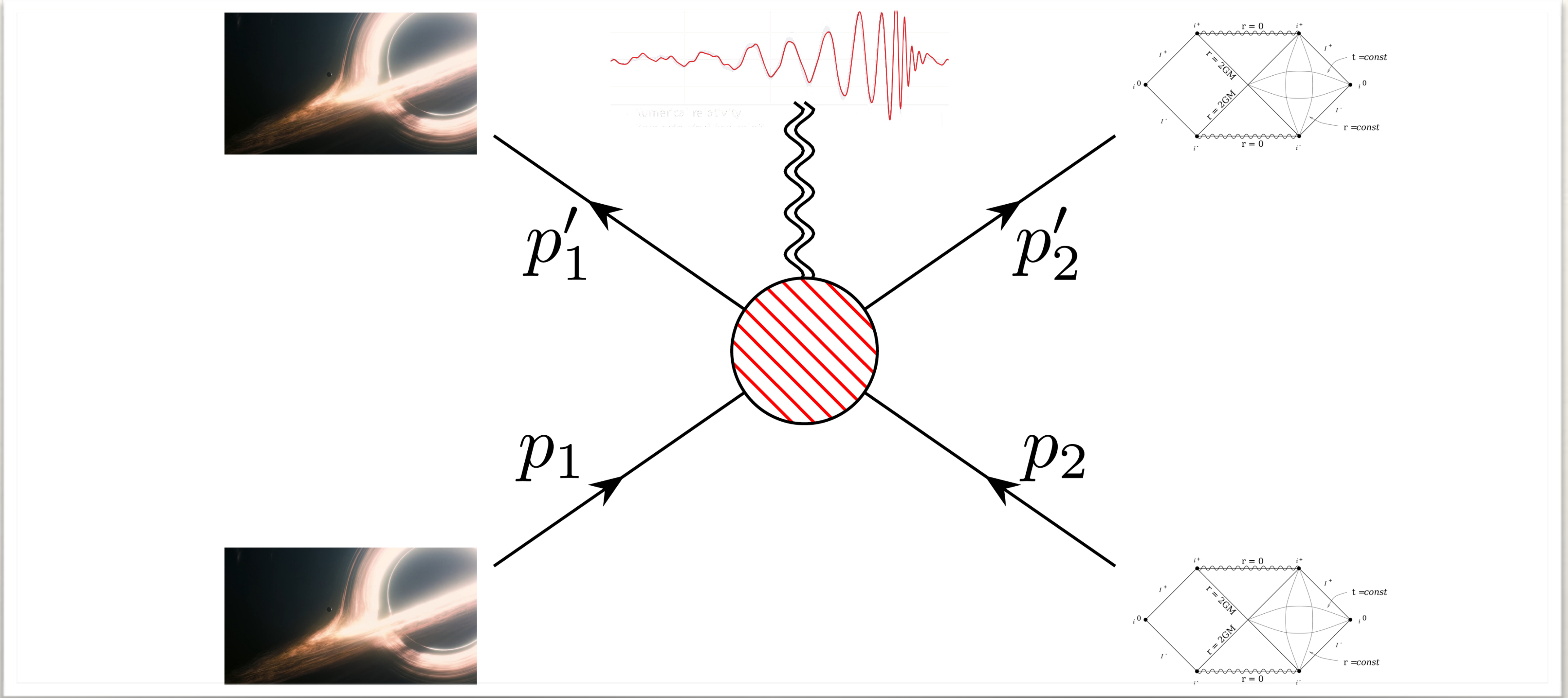
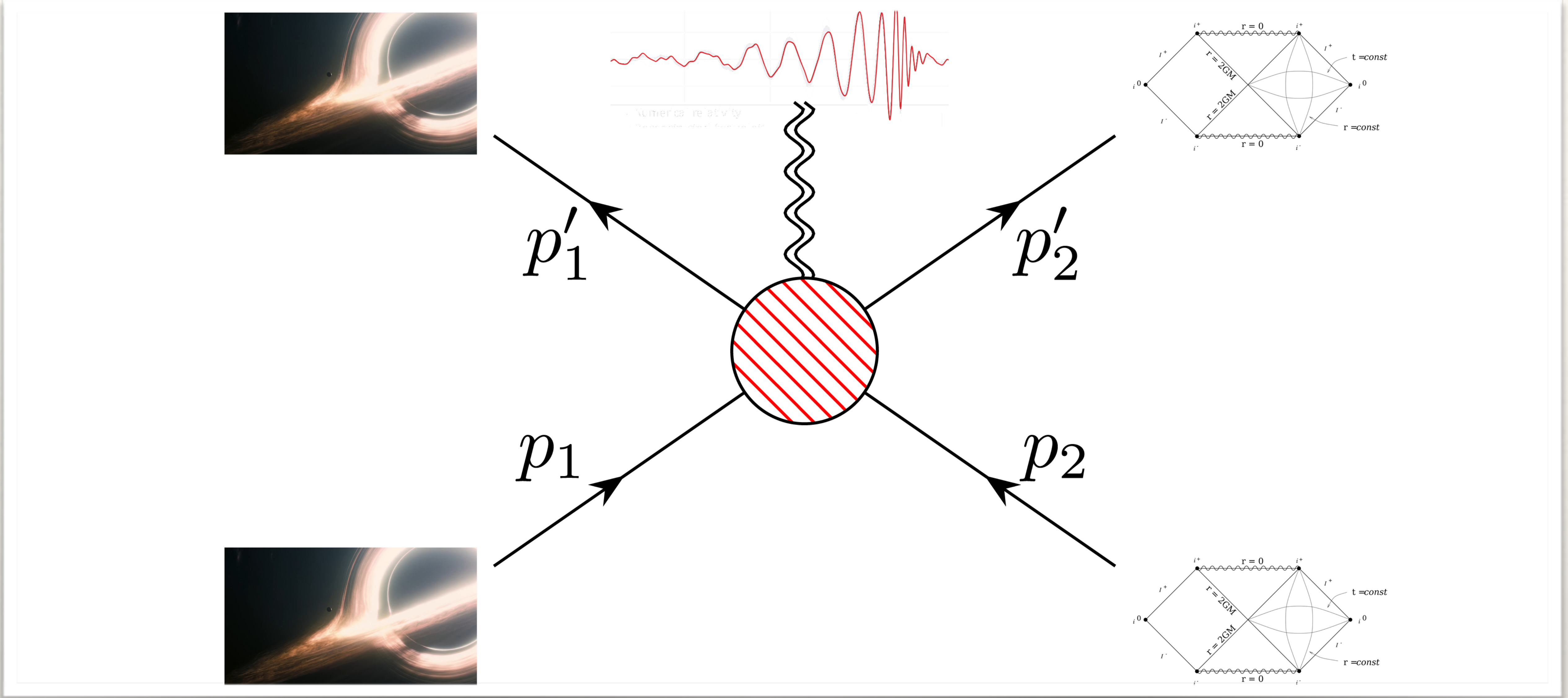


Quantum Amplitudes for Classical Physics

Donal O'Connell
University of Edinburgh

QCD meets gravity, Nordita, Dec 2018



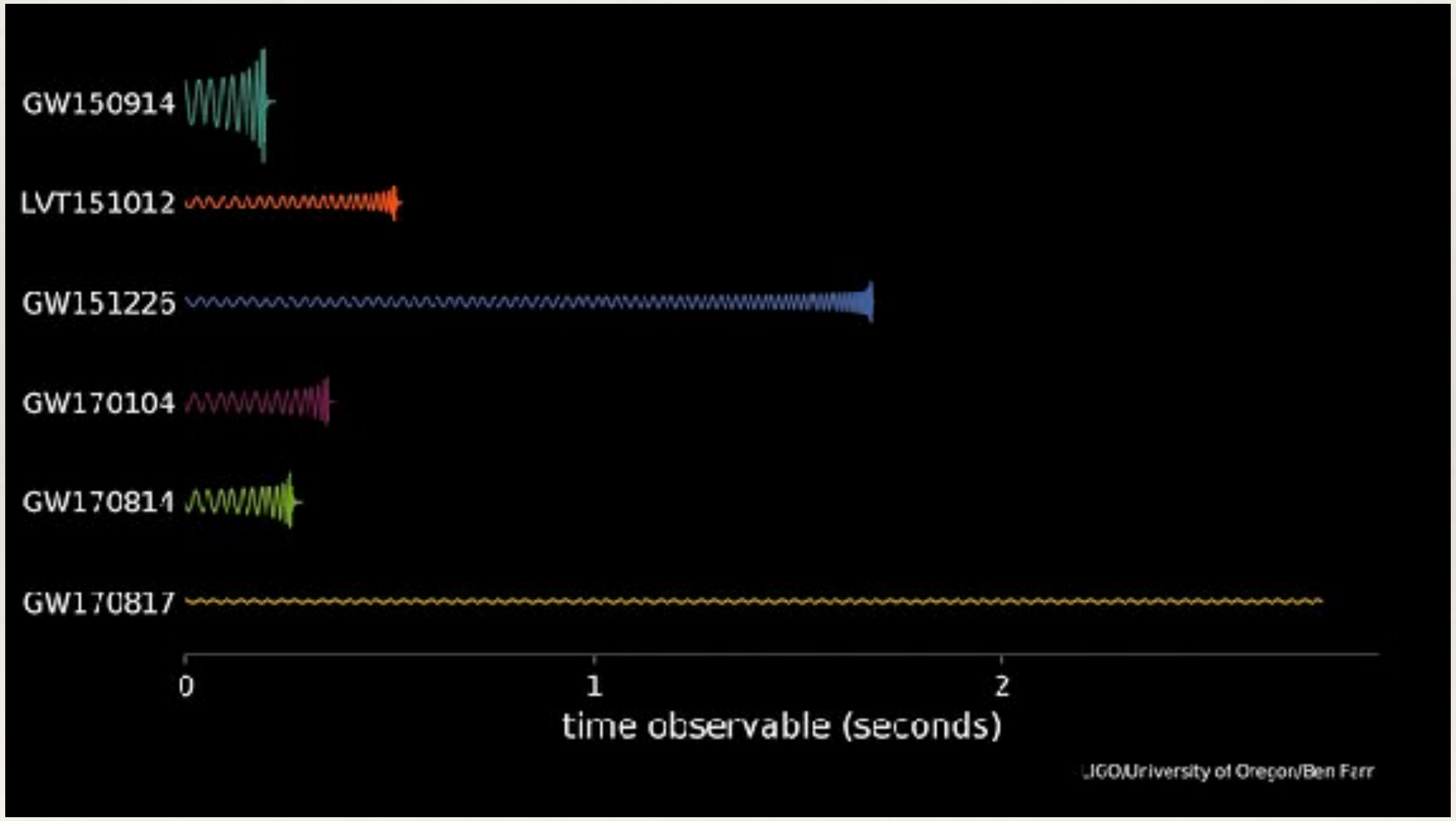
Quantum Amplitudes for Classical Physics

With David Kosower, Ben Maybee 1811.10950

QCD meets gravity, Nordita, Dec 2018

Invitation

- Last few years: great for gravitational physics
- LIGO & Virgo collaborations: gravitational wave data



