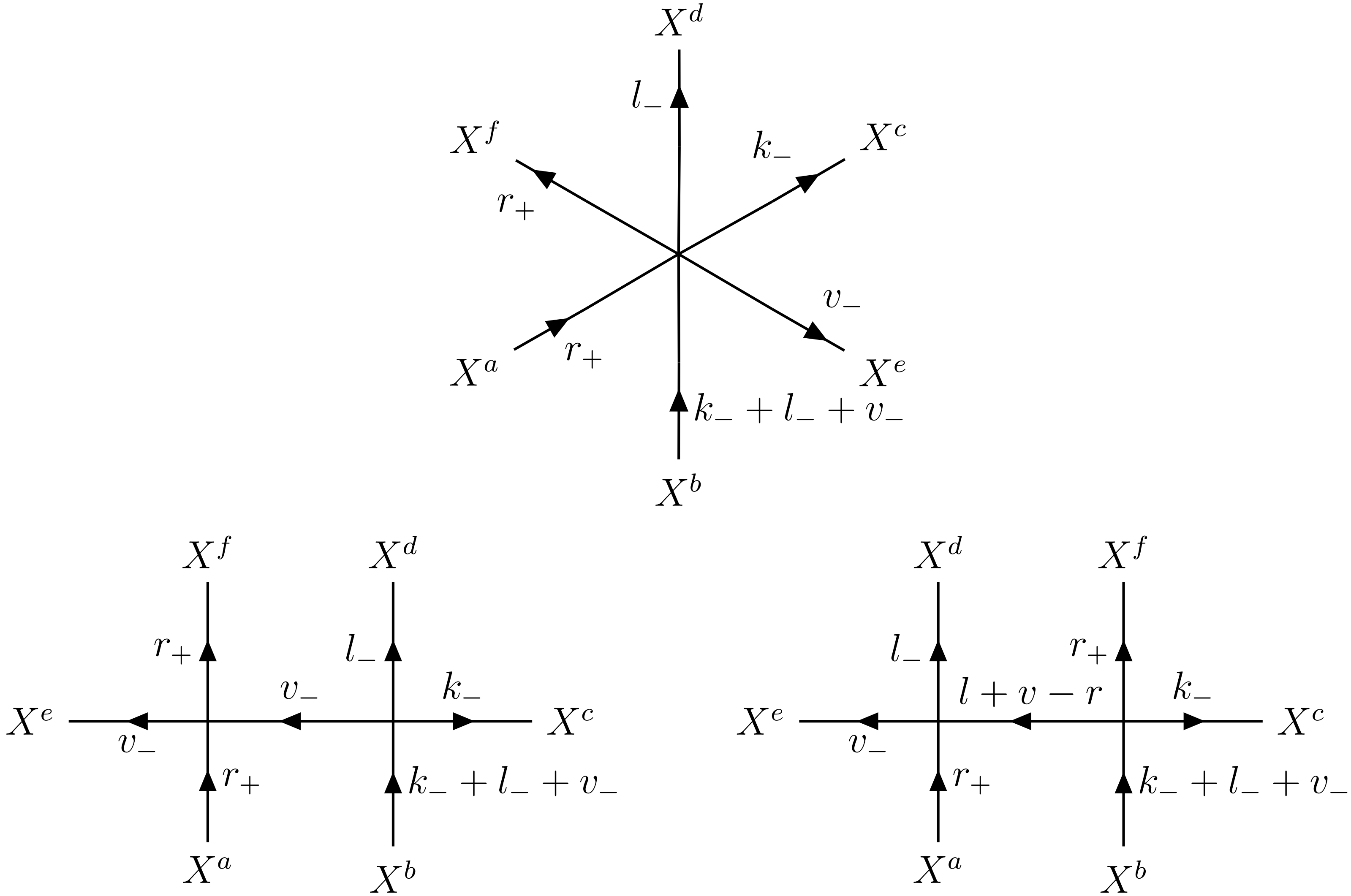
Higher-point amplitudes: particle production

