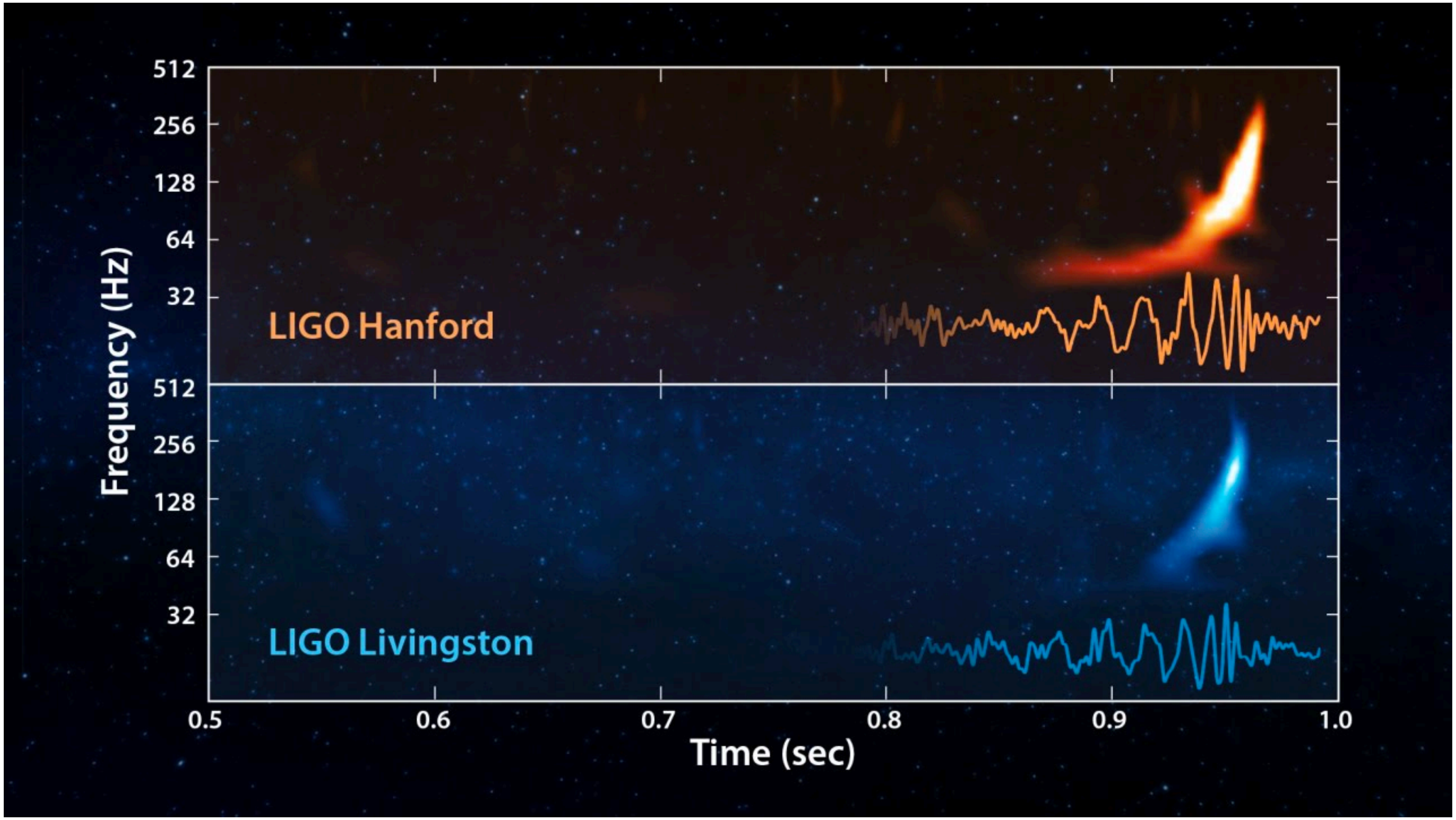
From Gluon Scattering to Black Hole Orbits