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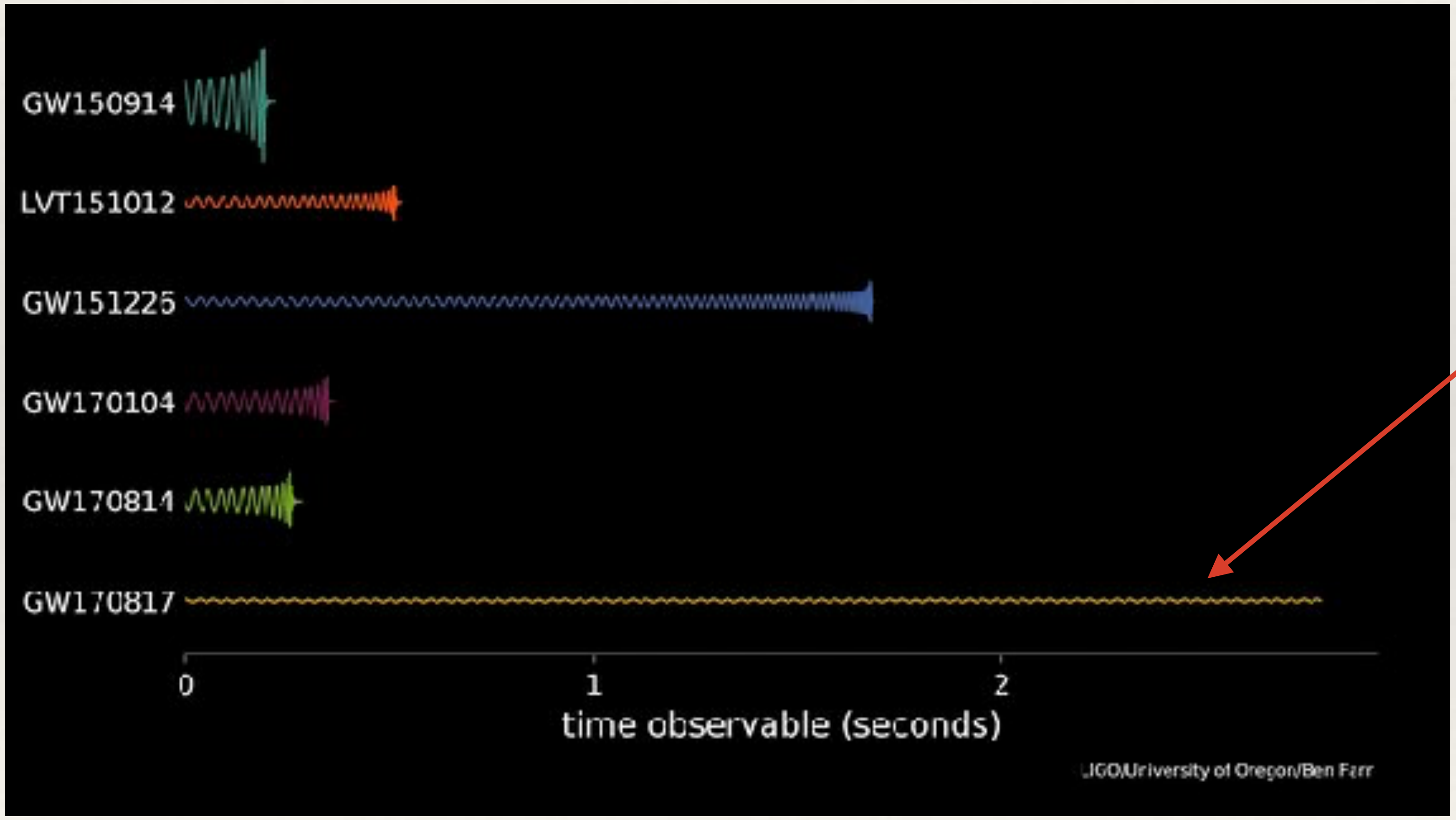
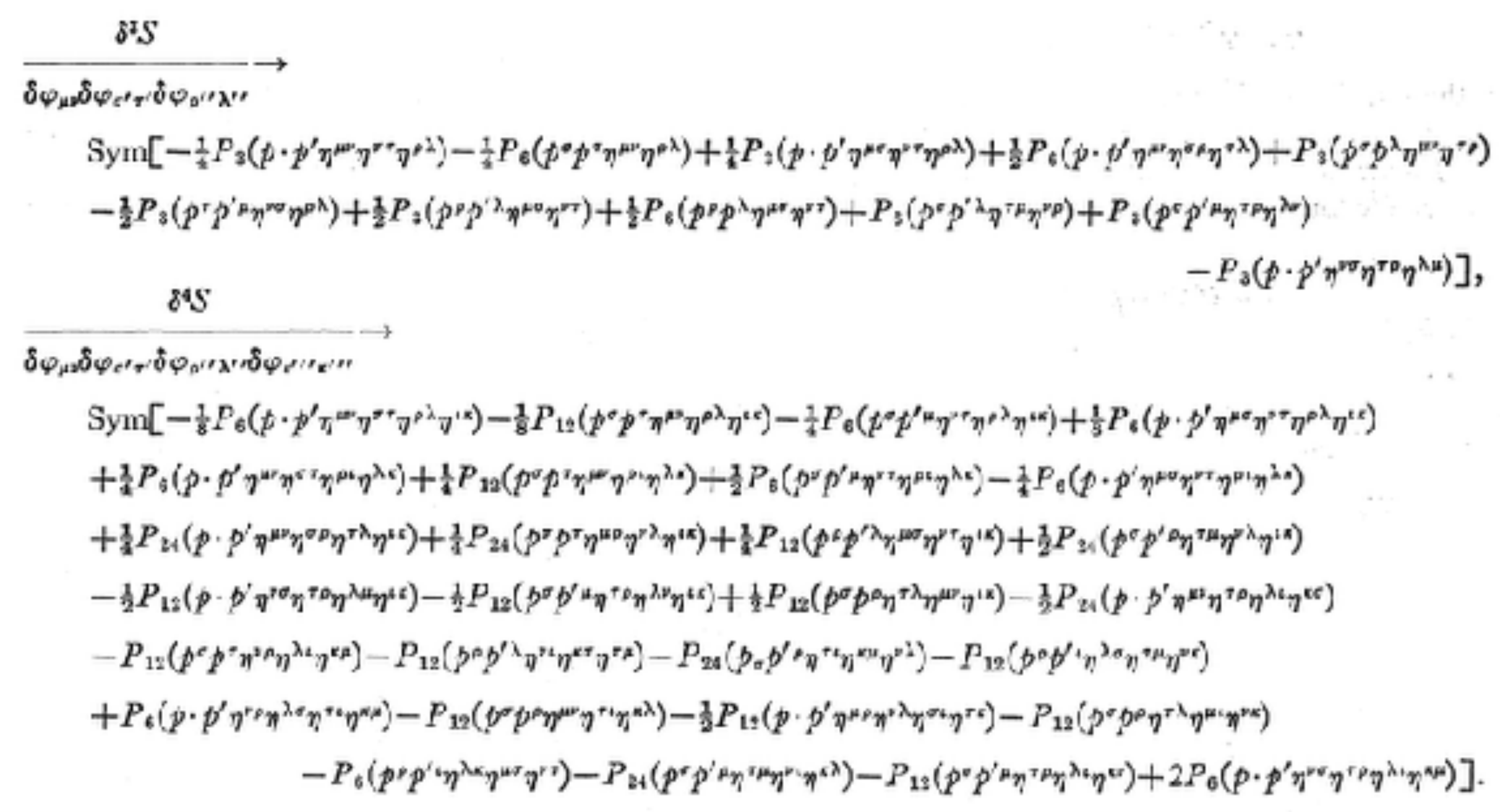


LIGO/University of Oregon/Ben Farr

Binary neutron star
~3000 cycles
~1 minute in band

Invitation

- Last few years: great for gravitational physics
- In parallel, major progress on theoretical quantum gravity
  - Quantum scattering amplitudes in gravity simpler than expected



Three point
vertex

Four point
vertex

de Witt

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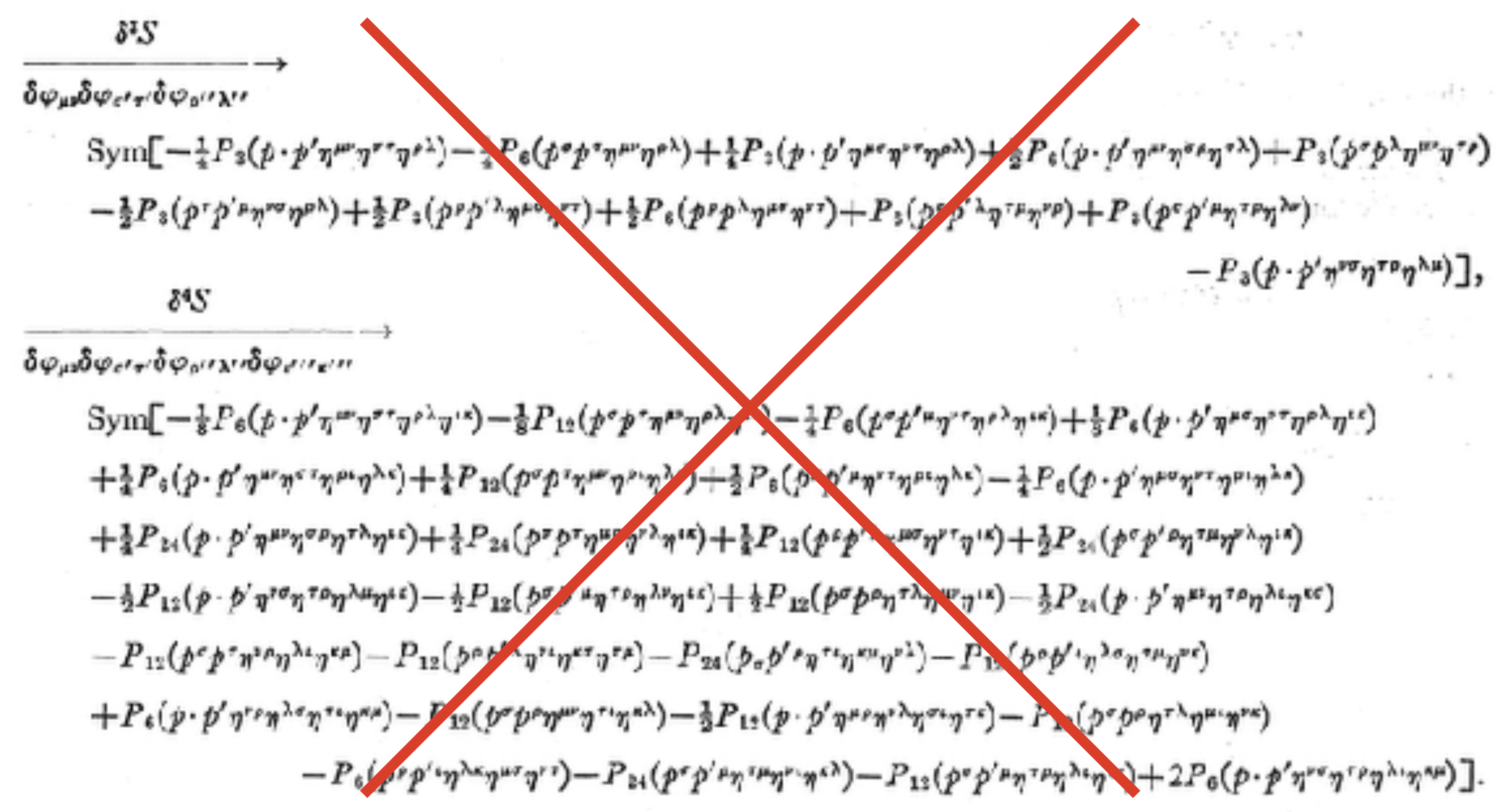


Three point vertex

Four point vertex

de Witt

Invitation

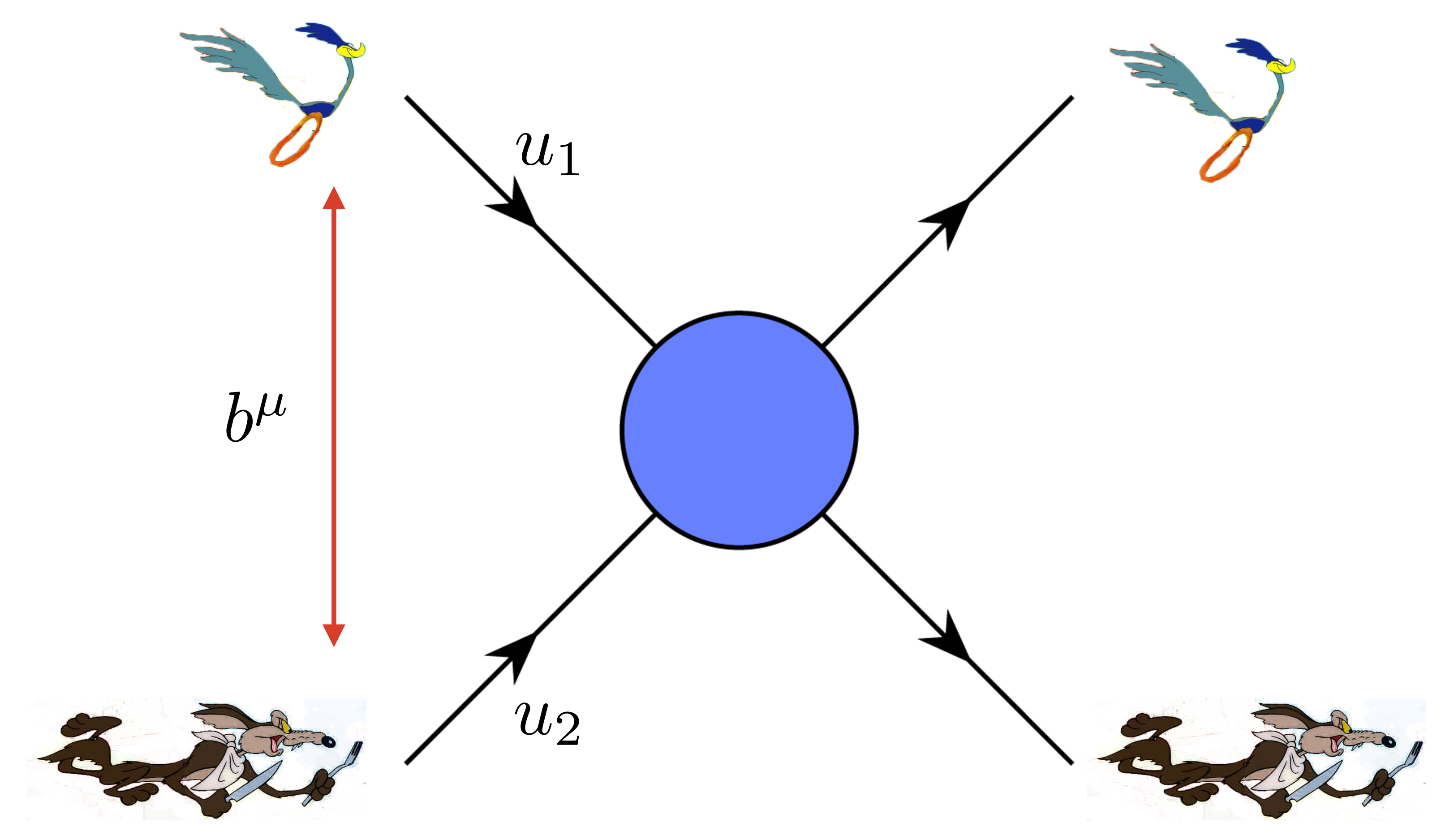
- Last few years: great for gravitational physics
- In parallel, major progress on theoretical quantum gravity
  - Quantum scattering amplitudes in gravity simpler than expected
- Sum of graphs = amplitude has special properties:
  - Obtain gravity amplitudes from much simpler YM amplitudes

→ *QCD meets gravity* conferences

- This talk: simple quantum gravity ⇒ simple classical gravity

Invitation

- This talk: simple quantum gravity ⇒ simple classical gravity
- Example: momentum deflection in classical scattering



Invitation

- This talk: simple quantum gravity ⇒ simple classical gravity
- Example: momentum deflection in classical fast scattering

$$\Delta p_1^\mu = -\frac{e^2}{2\pi}\frac{b^\mu}{b\cdot b}\frac{u_1\cdot u_2}{\sqrt{(u_1\cdot u_2)^2-1}} \qquad \Delta p_1^\mu = 4Gm_1m_2\frac{b^\mu}{b\cdot b}\frac{(u_1\cdot u_2)^2}{\sqrt{(u_1\cdot u_2)^2-1}}$$