• PCM, ZM and NAD (and S^N coset) classically integrable
but their massless tree level S-matrices fail to factorize
and contain non-zero particle production amplitudes:
standard lore about factorization in integrable models
fails in massless case (earlier indications [Nappi 80; Figueirido 89])

• $n > 4$ amplitudes have both type 1 and 2 IR ambiguities
similar non-zero results in $i\epsilon$- and massive regs

$2 \to 4$ amplitudes in PCM and $SO(N+1)/SO(N)$ model

• PCM for $G = SU(2)$, $f_{abc} = \epsilon_{abc}$ $(a, b = 1, 2, 3)$
particle-production: $+- \to - - -+$ and $+- \to - - ++$
contact term from 6-vertex + exchanges with two 4-vertices


X^d
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X^c
v_-
r_+
$k_- + l_- + v_-$
X^a
X^b

$$S[X^a(r_+)X^b(k_-+l_-+v_-)\to X^c(k_-)X^d(l_-)X^e(v_-)X^f(r_+)]$$
$$= \tfrac{i}{16}\lambda^4\, r_+\Big[-\tfrac{k_-l_-(k_-+l_-+2v_-)}{(k_-+v_-)(l_-+v_-)}\delta_{ab}\delta_{cd}\delta_{ef}+\tfrac{v_-(l_--k_-)(k_-+l_-+v_-)}{(k_-+v_-)(l_-+v_-)}\delta_{ad}\delta_{be}\delta_{cf}$$
$$-(a\leftrightarrow f)\Big]+(\text{cycle }\ k,c;\ l,d;\ v,e)$$

$$S[X^a(v_++r_+)X^b(k_-+l_-)\to X^c(k_-)X^d(l_-)X^e(v_+)X^f(r_+)]$$
$$= -\tfrac{i}{16}\lambda^4\left[v_+k_-\delta_{af}\delta_{bd}\delta_{ce}+(\text{cycle}\,k,c;l,d;-k-l,b)\right]+(\text{cycle}\,v,e;r,f;-$$

- same results for $S^N = SO(N+1)/SO(N)$ model

(e.g. $SU(2)$ PCM: $SU(2)\sim S^3\sim SO(4)/SO(3)$)

$$\mathcal{L}_{S^N} = -\tfrac{1}{2}\left[(\partial X^a)^2+(\partial X^{N+1})^2\right] = -\tfrac{1}{2}\Big[(\partial X^a)^2+\frac{\lambda^2(X^a\partial X^a)^2}{1-\lambda^2(X^a)^2}\Big]$$

e.g. for $(a,b,c,d,e,f) = (1,1,2,2,2,2)$

$$S[X^1(r_+)X^1(k_- + l_- + v_-) \to X^2(k_-)X^2(l_-)X^2(v_-)X^2(r_+)]$$
$$= -\tfrac{i}{16}\lambda^4\, r_+(k_- + l_- + v_-)\,,$$
$$S[X^1(r_+ + v_+)X^1(k_- + l_-) \to X^2(k_-)X^2(l_-)X^2(v_+)X^2(r_+)]$$
$$= -\tfrac{i}{16}\lambda^4\,(r_+ + v_+)(k_- + l_-)$$