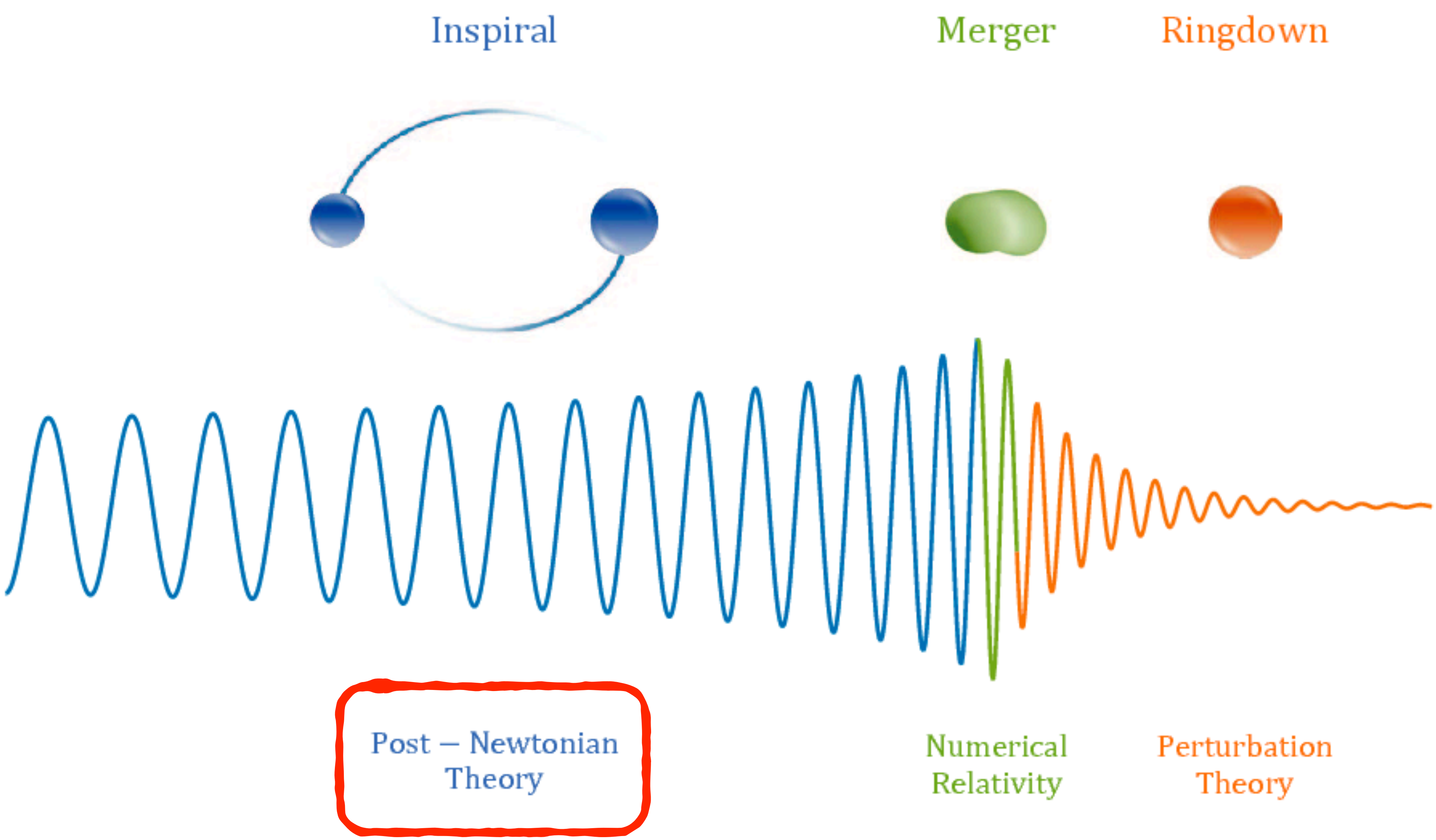
Clifford Cheung

CC, Rothstein, Solon (1808.02489)
Bern, CC, Roiban, Shen, Solon, Zeng (18xx.xxxxx)

Era of gravitational-wave astronomy begins!



Binary black hole merger in three phases:



I will focus on the conservative potential

(figure from 1610.03567)

The post-Newtonian (PN) approximation is an expansion in powers of

virial theorem

$$v^2 \sim \frac{GM}{r} \ll 1$$

which are tiny and perturbatively calculable during the inspiral phase.