EM Gravity

- Up to factors, just somehow square part of the EM expression
- We'll see this follows from a property of quantum amplitudes
  - Extends to all orders of perturbation theory, many observables

Invitation

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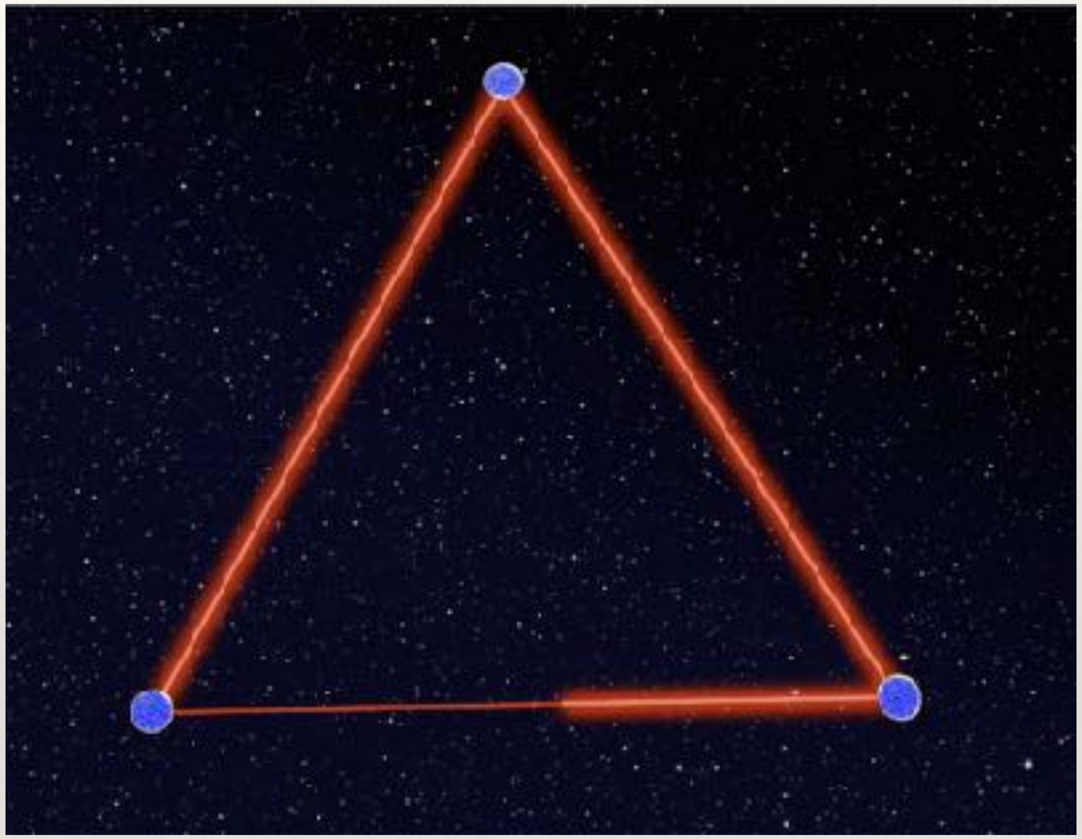
EM

Gravity

- Up to factors, just somehow square part of the EM expression
- We'll see this follows from a property of quantum amplitudes
  - Extends to all orders of perturbation theory, many observables

Invitation

- Theoretical progress is timely
- Bright future for gravitational wave observatories
    - Space-based: LISA
    - Ground-based: KAGRA, Voyager, Einstein Telescope,…
- High precision, large datasets
- We want high precision GR



Invitation

- Can extract interaction potential from scalar amplitudes *Donoghue, Bjerrum-Bohr, Damgaard, Torma, Holstein, Ross, Vanhove, Planté, Festuccia, Neill, Rothstein, Bern, Cheung, Roiban, Shen, Solon, Zeng*

  ⟶ Talks by Bern, Cheung

- Potential is flexible, but it requires some definition
- My philosophy: use on-shell, gauge invariant quantities
  - Experience with amplitudes is it pays to be on shell
  - Build formalism for computing classical observables which is valid to all orders

Quantum Amplitudes for Classical Physics

1. Background on amplitudes
2. Observables
3. Classical observables
4. Conclude

Amplitudes

- Time evolution operator from far past to far future is the S matrix:

$$U(-\infty, \infty) = S \qquad\qquad P(1 \to 2) = |\langle 2|S|1\rangle|^2$$

- Write $S = 1 + iT$

Amplitudes

- Time evolution operator from far past to far future is the S matrix:

$$U(-\infty, \infty) = S \qquad\qquad P(1 \to 2) = |\langle 2|S|1\rangle|^2$$

- Write $S = 1 + iT$

Nothing happens

Amplitudes

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"Transition matrix"

Amplitudes

- Time evolution operator from far past to far future is the S matrix:

$$U(-\infty, \infty) = S \qquad P(1 \to 2) = |\langle 2|S|1\rangle|^2$$

- Write $S = 1 + iT$

“Transition matrix”

- The matrix elements of T on momentum states are the amplitudes

$$\mathcal{A}(p_1 \ldots p_n \to q_1 \ldots q_m)\delta^4\left(\sum p_i - \sum q_j\right) = \langle q_1 \ldots q_m|T|p_1 \ldots p_n\rangle$$

- Compute eg with Feynman diagrams

Amplitudes

Kawai, Lewellen, Tye
Bern, Carrasco, Johansson

$$A_{\text{YM}} = \sum_{i \in \text{diagrams}} \frac{c_i n_i}{D_i}$$

Amplitudes

- Key idea to simplify gravity amplitudes:
  - “Double copy”: gravity = YM^2

Kawai, Lewellen, Tye
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Scalar propagator denominators

Amplitudes

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Kawai, Lewellen, Tye
Bern, Carrasco, Johansson

Gauge group colour factors: $f^{abe} f^{cde} \cdots$

$$A_{\mathrm{YM}} = \sum_{i \in \mathrm{diagrams}} \frac{\boxed{c_i} n_i}{D_i}$$

Amplitudes

- Key idea to simplify gravity amplitudes:
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Kawai, Lewellen, Tye
Bern, Carrasco, Johansson

Kinematic numerator: $\epsilon_1 \cdot (p_2 - p_3) \cdots$

$$A_{\text{YM}} = \sum_{i \in \text{diagrams}} \frac{c_i \boxed{n_i}}{D_i}$$

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- If numerators have property of colour-kinematics duality then

$$\Rightarrow A_{\text{GR}} = \sum_{i \in \text{diagrams}} \frac{n_i n_i}{D_i}$$

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$$c_i + c_j + c_k = 0$$

$$\Rightarrow n_i + n_j + n_k = 0$$

- Greatly simplifies calculation of gravity amplitudes

→ *Talks by Roiban, Edison, Kälin*

  - One issue: double copy introduces unwanted dilaton, axion

→ *Talks by Bern and Shen*

Amplitudes

- Now let's restore the factors of $\hbar$
- Easy to do this by dimensional analysis…
    - … once you know the right dimensions
- Consistent to choose dimension of amplitude

$$[\mathcal{A}_n] = M^{4-n}$$

(even when $\hbar \neq 1$)

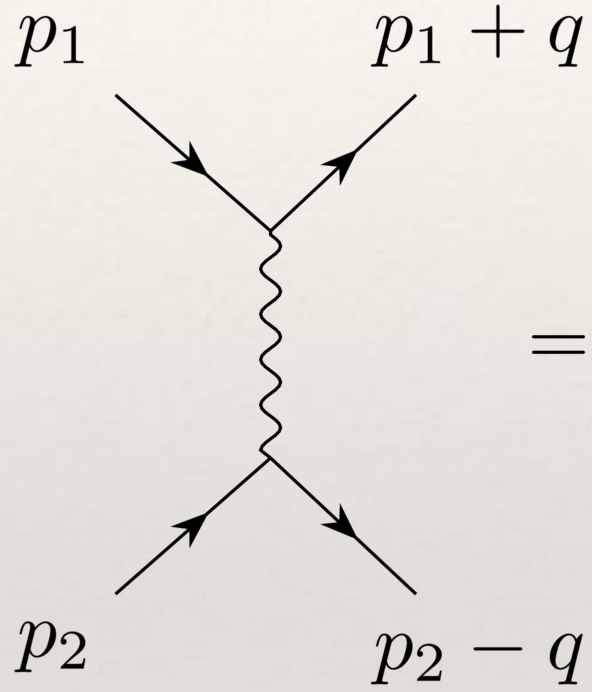
Dimension of mass

Amplitudes

- “Messenger”: photon or graviton, coupling g

$\hbar = 1$

$$i\mathcal{A}(p_1 p_2 \to p_1' p_2') =$$



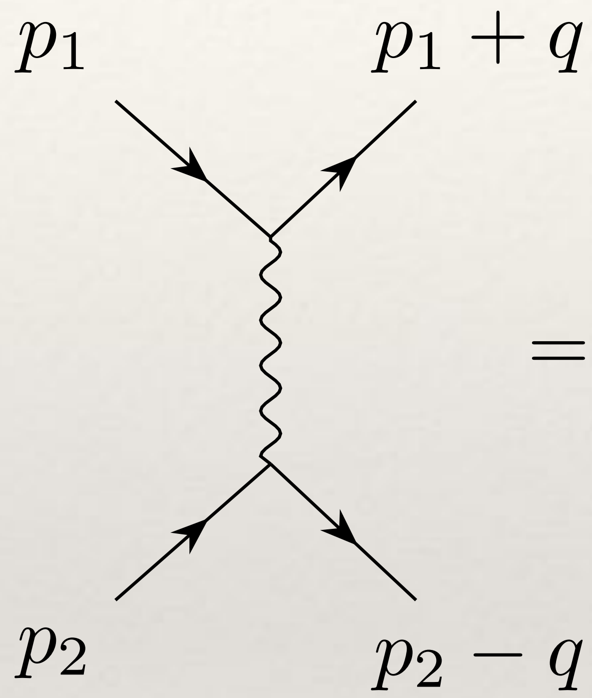
$$= i g^2 \frac{(4 p_1 \cdot p_2 + q^2)^n}{q^2} .$$

Amplitudes

- "Messenger": photon or graviton, coupling g

$\hbar = 1$

$$i\mathcal{A}(p_1 p_2 \to p_1' p_2') =$$



$$= ig^2 \frac{(4p_1 \cdot p_2 + q^2)^n}{q^2}.$$

- Dimensions of momenta, masses, polarisation vectors unchanged when $\hbar \neq 1$
- But coupling becomes dimensionful

Amplitudes

- “Messenger”: photon or graviton, coupling g

$\hbar \neq 1$

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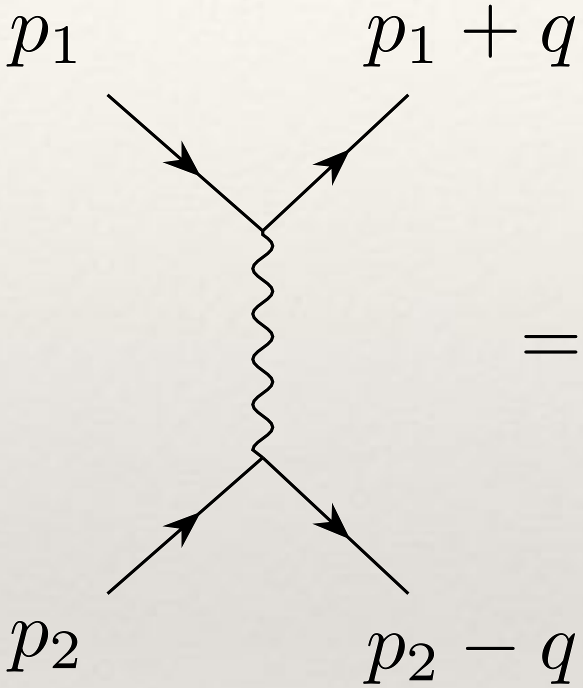

$$= i\frac{g^2}{\hbar}\frac{(4p_1\cdot p_2+q^2)^n}{q^2}.$$

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Amplitudes

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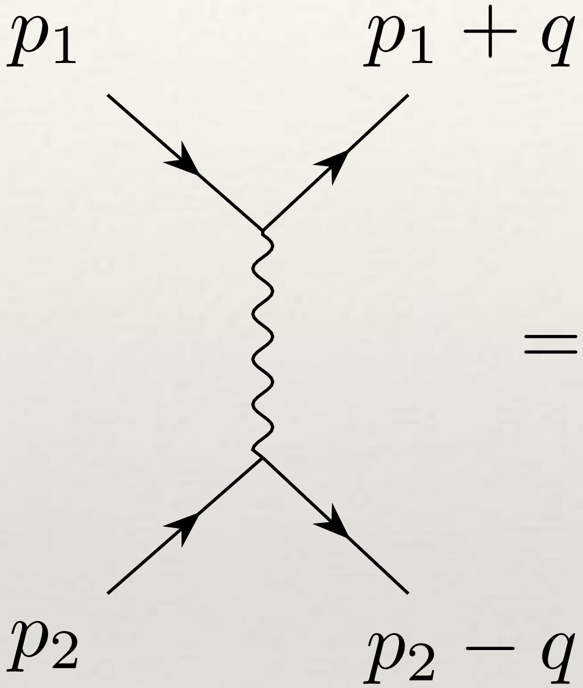


$$= i\frac{g^2}{\hbar}\frac{(4p_1 \cdot p_2 + q^2)^n}{q^2}.$$

Simply insert factor $1/\sqrt{\hbar}$ for each factor of g

- Dimensions of momenta, masses, polarisation vectors unchanged when $\hbar \neq 1$
- But coupling becomes dimensionful

Amplitudes

- So find

$$A_n^{(L)} \propto \frac{1}{\hbar^{\frac{n}{2}-1+L}}$$

- Well-known! QED: $g \to e \quad \alpha = e^2/(4\pi\hbar c)$
- Same in gravity $g \to \kappa/2 = \sqrt{8\pi G/\hbar}$

- But then how can we sensibly extract classical physics from an expansion in inverse powers of $\hbar$?