$2 \to 3$ amplitude in ZM_q model

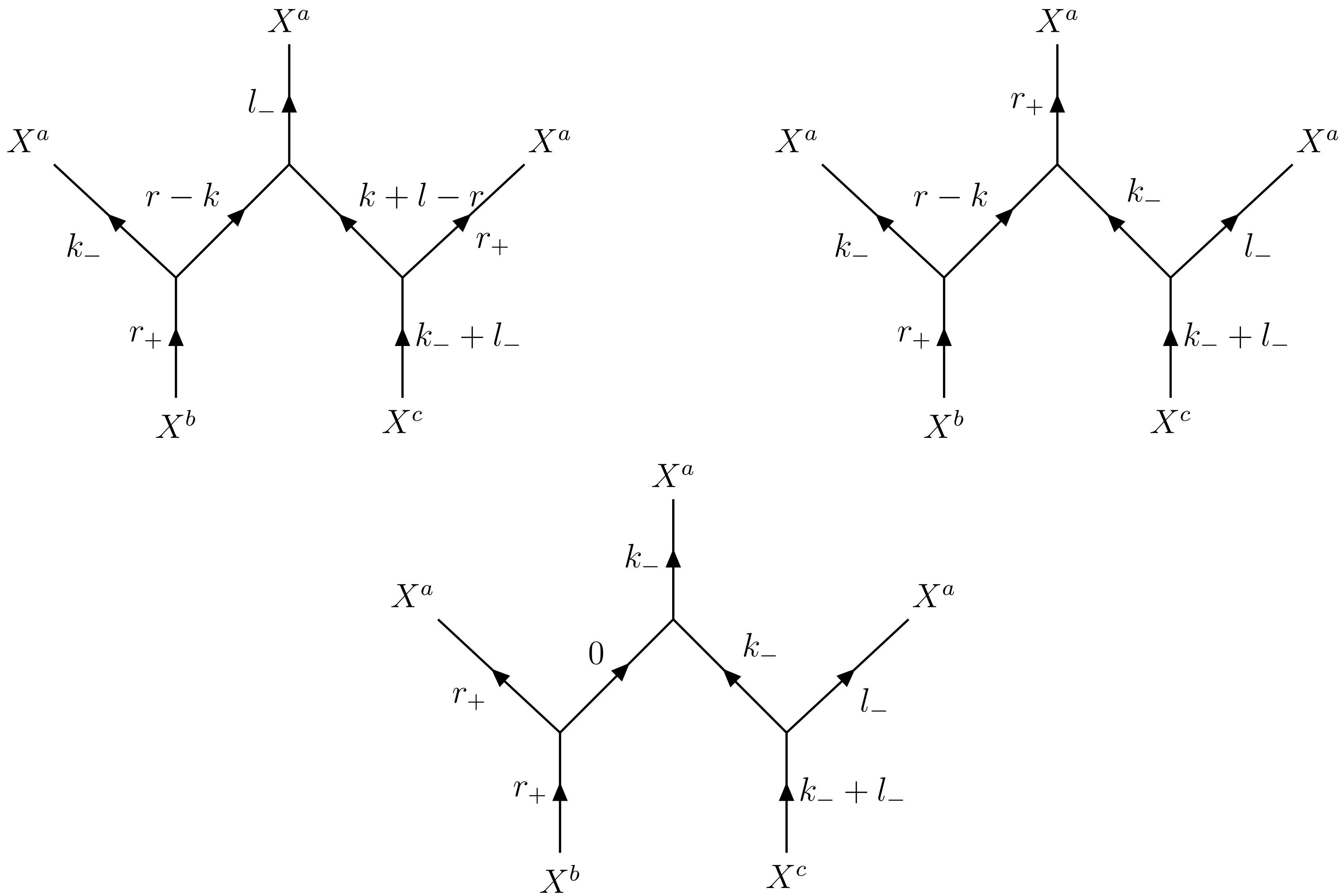
non-zero 5-point amplitude $+- \to +--$

exchanges with three 3-vertices and two internal propagators

outgoing particles with same $a = d = e$

$$S[X^b(r_+)X^c(k_- + l_-) \to X^a(r_+)X^a(k_-)X^a(l_-)] = A^{(1)} + A^{(2)} + A^{(3)}$$

$A^{(1)}$ – unambg.; $A^{(2)}$ – type 2; $A^{(3)}$ – type 1 and 2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$r-k$
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l_-
r_+
k_-+l_-
X^b
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k_-
X^a
X^a
0
k_-
r_+
l_-
r_+
k_-+l_-
X^b
X^c

• in massive regularization $A^{(1)} = A^{(2)},\;\; A^{(3)} = 0$

$$S[X^b(r_+)X^c(k_- + l_-) \to X^a(r_+)X^a(k_-)X^a(l_-)]$$
$$= -\tfrac{i}{64}\lambda^3\,(1-q^2)^3\, f^{abd} f^{ae}{}_d f^a{}_e{}^c\, r_+(k_- + l_-)$$

e.g. in $SU(2)$ case $f^{abd} f^{ae}{}_d f^a{}_e{}^c = 16\epsilon^{abc}$
non-zero except for $q = \pm 1$ when theory is free