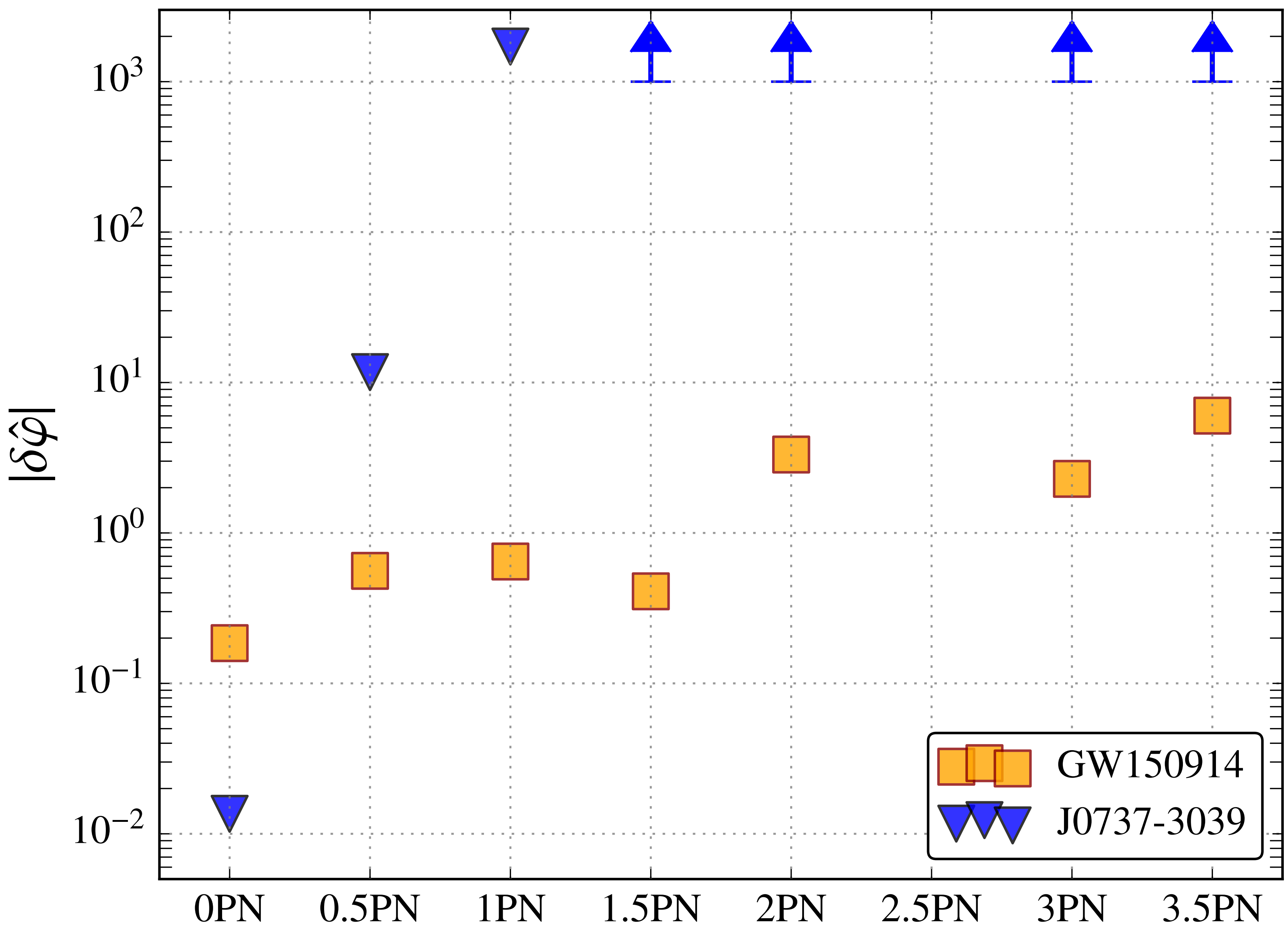
The post-Minkowskian (PM) expansion is an expansion in powers of G.

map of perturbative theory

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	7PN	
1PM	$(1$	$+v^2$	$+v^4$	$+v^6$	$+v^8$	$+v^{10}$	$+v^{12}$	$+v^{14}$	$+\cdots)\,G$
2PM		$(1$	$+v^2$	$+v^4$	$+v^6$	$+v^8$	$+v^{10}$	$+v^{12}$	$+\cdots)\,G^2$
3PM			$(1$	$+v^2$	$+v^4$	$+v^6$	$+v^8$	$+v^{10}$	$+\cdots)\,G^3$
4PM				$(1$	$+v^2$	$+v^4$	$+v^6$	$+v^8$	$+\cdots)\,G^4$
5PM					$(1$	$+v^2$	$+v^4$	$+v^6$	$+\cdots)\,G^5$