Amplitudes

- Haven't yet extracted all relevant factors of $\hbar$
- Will define observables which make sense in both classical and quantum theories
  - These will be gauge invariant, on-shell, well-defined quantities
  - See momenta of messengers in propagators $= \hbar$ wavenumbers
- Further cancellations occur inside observables

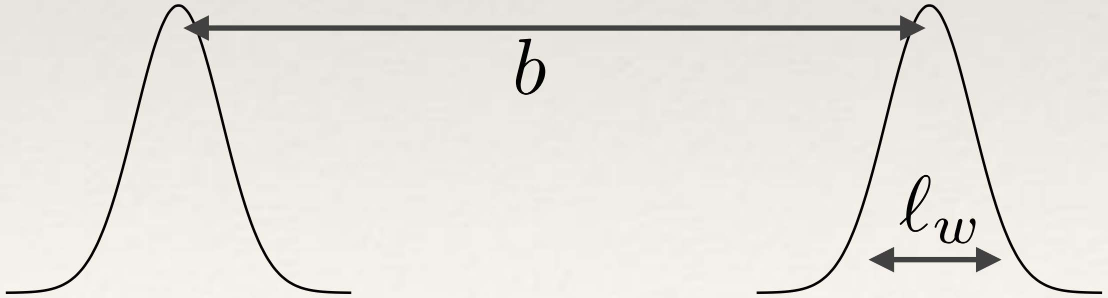
Observables

Incoming state

- Ultimately want to discuss classical point particles
- At quantum level: particles have uncertain positions, momenta

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

- Classical limit: $\hbar$ small
  - But can't take $\Delta p = 0$ … since then $\Delta x = \infty$
- Particles are in wavepackets



Incoming state

- Wavefunction has an intrinsic length scale ℓ_w

$$\psi(x) \sim \exp\left[-\frac{x^2}{\ell_w^2}\right] \xrightarrow{\text{Fourier}} \tilde{\psi}(p) \sim \exp\left[-\frac{\ell_w^2 p^2}{\hbar^2}\right]$$

- For impact parameter to make sense $\Delta x = \ell_w \ll b$

- For mass to make sense

$$\Delta p = \frac{\hbar}{\ell_w} \ll m \Rightarrow l_c = \frac{\hbar}{m} \ll \ell_w$$

- We require $l_c \ll \ell_w \ll b$

Incoming state

- Scatter different distinguishable, stable, scalar particles
- In the far past, prepare two well-separated localised states:

$$|\psi\rangle_{\text{in}} = \int Dp_1\, Dp_2\, e^{ib\cdot p_1} \phi_1(p_1)\phi_2(p_2)|p_1, p_2\rangle_{\text{in}}$$

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$$Dp_1 \equiv \frac{d^4 p_1}{(2\pi)^4}(2\pi)\delta(p_1^2 - m_1^2)$$

Integral over massive
on-shell phase space

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Particles displaced
by impact parameter, *b*

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Integral over massive on-shell phase space

Particles displaced by impact parameter, b

Wavefunctions: peaked at incoming momenta mu^μ.

Impulse

- First observable: impulse = time-integrated change in momentum of eg particle 1
- Can't measure the exact momenta… instead measure expectations

$$\langle p_1^\mu \rangle \equiv \langle \psi | \hat{P}_1^\mu | \psi \rangle \qquad \langle p_1'^\mu \rangle = \langle \psi | S^\dagger \hat{P}_1^\mu S | \psi \rangle$$

- Impulse is $\langle \Delta p_1^\mu \rangle = \langle \psi | S^\dagger \hat{P}_1^\mu S - \hat{P}_1^\mu | \psi \rangle$

$$= \langle \psi | i[\hat{P}_1^\mu, T] | \psi \rangle + \langle \psi | T^\dagger [\hat{P}_1^\mu, T] | \psi \rangle$$

- This is an all-orders definition

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Momentum operator

- Impulse is $$\langle \Delta p_1^\mu \rangle = \langle \psi | S^\dagger \hat{P}_1^\mu S - \hat{P}_1^\mu | \psi \rangle$$ $$= \langle \psi | i[\hat{P}_1^\mu, T] | \psi \rangle + \langle \psi | T^\dagger [\hat{P}_1^\mu, T] | \psi \rangle$$
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- Can't measure the exact momenta… instead measure expectations

$$\langle p_1^\mu\rangle \equiv \langle\psi|\hat{P}_1^\mu|\psi\rangle$$

Momentum operator

$$\langle p_1'^\mu\rangle = \langle\psi|S^\dagger \hat{P}_1^\mu \boxed{S|\psi\rangle}$$

Final state

- Impulse is $$\begin{aligned}\langle\Delta p_1^\mu\rangle &= \langle\psi|S^\dagger \hat{P}_1^\mu S - \hat{P}_1^\mu|\psi\rangle \\ &= \langle\psi|i[\hat{P}_1^\mu, T]|\psi\rangle + \langle\psi|T^\dagger[\hat{P}_1^\mu, T]|\psi\rangle\end{aligned}$$

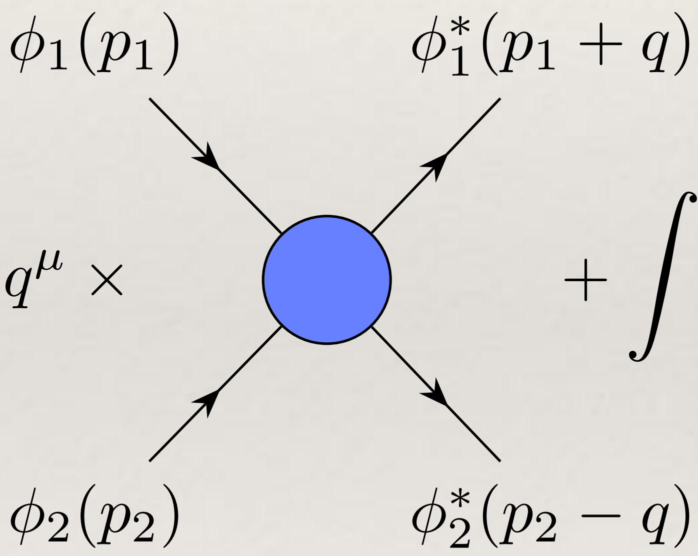
- This is an all-orders definition

Impulse

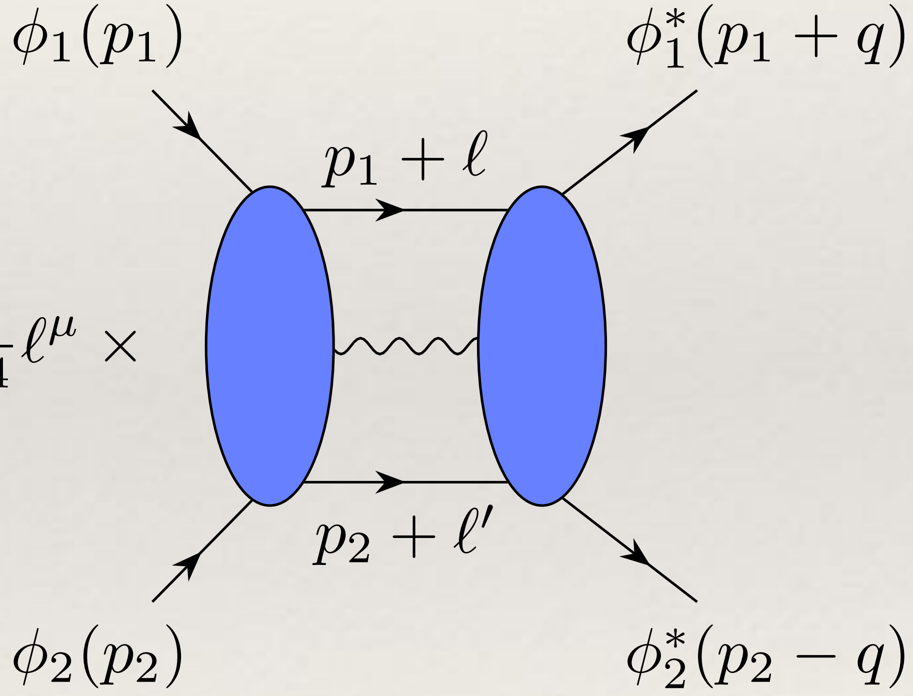
$$\langle \Delta p_1^\mu \rangle = \langle \psi | i[\hat{P}_1^\mu, T] | \psi \rangle + \langle \psi | T^\dagger [\hat{P}_1^\mu, T] | \psi \rangle$$

- Diagrammatically

$$\langle \Delta p_1^\mu \rangle = \int_{\text{on-shell}} e^{-ib\cdot q} \Bigg(q^\mu \times$$



$$+ \int \frac{d^4\ell}{(2\pi)^4} \ell^\mu \times$$



$$\Bigg).$$

Impulse

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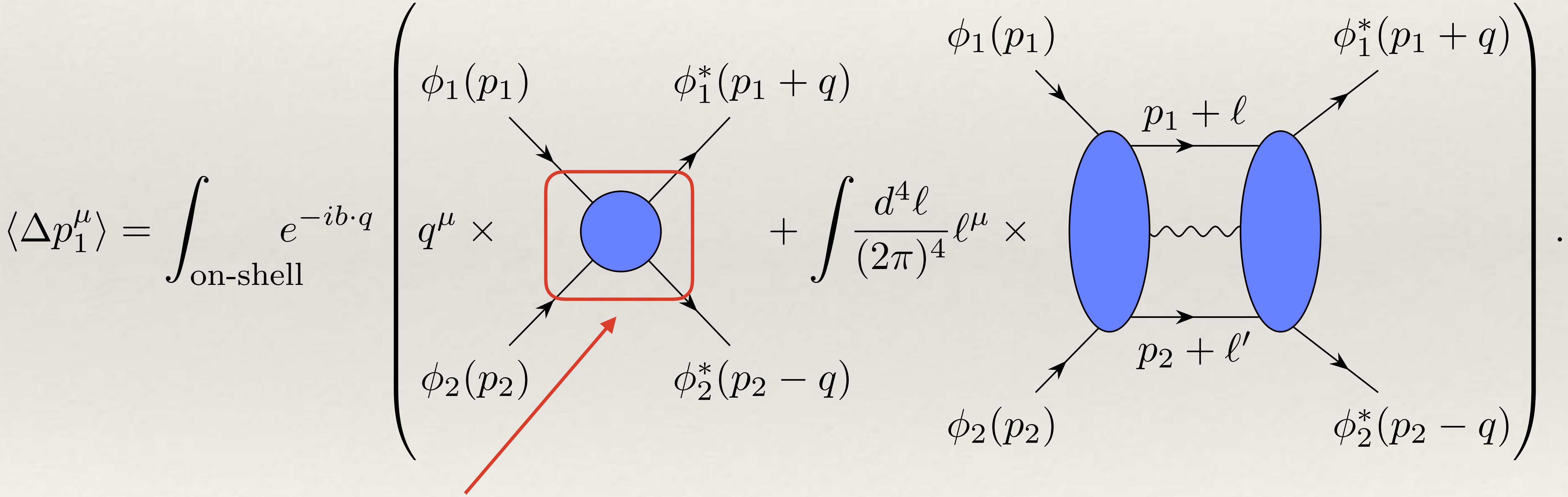
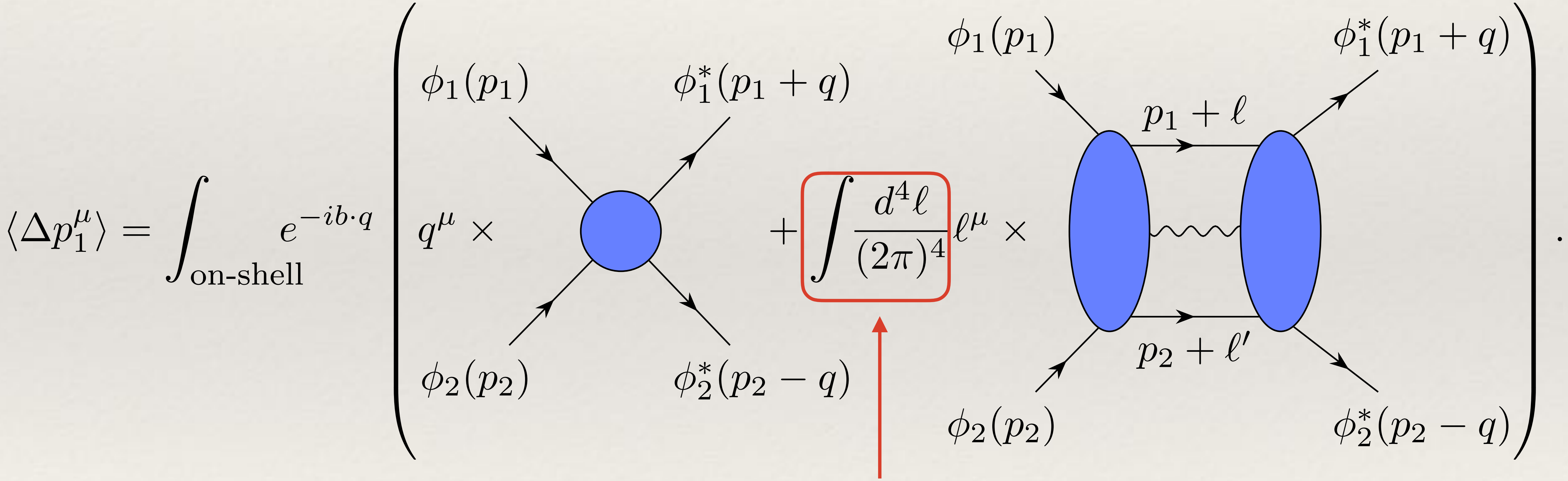


One term linear
in amplitude

Impulse

$$\langle \Delta p_1^\mu \rangle = \langle \psi | i[\hat{P}_1^\mu, T] | \psi \rangle + \langle \psi | T^\dagger [\hat{P}_1^\mu, T] | \psi \rangle$$