• in $i\epsilon$-regularization (apparently used in [Nappi 80])
all ambiguous contributions resolved as zero:
$A^{(2)} = 0 = A^{(3)}$ and thus $S_{i\epsilon} = A^{(1)} = \frac{1}{2} S_{\text{mass}}$

$2 \to 3$ amplitude in NAD model

$+- \to +--$ in $SU(2)$ case with outgoing $a = d = e$

3- and 4-vertices plus 5-point

$$\mathcal{L}^{(5)}_{\text{NAD}} = -\tfrac{1}{2}\lambda^3 \epsilon^{\mu\nu}\epsilon_{abc}X^d X^d X^a \partial_\mu X^b \partial_\nu X^c$$

using massive regularization:

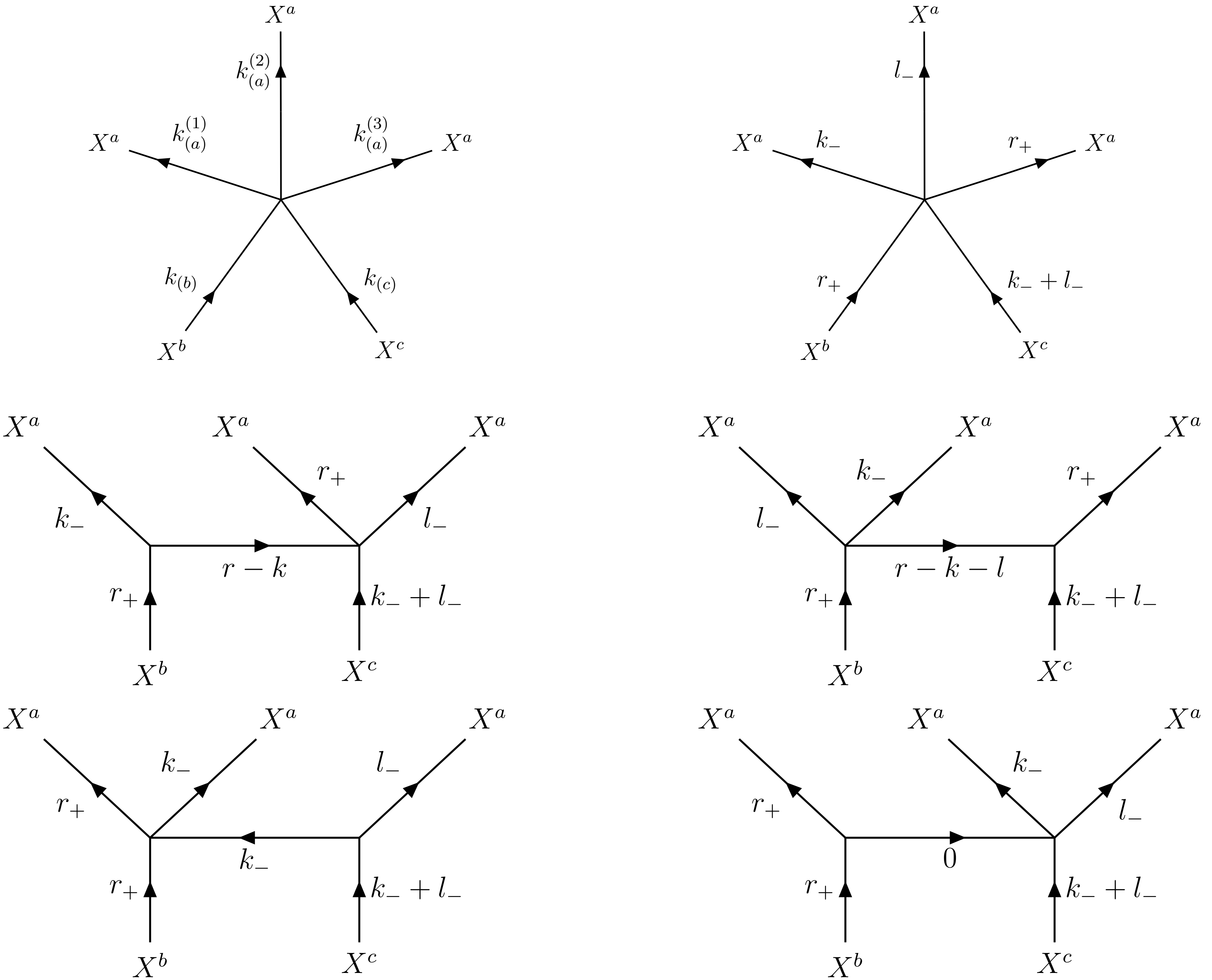
$$\begin{aligned} S[X^b(r_+)X^c(k_- + l_-) \to X^a(r_+)X^a(k_-)X^a(l_-)] \\ = A_{\text{3-v}} + A_{\text{cont}} + A_{\text{unamb}} + A_{\text{amb}} \\ = -\tfrac{i}{4}\lambda^3 \epsilon^{abc} r_+(k_- + l_-) \end{aligned}$$