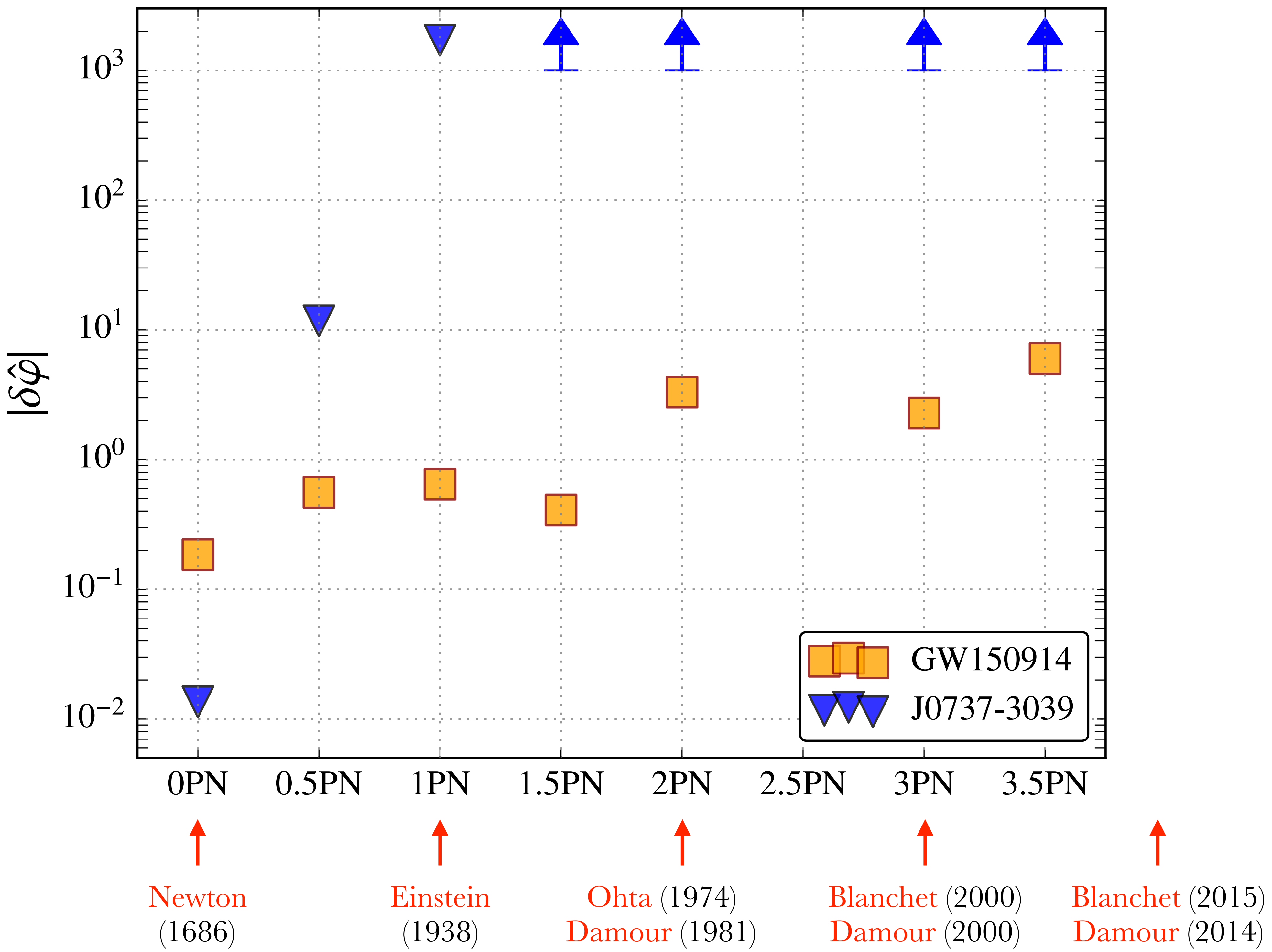
$$\vdots$$

LIGO will continue to test PN corrections.



"Tests of general relativity with GW150914" (1602.03841)

LIGO will continue to test PN corrections.



map of perturbative theory

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	7PN		
1PM	$(1$	$+\, v^2$	$+\, v^4$	$+\, v^6$	$+\, v^8$	$+\, v^{10}$	$+\, v^{12}$	$+\, v^{14}$	$+ \cdots)$	G
2PM		$(1$	$+\, v^2$	$+\, v^4$	$+\, v^6$	$+\, v^8$	$+\, v^{10}$	$+\, v^{12}$	$+ \cdots)$	G^2
3PM			$(1$	$+\, v^2$	$+\, v^4$	$+\, v^6$	$+\, v^8$	$+\, v^{10}$	$+ \cdots)$	G^3
4PM				$(1$	$+\, v^2$	$+\, v^4$	$+\, v^6$	$+\, v^8$	$+ \cdots)$	G^4
5PM					$(1$	$+\, v^2$	$+\, v^4$	$+\, v^6$	$+ \cdots)$	G^5

$$\vdots$$

map of perturbative theory

0PN 1PN 2PN 3PN 4PN 5PN 6PN 7PN

1PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + v^{14} + \cdots \right) G$$

2PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots \right) G^2$$

3PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \cdots \right) G^3$$

4PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + \cdots \right) G^4$$

5PM $$\left(1 + v^2 + v^4 + v^6 + \cdots \right) G^5$$

$$\vdots$$

map of perturbative theory

0PN 1PN 2PN 3PN 4PN 5PN 6PN 7PN

1PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + v^{14} + \cdots \right) G$$

2PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots \right) G^2$$

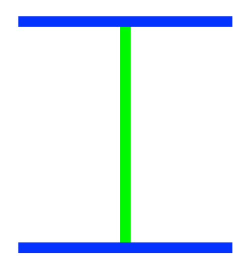
3PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \cdots \right) G^3$$

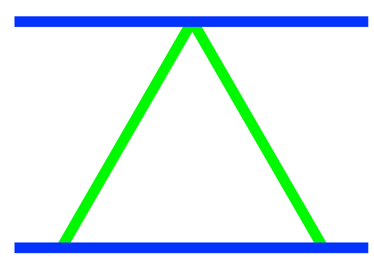
4PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + \cdots \right) G^4$$

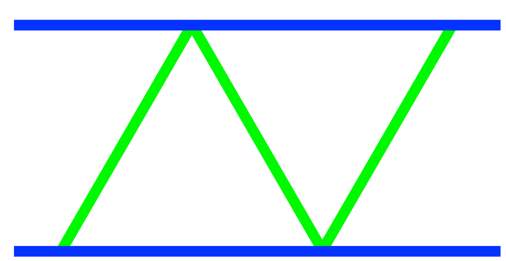
5PM $$\left(1 + v^2 + v^4 + v^6 + \cdots \right) G^5$$

$$\vdots$$

Crucially n-PN corresponds to up to n-loop!