- Diagrammatically



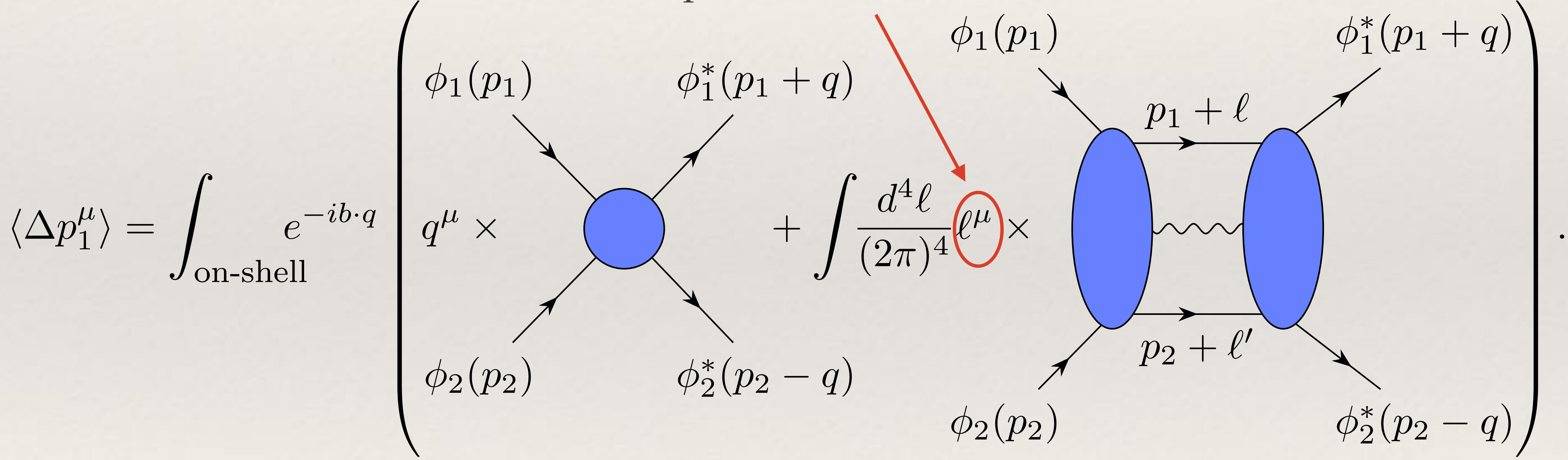
Integrate over undetermined cut momentum

Impulse

$$\langle \Delta p_1^\mu \rangle = \langle \psi | i[\hat{P}_1^\mu, T] | \psi \rangle + \langle \psi | T^\dagger [\hat{P}_1^\mu, T] | \psi \rangle$$

- Diagrammatically

Explicit factor of
cut loop momentum



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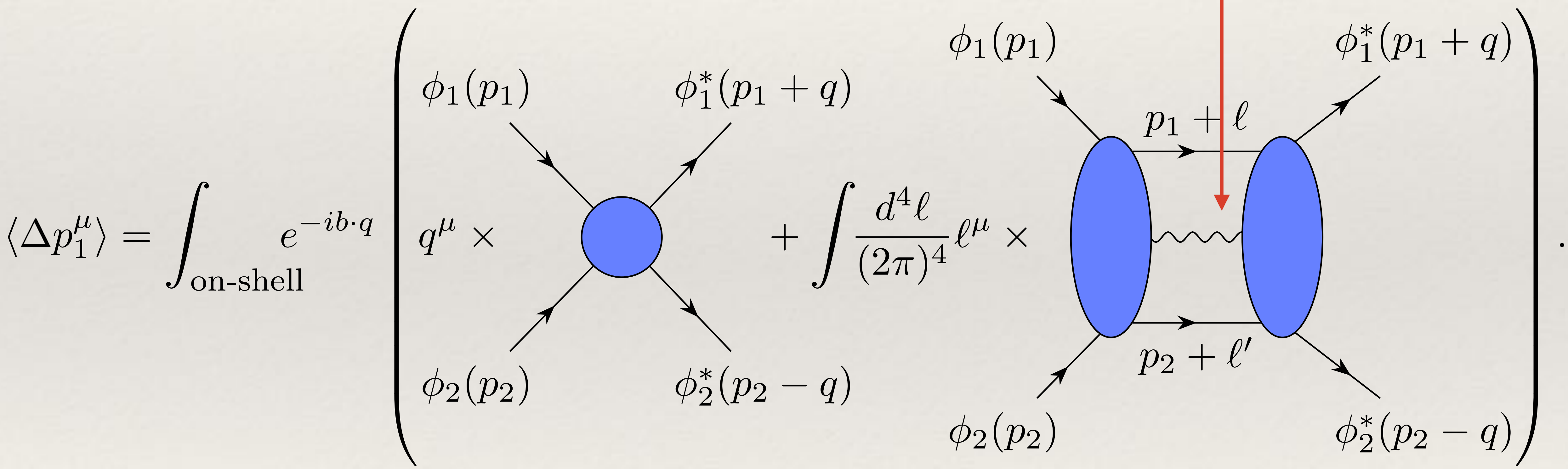
- Diagrammatically

Sum over all states



Radiated Momentum

- Always assume particles 1 and 2 present in final state
- Expectation of radiated momentum is

$$\langle k^\mu \rangle = \sum_X \int Dp_1 Dp_2 Dk \, k_X^\mu \, |\langle p_1 p_2 k X | S | \psi \rangle|^2$$

Radiated Momentum

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Time evolution operator

$$\langle k^\mu \rangle = \sum_X \int Dp_1 Dp_2 Dk \, k^\mu_X \, |\langle p_1 p_2 k X | \boxed{S} | \psi \rangle|^2$$

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Since $\langle p_1 p_2 k X | \psi \rangle = 0$,
can replace $S \to T$

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Messenger momentum

Since $\langle p_1 p_2 k X | \psi \rangle = 0$,
can replace $S \to T$

- An on-shell observable, in both classical and quantum theories

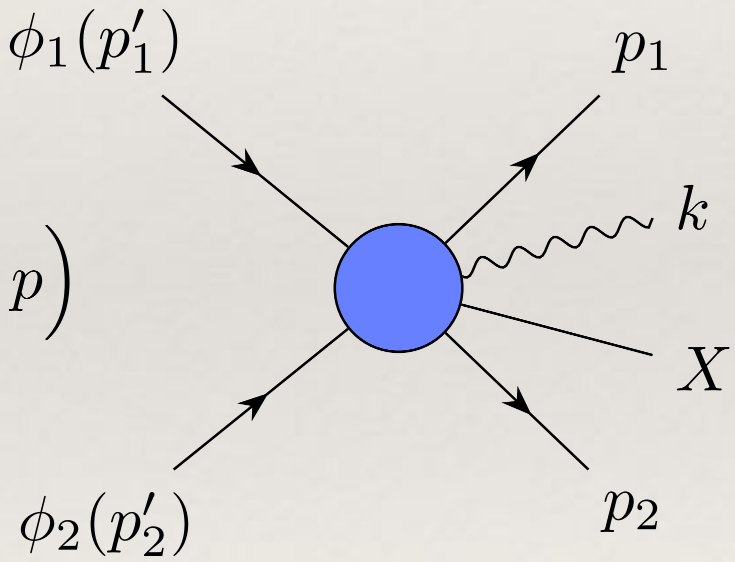
Radiated Momentum

$$\langle k^\mu \rangle = \sum_X \int Dp_1 Dp_2 Dk \, k^\mu_X \, |\langle p_1 p_2 k X | S | \psi \rangle|^2$$

- Insert the initial state $|\psi\rangle_{\text{in}} = \int Dp_1 \, Dp_2 \, e^{ib\cdot p_1} \phi_1(p_1) \phi_2(p_2) |p_1, p_2\rangle_{\text{in}}$

- Find:

$$\langle k^\mu \rangle = \sum_X \int Dk Dp_1 Dp_2 \, k^\mu_X \left| \int Dp_1' Dp_2' \, e^{ib\cdot p_1'} \, \delta^4 \left(\sum p \right) \right.$$



$$\left. \vphantom{\int} \right|^2$$

- Again, all-orders definition: can go to whatever order you want

Radiated Momentum

$$\langle k^\mu \rangle = \sum_X \int Dp_1 Dp_2 Dk \, k_X^\mu \, |\langle p_1 p_2 k X | S | \psi \rangle|^2$$

- Useful to rewrite this slightly:

$$\begin{aligned}\langle k^\mu \rangle &= \sum_X \int Dp_1 Dp_2 Dk \, \langle \psi | T^\dagger | p_1 p_1 k X \rangle \, k_X^\mu \, \langle p_1 p_2 k X | T | \psi \rangle \\ &= \langle \psi | T^\dagger \hat{K}^\mu T | \psi \rangle\end{aligned}$$

- Expectation value of messenger momentum operator at $t = +\infty$

Momentum balance

- Impulse and radiated momentum should balance
- Simple case: only fields are 1, 2 and messenger

$$\langle \Delta p_1^\mu \rangle + \langle \Delta p_2^\mu \rangle = \langle \psi | i[\hat{P}_1^\mu + \hat{P}_2^\mu, T] | \psi \rangle + \langle \psi | T^\dagger [\hat{P}_1^\mu + \hat{P}_2^\mu, T] | \psi \rangle$$

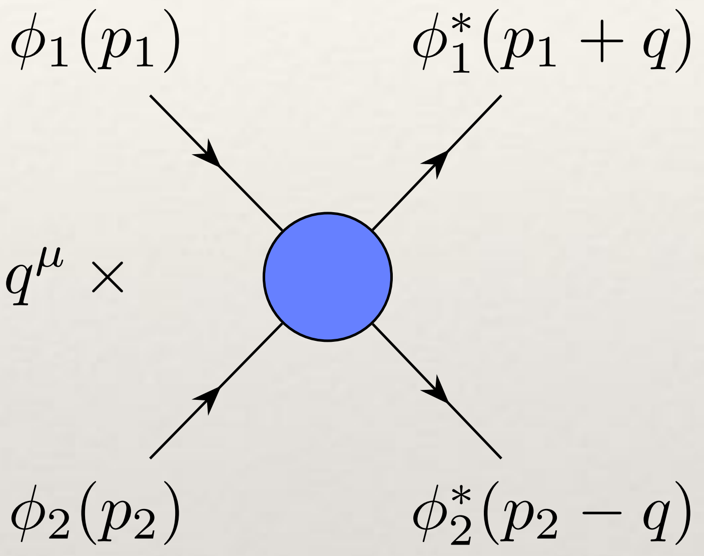
- But total momentum conserved $[\hat{P}_1^\mu + \hat{P}_2^\mu + \hat{K}^\mu, T] = 0$
- … and no messengers in initial state $\hat{K}^\mu | \psi \rangle = 0$

$$\Rightarrow \langle \Delta p_1^\mu \rangle + \langle \Delta p_2^\mu \rangle = -\langle \psi | T^\dagger [\hat{K}^\mu, T] | \psi \rangle = -\langle k^\mu \rangle$$

Classical Limit

- Already saw that $l_c \ll \ell_w \ll b$
- Focus on leading term in impulse

$$\langle \Delta p_1^\mu \rangle = \int_{\text{on-shell}} e^{-ib\cdot q/\hbar} q^\mu \times$$



- Momentum space wavepacket spread is $\Delta p = \dfrac{\hbar}{\ell_w} \Rightarrow q \lesssim \dfrac{\hbar}{\ell_w}$
- Write $q = \hbar\bar{q}$: $\bar{q}$ wavenumber associated with q
- More careful (and covariant) derivation in paper

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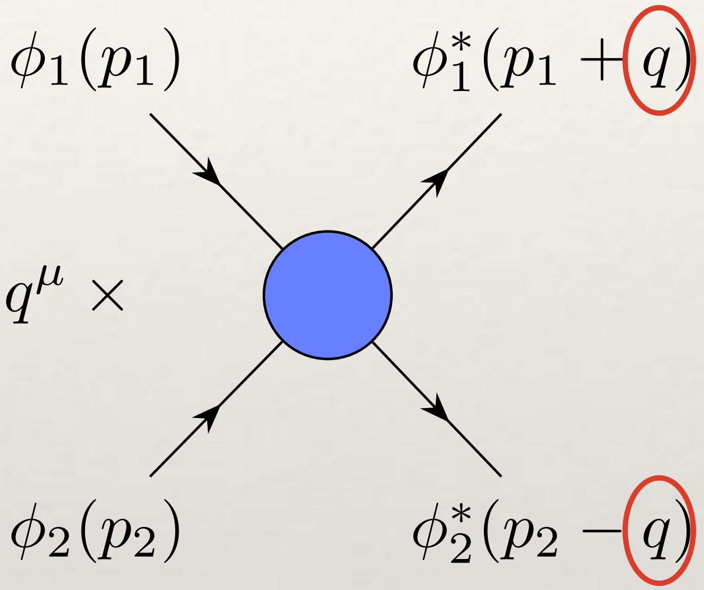

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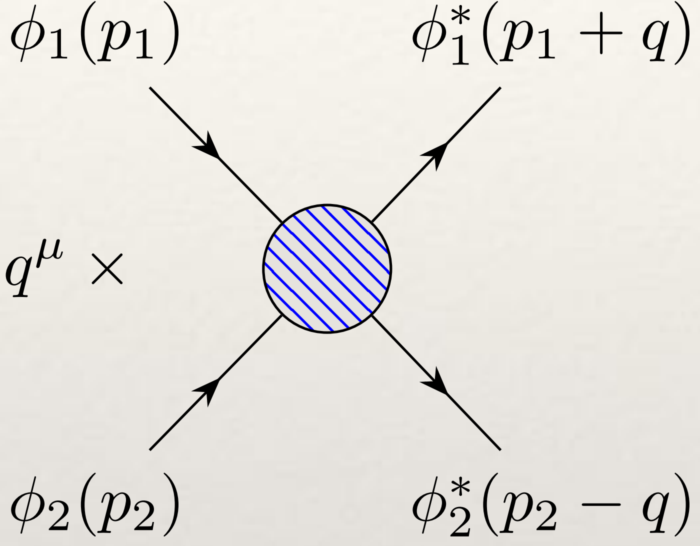
Classical Observables