• happens to coincide with same amplitude in ZM model (?)

• PCM vs NAD: different coeff. at 4-point level,

no 5-point in PCM (parity-invariant) but non-zero one in NAD:

path integral duality does not imply equiv. of massless S-matrices


X^a
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X^a X^a X^a
r_+ k_- l_-
0
r_+
$k_- + l_-$
X^b X^c

$2 \to 3$ amplitude in PCM_q

non-zero if WZ term present; in massive regularization

$$S[X^b(r_+)X^c(k_- + l_-) \to X^a(r_+)X^a(k_-)X^a(l_-)]$$
$$= -\tfrac{i}{4}q(q^2-1)\lambda^3\epsilon_{abc}r_+(k_- + l_-)$$

• particle production at the 5-point level unless $q \neq \pm 1$
(massless S-matrix of WZW trivial: left/right decouple)

• in $i\epsilon$-reg (used in [Figueirido 89]) result is $\frac{1}{2}$ of above

Massless S-matrix in doubled formalism

IR ambiguities in amplitudes of 2d chiral scalars:

may be alternative approach?

"doubled" sigma model: [AT 91; Roiban, AT 12]

treat left and right chiral scalars as independent off-shell fields

(relax off-shell 2d Lorentz inv)

$$G_{mn}\partial^\mu X^m \partial_\mu X^n + \epsilon^{\mu\nu} B_{mn}\partial_\mu X^m \partial_\nu X^n$$
$$= G_{mn}(\dot{X}^m \dot{X}^n - X'^m X'^n) - B_{mn}\dot{X}^m X'^n$$

$$\hat{\mathcal{L}}(X, X', P) = P\dot{X} - H(X, X', P), \ P_n = \frac{\partial \mathcal{L}}{\partial \dot{X}^n} \ \rightarrow \ P_n = \partial_1 \tilde{X}_n$$

doubled Lagrangian

$$\hat{\mathcal{L}} = \dot{X}^n \tilde{X}'_n - \tfrac{1}{2}(G_{mn} - B_{mk}G^{kl}B_{ln})X'^m X'^n - \tfrac{1}{2}G^{mn}\tilde{X}'_m \tilde{X}'_n + B_{mk}G^{kn}X'^m \tilde{X}'_n$$

$$\hat{S}(X, \tilde{X}) = \tfrac{1}{2}\int d^2\sigma \big(\Omega_{IJ}\dot{\mathrm{X}}^I \mathrm{X}'^J - M_{IJ}\mathrm{X}'^I \mathrm{X}'^J\big)\,, \qquad \mathrm{X}^I = (X^m, \tilde{X}^m)$$

$$\Omega = (0, I; I, 0), \quad M = (G - BG^{-1}B;\ BG^{-1};\ -G^{-1}B;\ G^{-1})$$

expand $G_{mn} = \delta_{mn} + H_{mn}(X)$ and introduce $X_\pm$

$$X^m = X_+^m + X_-^m \,, \qquad \tilde{X}^m = X_+^m - X_-^m \,, \qquad\qquad X_\pm^m = \tfrac{1}{2}(X^m \pm \tilde{X}^m)$$