$A(p, q) \sim \{$  $\}$ $G(1 + v^2 + \cdots)$

$+ \{$  $+ \cdots \}$ $G^2(1 + v^2 + \cdots)$

$+ \{$  $+ \cdots \}$ $G^3(1 + v^2 + \cdots)$

Loop $\neq$ quantum due to $e^{iqr/\hbar}$ Fourier factor.

Can amplitudes methods give an efficient and scaleable path to higher PN? Several issues:

- massive black hole $\neq$ SYM gluon

  → but BH is a point-like particle at long distances

- scattering amplitudes $\neq$ potential

  → but EFT methods solved this in NRQCD

- evaluating high loop integrals = hard

  → but potential is *simple* so the integrals should be too!

full theory effective theory

full theory

effective theory

amplitudes methods ↓

BH / graviton tree amplitudes: A_{tree}

generalized unitarity ↓

integral representation:

$$A = \sum_i d^{(i)} I^{(i)}$$

multi-loop integration ↓

full loop amplitude: $A(p, q)$

full theory		effective theory	
	↓ amplitudes methods		↓ build ansatz
BH / graviton tree amplitudes	A_{tree}	$V(p,q)$	effective BH Lagrangian
	↓ generalized unitarity		↓ Feynman diagrams
integral representation	$A = \sum_i d^{(i)} I^{(i)}$	$A_{\text{EFT}} = \sum_i d^{(i)}_{\text{EFT}} I^{(i)}$	integral representation
	↓ multi-loop integration		↓ multi-loop integration
full loop amplitude	$A(p,q)$	$A_{\text{EFT}}(p,q)$	EFT loop amplitude

full theory

amplitudes methods

BH / graviton tree amplitudes: A_{tree}

generalized unitarity

integral representation: $$A = \sum_i d^{(i)} I^{(i)}$$

multi-loop integration

full loop amplitude: $A(p, q)$

effective theory

build ansatz

effective BH Lagrangian: $V(p, q)$

Feynman diagrams

integral representation: $$A_{\text{EFT}} = \sum_i d^{(i)}_{\text{EFT}} I^{(i)}$$

multi-loop integration

EFT loop amplitude: $A_{\text{EFT}}(p, q)$

identical physics

$$A(p, q) = A_{\text{EFT}}(p, q)$$

I will present our results at 2PM and 3PM, focusing on certain aspects of:

- calculating the scattering amplitudes

  (newly developed methods for efficient integration)

- extracting the conservative potential

  (extending EFT matching to all orders in velocity)

- analyzing the results for consistency

  (old and new diagnostics on the final answer)

map of perturbative theory

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	7PN		
1PM	$\big(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ v^{14}$	$+\cdots\big)$	G
2PM		$\big(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\cdots\big)$	G^2
3PM			$\big(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\cdots\big)$	G^3
4PM				$\big(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\cdots\big)$	G^4
5PM					$\big(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\cdots\big)$	G^5

$$\vdots$$

map of perturbative theory

0PN 1PN 2PN 3PN 4PN 5PN 6PN 7PN

1PM
$$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + v^{14} + \cdots \right) G$$

2PM
$$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots \right) G^2$$

3PM
$$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \cdots \right) G^3$$

4PM
$$\left(1 + v^2 + v^4 + v^6 + v^8 + \cdots \right) G^4$$

5PM
$$\left(1 + v^2 + v^4 + v^6 + \cdots \right) G^5$$

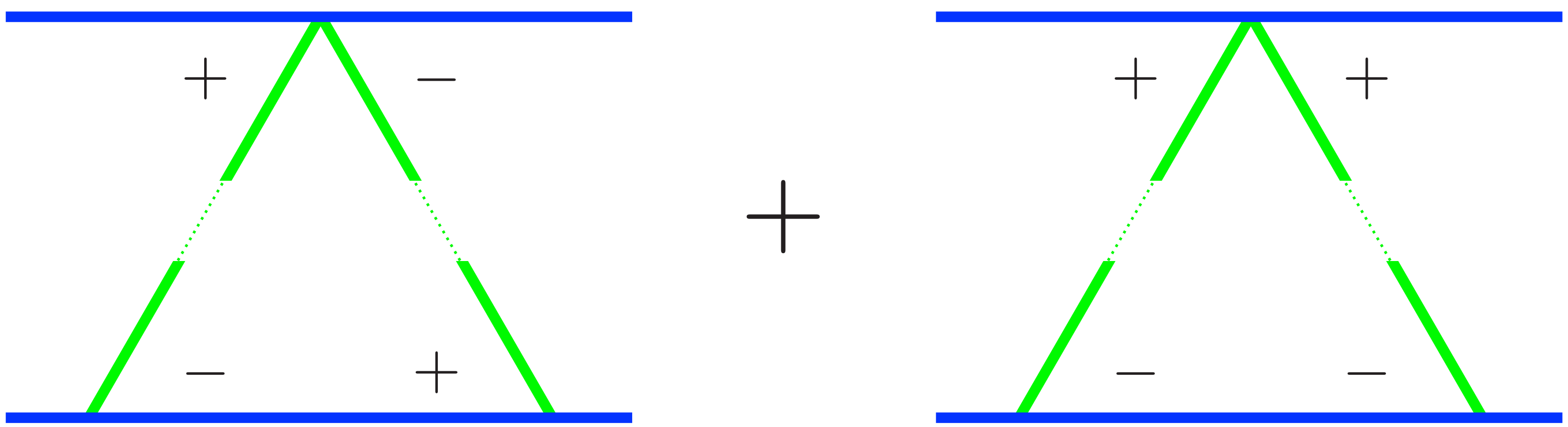
$$\vdots$$

1808.02489

18xx.xxxx

i) calculating the amplitude

The loop integrands of amplitudes are built by matching singularities to tree amplitudes.