1. LO Impulse

- See how this works: simple case, LO impulse

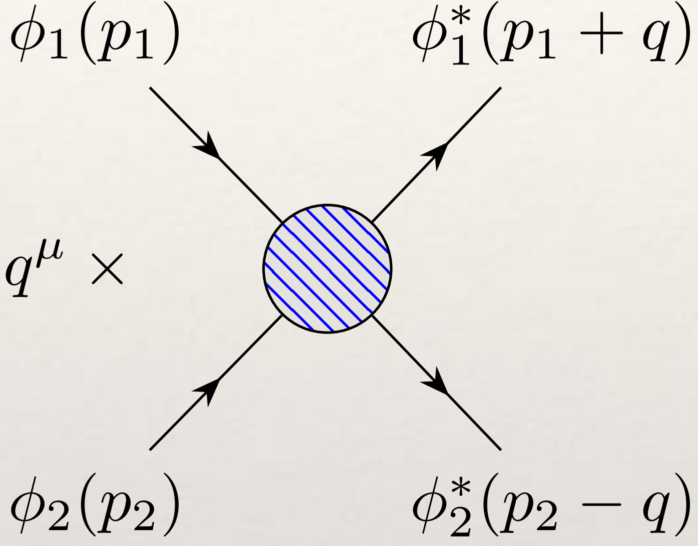
$$\langle \Delta p_1^\mu \rangle = \int_{\text{on-shell}} e^{-ib\cdot q/\hbar} q^\mu \times$$



1. LO Impulse

- See how this works: simple case, LO impulse

$$\langle \Delta p_1^\mu \rangle = \int_{\text{on-shell}} e^{-ib\cdot q/\hbar} q^\mu \times$$



$$= i \int Dp_1 Dp_2 d^4q \; \delta(2p_1\cdot q + q^2)\delta(2p_2\cdot q - q^2)$$

$$\times\, e^{-ib\cdot q/\hbar}\; \phi_1(p_1)\phi_2(p_2)\phi_1^*(p_1+q)\phi_2^*(p_2-q)\; q^\mu\; \mathcal{A}(q)$$

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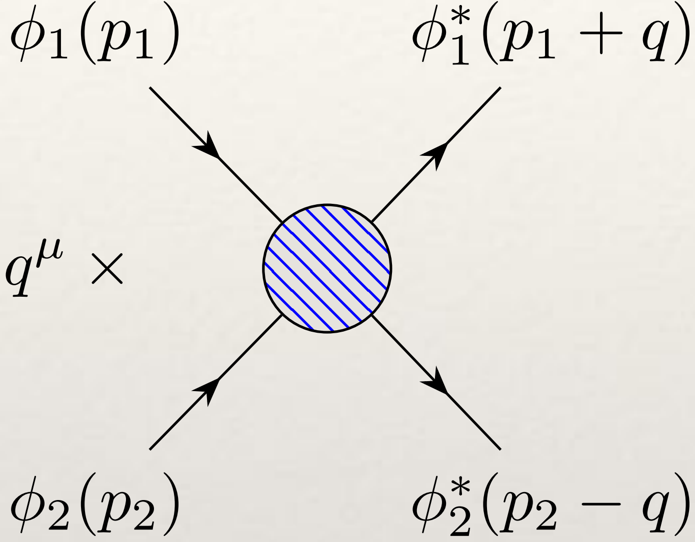

$q = \hbar\bar{q}$: neglect

$$= i\int Dp_1 Dp_2 d^4q\ \delta(2p_1\cdot q + \boxed{q^2})\delta(2p_2\cdot q - \boxed{q^2})$$

$$\times\ e^{-ib\cdot q/\hbar}\ \phi_1(p_1)\phi_2(p_2)\phi_1^*(p_1 + \boxed{q})\phi_2^*(p_2 - \boxed{q})\ q^\mu\ \mathcal{A}(q)$$

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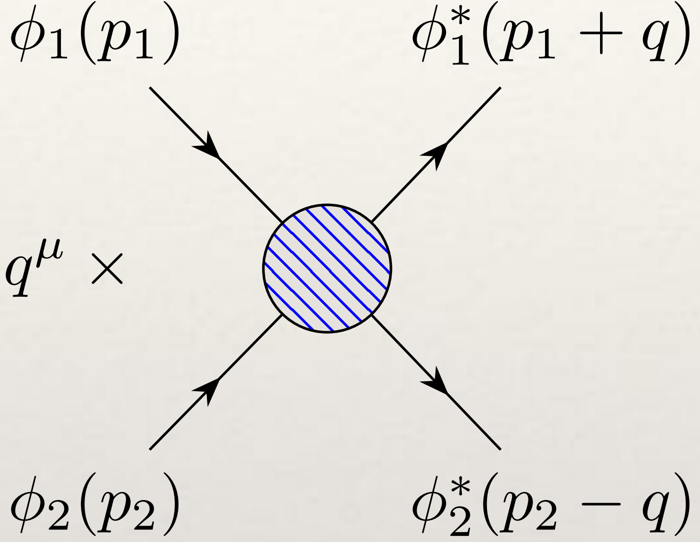

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- Wavefunctions peaked at $p_1 = m_1 u_1, p_2 = m_2 u_2$

$$\to i\int d^4\bar{q}\; \frac{\delta(u_1\cdot\bar{q})}{2m_1}\frac{\delta(u_2\cdot\bar{q})}{2m_2}\; e^{-ib\cdot\bar{q}}\; \bar{q}^\mu\; \hbar^3\mathcal{A}(p_1p_2 \to p_1+q, p_2-q)$$

1. LO Impulse

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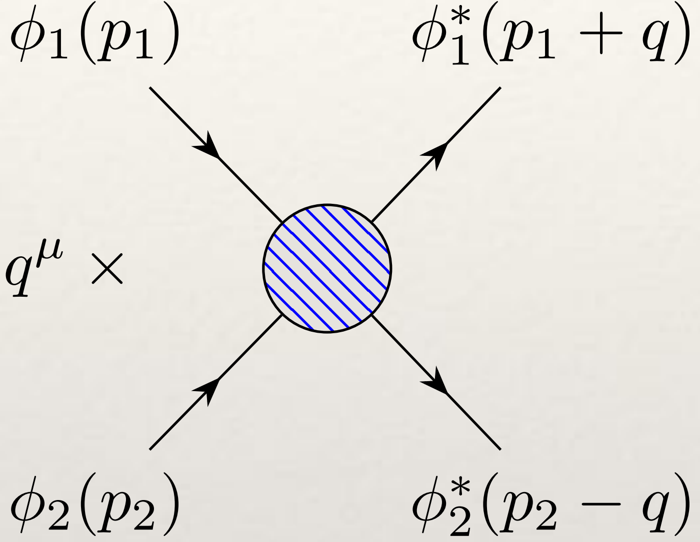

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- Wavefunctions peaked at $p_1 = m_1 u_1, p_2 = m_2 u_2$

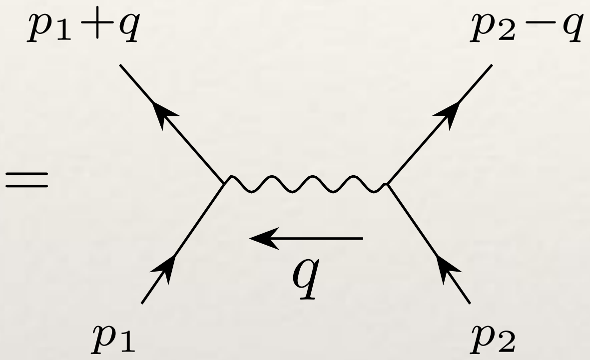
3 powers of $\hbar$

$$\to i \int d^4\bar{q} \, \frac{\delta(u_1 \cdot \bar{q})}{2m_1} \frac{\delta(u_2 \cdot \bar{q})}{2m_2} \, e^{-ib\cdot\bar{q}} \; \bar{q}^\mu \boxed{\hbar^3} \mathcal{A}(p_1 p_2 \to p_1 + q, p_2 - q)$$

1. LO Impulse

- More specifically, take scalar QED amplitude

$\mathcal{A} =$



$$= \frac{e^2}{\hbar}\frac{(2p_1+q)\cdot(2p_2-q)}{q^2} = \frac{e^2}{\hbar}\frac{4p_1\cdot p_2+\hbar^2\bar{q}^2}{\hbar^2\bar{q}^2}$$

1. LO Impulse

- More specifically, take scalar QED amplitude

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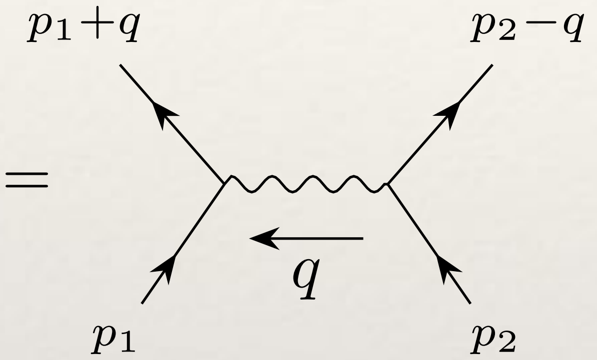

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3 powers of $\hbar$

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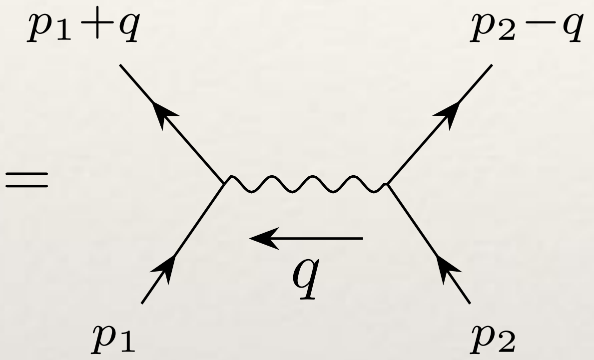


$$= \frac{e^2}{\hbar}\frac{(2p_1+q)\cdot(2p_2-q)}{q^2} = \frac{e^2}{\hbar}\frac{4p_1\cdot p_2 + \overset{\text{Negligible}}{\boxed{\hbar^2\bar{q}^2}}}{\hbar^2\bar{q}^2}$$

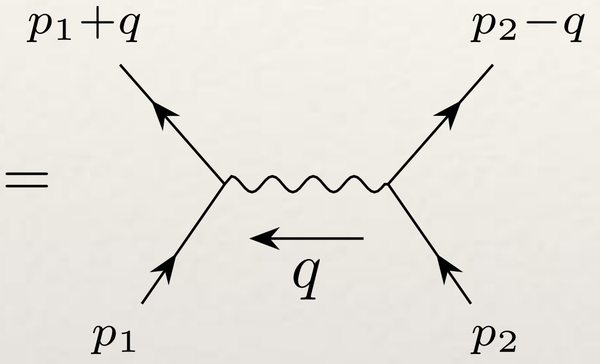
3 powers of $\hbar$

1. LO Impulse

- More specifically, take scalar QED amplitude

Negligible

$\mathcal{A} =$



$$= \frac{e^2}{\hbar}\,\frac{(2p_1+q)\cdot(2p_2-q)}{q^2} = \frac{e^2}{\hbar}\,\frac{4p_1\cdot p_2 + \boxed{\hbar^2\bar{q}^2}}{\hbar^2\bar{q}^2}$$

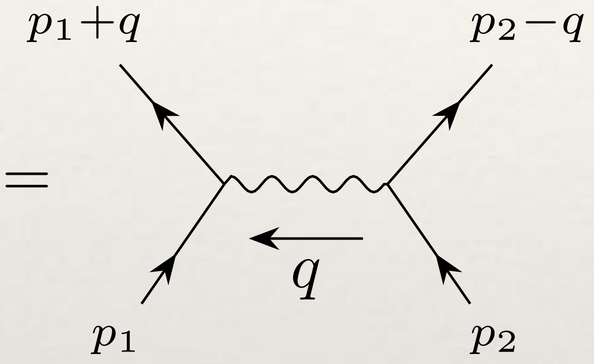
3 powers of $\hbar$

$$\Rightarrow \Delta p_1^\mu = i\int d^4\bar{q}\;\delta(u_1\cdot\bar{q})\delta(u_2\cdot\bar{q})\;e^{-ib\cdot\bar{q}}\;\bar{q}^\mu\;e^2\,\frac{u_1\cdot u_2}{\bar{q}^2}$$

1. LO Impulse

- More specifically, take scalar QED amplitude

$$\mathcal{A} =$$



Negligible

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$$= -\frac{e^2}{2\pi}\,\frac{b^\mu}{b\cdot b}\,\frac{u_1\cdot u_2}{\sqrt{(u_1\cdot u_2)^2-1}}$$

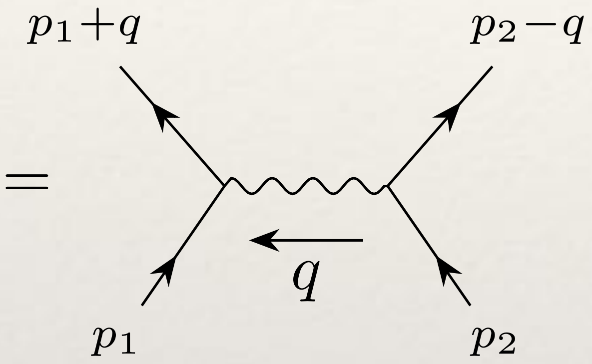
Integral straightforward

1. LO Impulse

In progress with Ben Maybee & Justin Vines

- Double copy:

$$\mathcal{M} =$$



$$= -\frac{\kappa^2}{16\hbar}\frac{[(2p_1+q)\cdot(2p_2-q)]^2}{q^2} \simeq -\frac{\kappa^2}{\hbar}\frac{(p_1\cdot p_2)^2}{\hbar^2\bar{q}^2}$$