free action ($\partial_\pm = \pm\partial_0 + \partial_1$)

$$\hat{\mathcal{L}}_0(X_+, X_-) = -\partial_1 X_+^n \partial_- X_+^n - \partial_1 X_-^n \partial_+ X_-^n \,, \qquad \partial_\pm = \pm\partial_0 + \partial_1$$

classical eqs: $\partial_1 \partial_- X_+ = 0 \to \partial_- X_+ = 0$

$$\partial_- X_+^n = 0 \,, \qquad \partial_+ X_-^n = 0 \,, \qquad \partial_\mp X_\pm^n \Big|_{|\sigma| \to \infty} = 0$$

natural for scattering of chiral scalars

$$\hat{\mathcal{L}} = -\, \partial_1 X_+^a \partial_- X_+^a - \partial_1 X_-^a \partial_+ X_-^a$$
$$- \, H_{ab}(X) \partial_1 X_+^a \partial_1 X_-^b - B_{ab}(X) \partial_1 X_+^a \partial_1 X_-^b + \mathcal{O}(B^2,\ H^2,\ HB)$$

on-shell S-matrix for X_+, X_- is Lorentz invariant

• linear order in H, B: no "chiral" vertices with only X_+ or X_-
no type 1 or type 2 ambiguities in simple exchange diagrams
with just one internal line
• T-duality (2d scalar-scalar duality):
is manifest symmetry in doubled formulation
Example: $G = 1 + \lambda^2 X^2$
$\mathcal{L} = -\frac{1}{2}(\partial X)^2 - \frac{1}{2}G(X)(\partial Y)^2 \,, \qquad \tilde{\mathcal{L}} = -\frac{1}{2}(\partial X)^2 - \frac{1}{2}G^{-1}(X)(\partial \tilde{Y})^2$
doubled Lagrangians are equivalent: $Y \leftrightarrow \tilde{Y}, \;\; G \to G^{-1}$

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_0 - \tfrac{1}{2}G(X)(\partial_1 Y)^2 - \tfrac{1}{2}G^{-1}(X)(\partial_1 \tilde{Y})^2$$
$$\hat{\mathcal{L}}_0 \equiv \tfrac{1}{2}\big(\partial_0 X \partial_1 \tilde{X} + \partial_1 X \partial_0 \tilde{X} + \partial_0 Y \partial_1 \tilde{Y} + \partial_1 Y \partial_0 \tilde{Y}\big)$$

in chiral basis $Y_\pm = \frac{1}{2}(Y \pm \tilde{Y})$ symmetry is [Roiban, AT 12]

$$\tilde{S} = (-1)^{n_-}\, S \,, \qquad Y_+ \to Y_+ \,, \qquad Y_- \to -Y_-, \quad X \to X$$

n_-= number of Y_- legs in amplitude

• compute PCM, etc., amplitudes in doubled formulation

$++ \to --$ for "interpolating" Lagrangian $\mathcal{L}_{\rm p,q}$:

coefficient $\kappa = {\rm p} - \frac{13}{9}{\rm q}^2$ instead of ${\rm p} - {\rm q}^2$

in standard approach with massive regularization:

$2 \to 4$ amplitude in $SU(2)$ PCM: $+- \to ---+$ is same

but $+- \to --++$ is different by $-\frac{5}{4}$

• why different? related to type 1 ambigs in standard approach

reason: non-local field-dependent transformation

between fields in standard and doubled actions

(cf. $\partial_a Y \to \epsilon_{ab} G^{-1}(X) \partial^b \tilde{Y}$ in T-duality case):

effectively different ways of how IR ambiguities

appear and are resolved

Open questions:

• tree-level massless scattering – IR ambiguities
particle creation in integrable models:
cannot use massless S-matrix in search for
new integrable σ-models ?
• massive regularization and $i\epsilon$
break usual link to integrability but
could there be a prescription consistent with integrability?
• massless particle-production amplitudes that are
free from IR ambiguities vanish in integrable models?
• is it possible to relax standard factorization condition
into some modified criterion?
(cf. no particle creation in partial colour ordered amplitudes)