See Zvi's talk for details on the integrands.

Integration is highly nontrivial, requiring the new tools developed in *1808.02489*.

$$A = \int d^4\ell_1 d^4\ell_2 \, \mathcal{I}_{\mathrm{4d}}$$

↓ evaluate energy integrals via residues

$$= \int d^3\ell_1 d^3\ell_2 \, \mathcal{I}_{\mathrm{3d}}$$

↓ expand in large mass

$$= \int d^3\ell_1 d^3\ell_2 \, \left(\mathcal{I}_{\mathrm{3d}}^{(0)} + \mathcal{I}_{\mathrm{3d}}^{(1)} + \mathcal{I}_{\mathrm{3d}}^{(2)} + \cdots \right)$$

↓ evaluate 3d integrals

the answer

$$A = \int d^4\ell_1 d^4\ell_2 \, \mathcal{I}_{\text{4d}}$$

↓ evaluate energy integrals via residues

$$= \int d^3\ell_1 d^3\ell_2 \, \mathcal{I}_{\text{3d}}$$

(this is our slowest step!) ↓ expand in large mass

$$= \int d^3\ell_1 d^3\ell_2 \left(\mathcal{I}_{\text{3d}}^{(0)} + \mathcal{I}_{\text{3d}}^{(1)} + \mathcal{I}_{\text{3d}}^{(2)} + \cdots \right)$$

↓ evaluate 3d integrals

the answer

After 3d reduction, the vast majority of terms are iterated bubble integrals of the form

$$\int d^3\ell \, \frac{\ell_{i_1}\ell_{i_2}\cdots\ell_{i_m}}{\ell^\alpha(\ell+q)^\beta} \quad = \quad \text{(A.10) of Smirnov's "Feynman Integral Calculus"}$$

The 3d integrals which appear exactly mirror the simplicity of those in NRGR.

We also applied several highly nontrivial 4d checks using IBPs and differential equations.

For the nonrelativistic scattering amplitude for spinless particles at $O(G^3 v^n)$ we obtain

$$A_{3\mathrm{PM}} = 4\pi G^3 M^4 \Big\{ \left(-\frac{17\nu}{4} + \frac{5\nu^2}{2}\right) + u^2 \left(-\frac{17\nu}{4} + \frac{5\nu^2}{2}\right) + u^4 \left(\frac{21\nu}{32} - \frac{539\nu^2}{160} + \frac{309\nu^3}{32} + \frac{31\nu^4}{16}\right)$$
$$+u^6 \left(-\frac{23\nu}{64} + \frac{17553\nu^2}{4480} - \frac{5193\nu^3}{640} + \frac{113\nu^4}{8} + \frac{233\nu^5}{128}\right)$$
$$+u^8 \left(\frac{125\nu}{512} - \frac{124937\nu^2}{32256} + \frac{254941\nu^3}{17920} - \frac{6463\nu^4}{320} + \frac{8919\nu^5}{512} + \frac{445\nu^6}{256}\right)$$
$$+u^{10} \left(-\frac{189\nu}{1024} + \frac{101123\nu^2}{26880} - \frac{1279241\nu^3}{64512} + \frac{298383\nu^4}{7168} - \frac{43181\nu^5}{1024} + \frac{10077\nu^6}{512} + \frac{1717\nu^7}{1024}\right) + \cdots \Big\}$$

We checked its gauge invariance explicitly.

We have resummed the series expansion into an analytic all orders in velocity expression.

ii) extracting the potential

Define a general Lagrangian for the EFT,

$$L_{\text{kin}} = \int_p A^\dagger(-p)\left(i\partial_t - \sqrt{p^2 + m_A^2}\right)A(p)$$

$$+ \int_p B^\dagger(-p)\left(i\partial_t - \sqrt{p^2 + m_B^2}\right)B(p)$$

$$L_{\text{int}} = -\int_{p,p'} V_{\text{int}}(p,p')\, A^\dagger(p')A(p)B^\dagger(-p')B(-p)$$

which describes the dynamics of two spinless, nonrelativistic particles denoted A and B.