- Therefore the momentum transfer in dilaton gravity is

$$\Delta p_1^\mu = i\int d^4\bar{q}\;\delta(u_1\cdot\bar{q})\delta(u_2\cdot\bar{q})\;e^{-ib\cdot\bar{q}}\;\bar{q}^\mu\;\frac{\kappa^2 m_1 m_2}{4}\;\frac{(u_1\cdot u_2)^2}{\bar{q}^2}$$

$$= 4Gm_1m_2\frac{b^\mu}{b\cdot b}\frac{(u_1\cdot u_2)^2}{\sqrt{(u_1\cdot u_2)^2-1}}$$

1. LO Impulse

In progress with Ben Maybee & Justin Vines

- The momentum transfer in dilaton gravity is

$$\Delta p_1^\mu = 4Gm_1m_2\frac{b^\mu}{b\cdot b}\frac{(u_1\cdot u_2)^2}{\sqrt{(u_1\cdot u_2)^2-1}}$$

- It's easy to remove the dilaton to find the GR result

$$\Delta p_1^\mu = 4Gm_1m_2\frac{b^\mu}{b\cdot b}\frac{(u_1\cdot u_2)^2-\frac{1}{2}}{\sqrt{(u_1\cdot u_2)^2-1}}$$

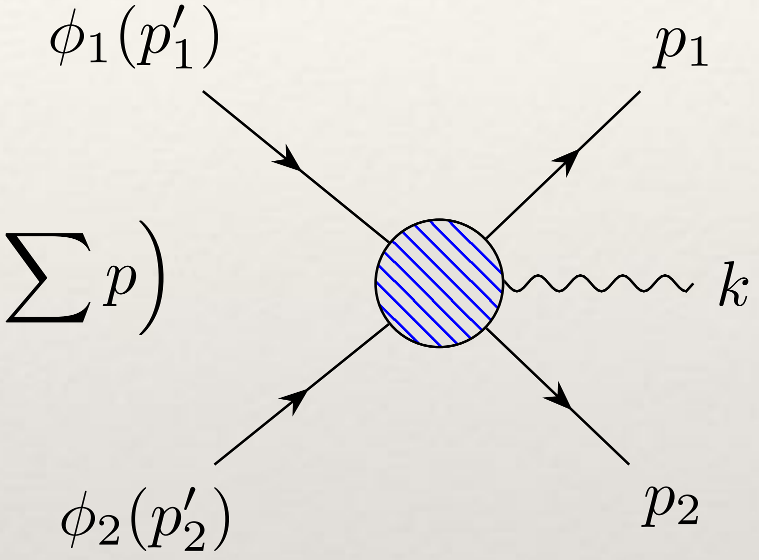
Portilla;
Westpfahl, Goller;
Ledvinka, Schäfer, Bičák; Damour

- Definitely attractive: follows from the double copy
- Extract scattering angle easily

2. LO Radiation

- Leading classical limit of radiated momentum

$$\langle k^\mu \rangle = \int DkDp_1Dp_2\, k^\mu \left| \int Dp_1'\,Dp_2'\, e^{ib\cdot p_1'}\, \delta^4\left(\sum p\right) \right.$$



$$\left. \vphantom{\int} \right|^2$$

$$\to \int D\bar{k}\; \bar{k}^\mu\; \hbar^7 \left| \int d^4\bar{q}_1 d^4\bar{q}_2 \delta(u_1\cdot\bar{q}_1)\delta(u_2\cdot\bar{q}_2)\; e^{ib\cdot\bar{q}_1}\delta^4(\bar{q}_1+\bar{q}_2-\bar{k}) \right.$$

$$\left. \times \frac{1}{4m_1m_2}\mathcal{A}(p_1+\hbar\bar{q}_1, p_2+\hbar\bar{q}_2 \to p_1p_2k) \right|^2$$

2. LO Radiation

- Unpack notation a little more:

$$\langle k^\mu \rangle = \int d^4\bar{k}\delta(\bar{k}^2)\theta(k^0)\ \bar{k}^\mu\ \hbar^7 \left| \int d^4\bar{q}_1 d^4\bar{q}_2 \delta(u_1 \cdot \bar{q}_1)\delta(u_2 \cdot \bar{q}_2)\ e^{ib\cdot\bar{q}_1}\delta^4(\bar{q}_1 + \bar{q}_2 - \bar{k}) \right.$$

$$\left. \times \frac{1}{4m_1 m_2}\mathcal{A}(p_1 + \hbar\bar{q}_1, p_2 + \hbar\bar{q}_2 \to p_1 p_2 k) \right|^2$$

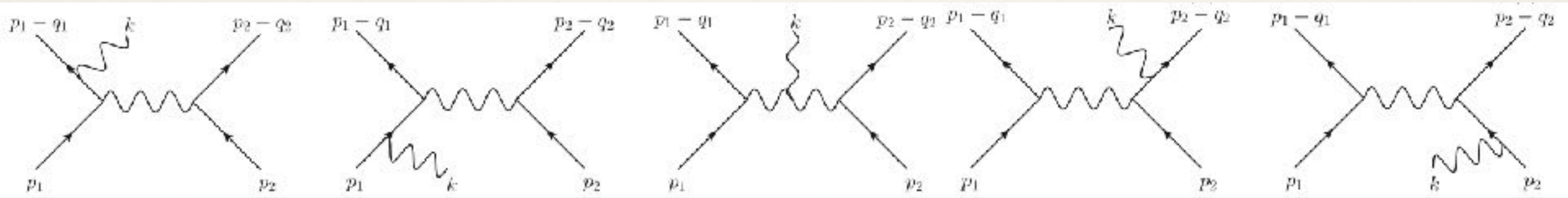
$$= \int d^4\bar{k}\delta(\bar{k}^2)\theta(k^0)\ \bar{k}^\mu\ |\mathcal{J}|^2$$

- This classical formula was used by Goldberger and Ridgway
- Quantity $\mathcal{J} = \epsilon_\mu \partial^2 A^\mu$ in terms of classical gauge field
- Amplitudes double copy $\Rightarrow$ double copy for classical radiation

$\longrightarrow$ *Talks by Shen, Carrillo González*

2. LO Radiation

- Simplest case: 5 point tree amplitude in Yang-Mills theory



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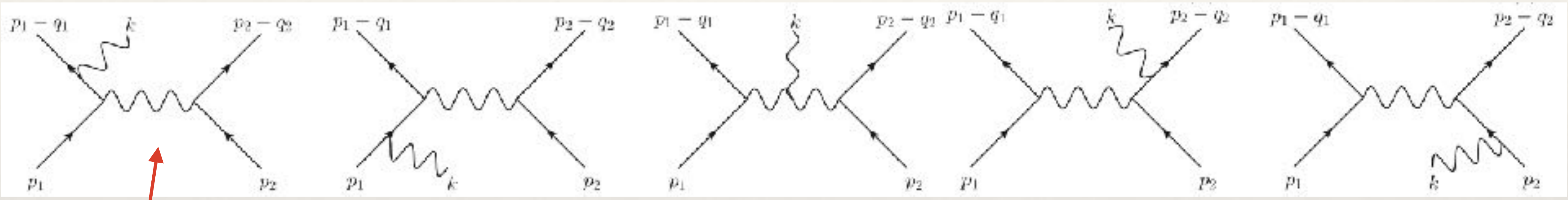
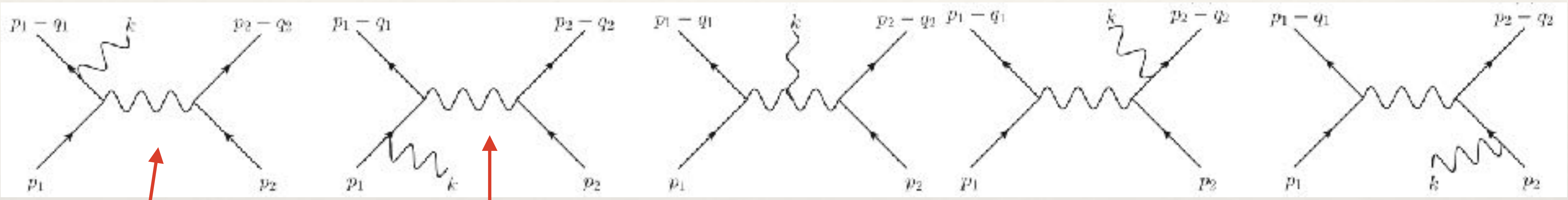


$$n_A = (2p_1 + q_2) \cdot (2p_2 - q_2)\, \epsilon \cdot (2p_1 + 2q_2) - (2p_1 \cdot q_2 + q_2^2)\, \epsilon \cdot (2p_2 - q_2)$$

2. LO Radiation

- Simplest case: 5 point tree amplitude in Yang-Mills theory



$$n_B = (2p_1 - k - q_1) \cdot (2p_2 - q_2)\, 2\epsilon \cdot p_1 + 2p_1 \cdot k\, \epsilon \cdot (2p_2 - q_2)$$

$$n_A = (2p_1 + q_2) \cdot (2p_2 - q_2)\, \epsilon \cdot (2p_1 + 2q_2) - (2p_1 \cdot q_2 + q_2^2)\, \epsilon \cdot (2p_2 - q_2)$$

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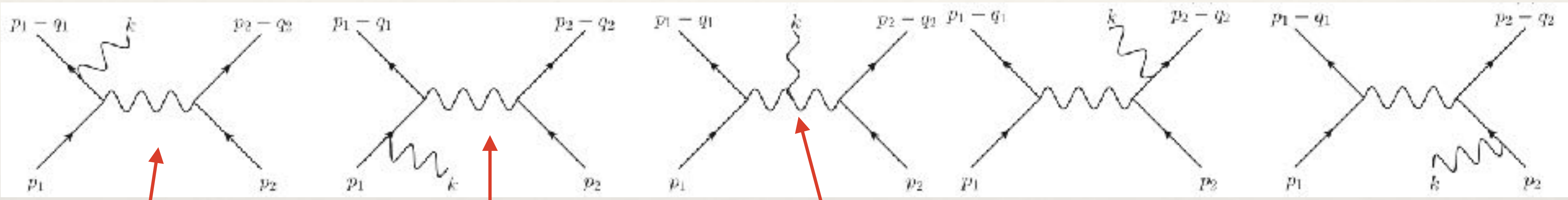

$$n_C = n_A - n_B$$

$$n_B = (2p_1 - k - q_1) \cdot (2p_2 - q_2)\, 2\epsilon \cdot p_1 + 2p_1 \cdot k\ \epsilon \cdot (2p_2 - q_2)$$

$$n_A = (2p_1 + q_2) \cdot (2p_2 - q_2)\ \epsilon \cdot (2p_1 + 2q_2) - (2p_1 \cdot q_2 + q_2^2)\ \epsilon \cdot (2p_2 - q_2)$$

2. LO Radiation

- Now expand in $\hbar$
  - Leading order term vanishes! (Singular in $\hbar$)
  - To get classical term, need to expand propagators…

$$\frac{(n_0+\delta n)^2}{d_0+\delta d}=\frac{n_0^2}{d_0}+\frac{2n_0\,\delta n}{d_0}-\frac{n_0^2\,\delta d}{d_0^2}$$

- Generate funny denominators, hard to see Jacobi relation
  - Double copy more obscure in classical case: quantum > classical

2. LO Radiation

$$P_{12}^{\mu} \equiv k \cdot u_1 \, u_2^{\mu} - k \cdot u_2 \, u_1^{\mu}$$

$$Q_{12}^{\mu} \equiv (q_1 - q_2)^{\mu} - \frac{q_1^2}{k \cdot u_1} u_1^{\mu} + \frac{q_2^2}{k \cdot u_2} u_2^{\mu}$$

2. LO Radiation

- Compute amplitude to find

$$\bar{\mathcal{M}} = 16 m_1^2 m_2^2 \, \epsilon_\mu \epsilon_\nu \left[4 \frac{P_{12}^\mu P_{12}^\nu}{q_1^2 q_2^2} + 2 \frac{u_1 \cdot u_2}{q_1^2 q_2^2} \left(Q_{12}^\mu P_{12}^\nu + Q_{12}^\nu P_{12}^\mu \right) \right. \\ \left. + (u_1 \cdot u_2)^2 \left(\frac{Q_{12}^\mu Q_{12}^\nu}{q_1^2 q_2^2} - \frac{P_{12}^\mu P_{12}^\nu}{(k \cdot u_1)^2 (k \cdot u_2)^2} \right) \right] .$$

Luna, Nicholson, White, DOC

$$P_{12}^\mu \equiv k \cdot u_1 \, u_2^\mu - k \cdot u_2 \, u_1^\mu$$

$$Q_{12}^\mu \equiv (q_1 - q_2)^\mu - \frac{q_1^2}{k \cdot u_1} u_1^\mu + \frac{q_2^2}{k \cdot u_2} u_2^\mu$$

2. LO Radiation

- Compute amplitude to find
- By investigating factorisation channels, easy to remove dilaton:

$$\bar{\mathcal{M}}_{\rm GR} = 16 m_1^2 m_2^2 \, \epsilon_\mu \epsilon_\nu \left[4 \frac{P_{12}^\mu P_{12}^\nu}{q_1^2 q_2^2} + 2 \frac{u_1 \cdot u_2}{q_1^2 q_2^2} \left(Q_{12}^\mu P_{12}^\nu + Q_{12}^\nu P_{12}^\mu \right) + \left((u_1 \cdot u_2)^2 - \frac{1}{2} \right) \left(\frac{Q_{12}^\mu Q_{12}^\nu}{q_1^2 q_2^2} - \frac{P_{12}^\mu P_{12}^\nu}{(k \cdot u_1)^2 (k \cdot u_2)^2} \right) \right] .$$

Luna, Nicholson, White, DOC

$$P_{12}^\mu \equiv k \cdot u_1 \, u_2^\mu - k \cdot u_2 \, u_1^\mu$$

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Luna, Nicholson, White, DOC

$$P_{12}^\mu \equiv k \cdot u_1 \, u_2^\mu - k \cdot u_2 \, u_1^\mu$$

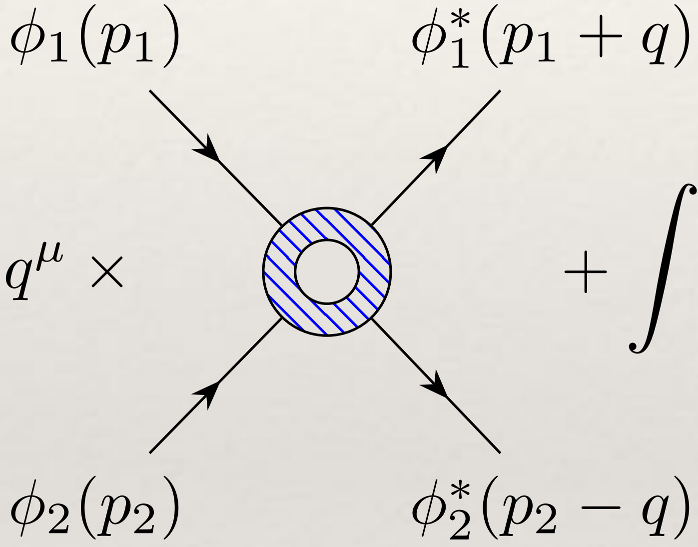
$$Q_{12}^\mu \equiv (q_1 - q_2)^\mu - \frac{q_1^2}{k \cdot u_1} u_1^\mu + \frac{q_2^2}{k \cdot u_2} u_2^\mu$$

- Insert into radiation formula; matches direct GR calculation

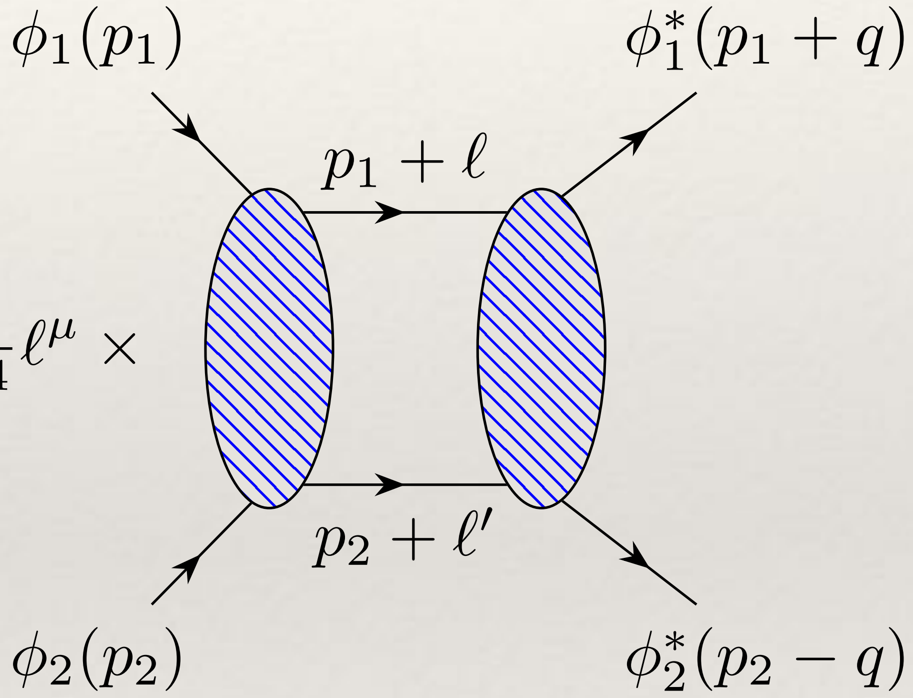
3. NLO Impulse

- As well as term linear in amplitude, now encounter cut

$$\langle \Delta p_1^\mu \rangle = \int_{\text{on-shell}} e^{-ib\cdot q} \Bigg(q^\mu \times$$



$$+ \int \frac{d^4\ell}{(2\pi)^4} \ell^\mu \times$$



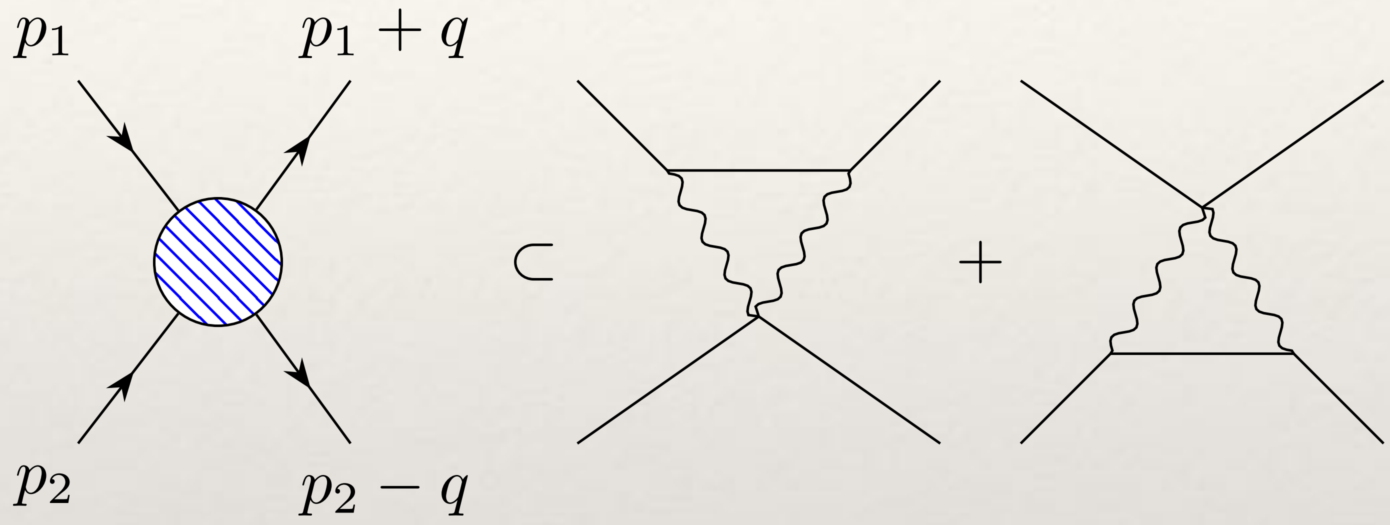
$$\Bigg) .$$

- Singularities in $\hbar$ cancel between these terms
- Interesting to see eg how massive propagators get cut

3. NLO Impulse

- Eg one loop term

$$i\mathcal{A}(p_1 p_2 \to p_1' p_2') =$$



- Extract $\hbar$

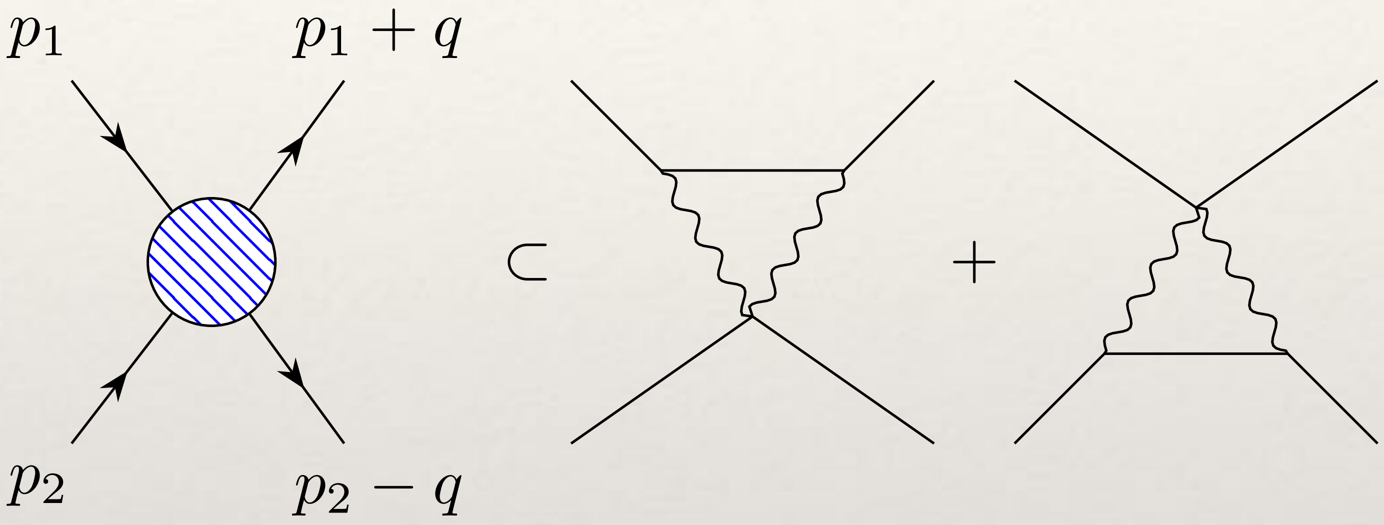
$$\frac{e^4}{\hbar^2} \int \frac{d^4\ell}{\ell^2(\ell - \hbar\bar{q})^2} \frac{1}{2p_1 \cdot \ell + \ell^2 + i\epsilon} \simeq \frac{e^4}{\hbar^2} \int \frac{d^4\ell}{\ell^2(\ell - \hbar\bar{q})^2} \frac{1}{2p_1 \cdot \ell + i\epsilon}$$

$$= \frac{e^4}{\hbar^3} \int \frac{d^4\bar{\ell}}{\bar{\ell}^2(\bar{\ell} - \bar{q})^2} \frac{1}{2p_1 \cdot \bar{\ell} + i\epsilon}$$

3. NLO Impulse

- Eg one loop term

$i\mathcal{A}(p_1 p_2 \to p_1' p_2') =$



- Exploit symmetry $\bar{\ell} \to \bar{\ell}' = \bar{q} - \bar{\ell} \Rightarrow \bar{\ell} - \bar{q} \to -\bar{\ell}'$

$$\frac{e^4}{\hbar^3} \int \frac{d^4\bar{\ell}}{\bar{\ell}^2(\bar{\ell} - \bar{q})^2} \frac{1}{2p_1 \cdot \bar{\ell} + i\epsilon} = -\frac{e^4}{\hbar^3} \int \frac{d^4\bar{\ell}}{\bar{\ell}^2(\bar{\ell} - \bar{q})^2} \frac{1}{2p_1 \cdot \bar{\ell} - i\epsilon}$$

$$= -\frac{i\pi\, e^4}{2\hbar^3} \int \frac{d^4\bar{\ell}}{\bar{\ell}^2(\bar{\ell} - \bar{q})^2} \delta(p_1 \cdot \bar{\ell})$$

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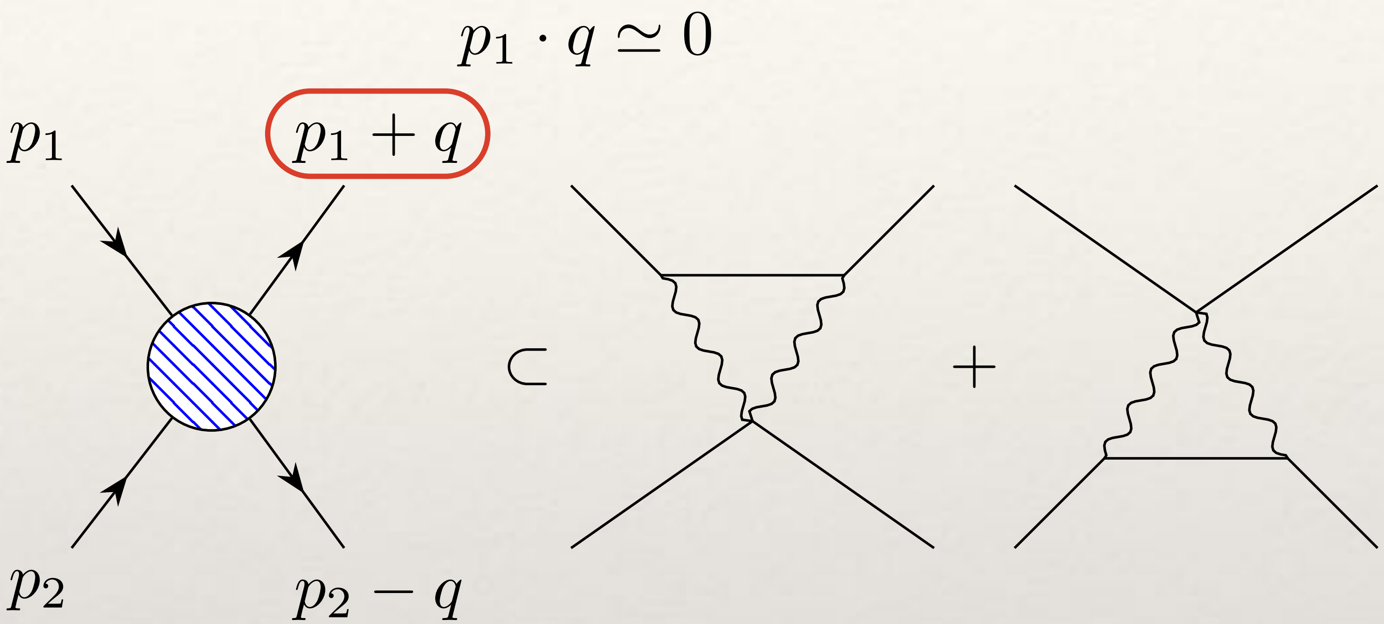

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Conclusions

- Lots of motivation to calculate in classical perturbative GR
  - On-shell methods and scattering amplitudes have promise
  - Can use standard double copy to get radiation, impulse

- Really understanding quantum theory: new computational power
  - God didn't make quantum field theory out of spite
  - We still don't understand colour-kinematics duality