potential
(position space)

$$V(p,r) = \sum_{i=1}^{\infty} c_i(p^2) \left(\frac{\kappa}{4\pi r}\right)^i$$

Fourier transform

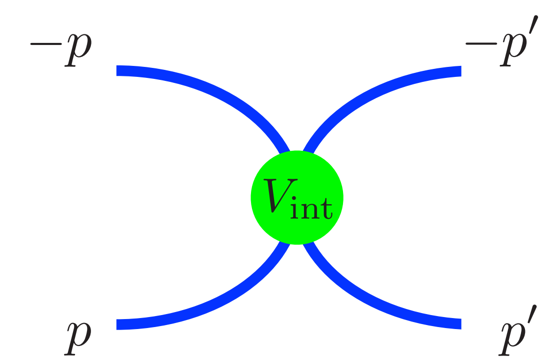
potential
(momentum space)

$$\widetilde{V}(p,q) = \frac{\kappa c_1(p^2)}{q^2} + \frac{\kappa^2 c_2(p^2)}{8q} - \frac{\kappa^3 c_3(p^2)\log q}{16\pi^2} + \cdots$$

off-shell continuation

interaction vertex

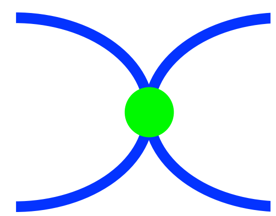
$$V_{\text{int}}(p,p') = \widetilde{V}\left(\sqrt{\tfrac{p^2+p'^2}{2}}, |p-p'|\right) =$$



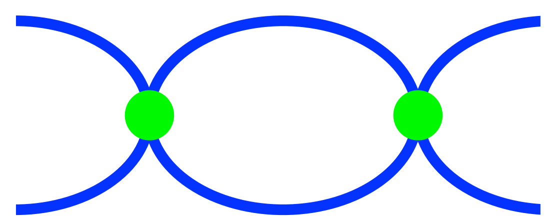
Scattering amplitudes in the EFT trivial to compute using Feynman diagrams.

$$A_{\mathrm{EFT}} = \sum_{i=1}^{\infty} A_{\mathrm{EFT}}^{(i)}$$ ← PM expansion

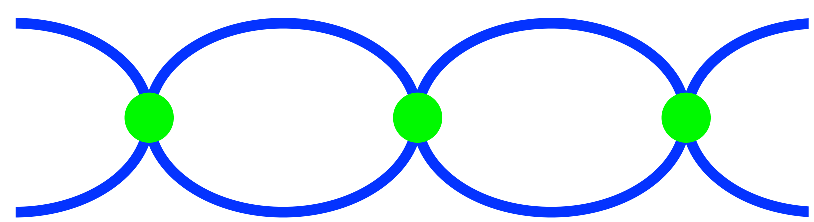
$=$



$+$



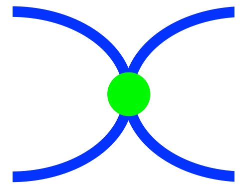
$+$



$+ \cdots$

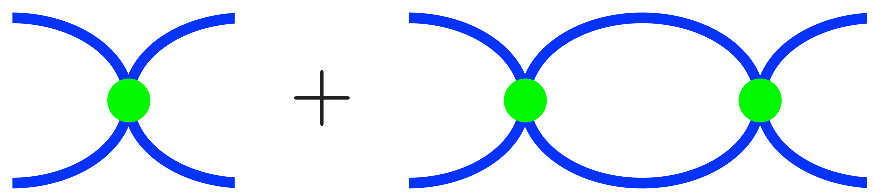
Each PM order is not homogenous in loops.

$A^{(1)}_{\mathrm{EFT}} =$



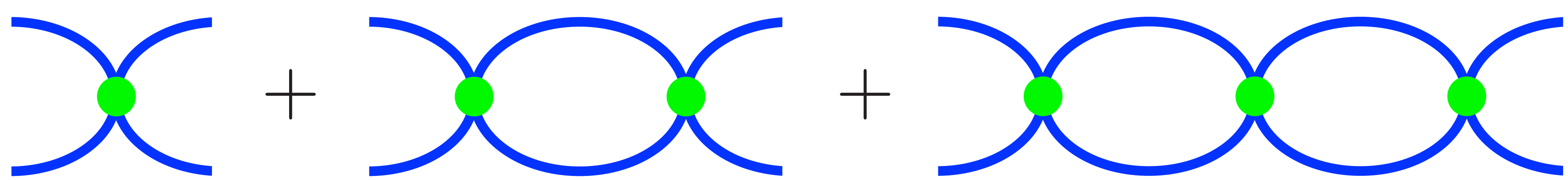
c_1

$A^{(2)}_{\mathrm{EFT}} =$



c_2 + c_1^2

$A^{(3)}_{\mathrm{EFT}} =$



c_3 + $c_1 c_2$ + c_1^3

We solve for the potential by setting the full amplitude equal to the EFT amplitude:

we have this at 3PM → $$A = A_{\text{EFT}}$$ ← we have this at 5PM

Key point: even though both the full and EFT amplitudes are IR divergent, we can compute their IR finite difference.

$$A^{(2)}_{\text{EFT}} - A^{(2)} = 0 = \kappa^2 \left[-\frac{c_2(p^2)}{q} + \int d^3\ell \, (\mathcal{I}_{\text{EFT}} - \mathcal{I}) \right]$$

separately non-singular in velocity

From the one-loop amplitude, we obtain an analytic form for $O(G^2v^n)$ ~ 2PM potential:

$$c_1(p^2) = \frac{1}{E^2\xi}\left[m_A^2 m_B^2 - 2(p_1 \cdot p_2)^2\right]$$

$$c_2(p^2) = \frac{1}{32E^2\xi}\Big[2E(\xi - 1)c_1^2 - 16E(p_1 \cdot p_2)c_1$$

$$+3(m_A + m_B)(m_A^2 m_B^2 - 5(p_1 \cdot p_2)^2\Big]$$

Note the amplitude can be calculated via any method: Feynman diagrams or modern tools.

From the two-loop amplitude, we obtain an expression for the $O(G^3 v^n)$ potential:

$$\frac{V_{3\mathrm{PM}}}{\mu} = \frac{1}{\rho^3}\Bigg\{\left(-\frac{1}{4}-\frac{3\nu}{2}\right)+u^2\left(-\frac{25}{8}-\frac{65\nu}{4}-\frac{107\nu^2}{8}\right)+u^4\left(\frac{105}{32}-\frac{4049\nu}{160}-\frac{2589\nu^2}{32}-\frac{487\nu^3}{16}\right)$$
$$+u^6\left(-\frac{273}{64}+\frac{178553\nu}{4480}-\frac{30993\nu^2}{640}-\frac{1527\nu^3}{8}-\frac{6607\nu^4}{128}\right)$$
$$+u^8\left(\frac{2805}{512}-\frac{1947527\nu}{32256}+\frac{3093791\nu^2}{17920}+\frac{5787\nu^3}{320}-\frac{168131\nu^4}{512}-\frac{19425\nu^5}{256}\right)$$
$$+u^{10}\left(-\frac{7007}{1024}+\frac{2354633\nu}{26880}-\frac{23190389\nu^2}{64512}+\frac{3013571\nu^3}{7168}+\frac{340279\nu^4}{1024}-\frac{237639\nu^5}{512}-\frac{104655\nu^6}{1024}\right)+\cdots\Bigg\}$$

These have terms far beyond 4PN, for which we are unaware of any existing calculation.

So how do we check it…?

iii) checking the answer

Check #1: In the probe limit, $m_A \ll m_B$, this describes a particle in a black hole geometry.

$$\frac{V_{3\mathrm{PM}}}{\mu} = \frac{1}{\rho^3}\left\{\left(-\frac{1}{4}-\frac{3\nu}{2}\right)+u^2\left(-\frac{25}{8}-\frac{65\nu}{4}-\frac{107\nu^2}{8}\right)+u^4\left(\frac{105}{32}-\frac{4049\nu}{160}-\frac{2589\nu^2}{32}-\frac{487\nu^3}{16}\right)\right.$$
$$+u^6\left(-\frac{273}{64}+\frac{178553\nu}{4480}-\frac{30993\nu^2}{640}-\frac{1527\nu^3}{8}-\frac{6607\nu^4}{128}\right)$$
$$+u^8\left(\frac{2805}{512}-\frac{1947527\nu}{32256}+\frac{3093791\nu^2}{17920}+\frac{5787\nu^3}{320}-\frac{168131\nu^4}{512}-\frac{19425\nu^5}{256}\right)$$
$$\left.+u^{10}\left(-\frac{7007}{1024}+\frac{2354633\nu}{26880}-\frac{23190389\nu^2}{64512}+\frac{3013571\nu^3}{7168}+\frac{340279\nu^4}{1024}-\frac{237639\nu^5}{512}-\frac{104655\nu^6}{1024}\right)+\cdots\right\}$$

Our result agrees with Schwarzschild metric.

$$\frac{H_{\mathrm{Sch}}}{\mu} = \left(1-\frac{1}{2\rho}\right)\left(1+\frac{1}{2\rho}\right)^{-1}\sqrt{1+\left(1+\frac{1}{2\rho}\right)^{-4}u^2}-1$$
$$= \cdots+\frac{1}{\rho^3}\left\{-\frac{1}{4}-\frac{25u^2}{8}+\frac{105u^4}{32}-\frac{273u^6}{64}+\frac{2805u^8}{512}-\frac{7007u^{10}}{1024}+\cdots\right\}$$

Check #2: Physically equivalent potentials generate the same energy for a circular orbit,

angular frequency

$$\varepsilon(x) = -\frac{x}{2}\left(1 - \frac{x}{12}(9+\nu) - \frac{x^2}{24}(81 - 57\nu + \nu^2) + \cdots\right)$$

energy per reduced mass

written in terms of the appropriate quantities.

We find agreement at the highest order for which our results apply to a virialized system, which is $O(G^3v^0)$, *i.e.* 2PN.

Check #3: Physically equivalent potentials produce the exact same scattering amplitudes.

our potential

state of the art
4PN potential
from literature

V_1 ⟷ V_2

are these physically
equivalent ???

$V_1 \downarrow$ $\quad$ $V_2 \downarrow$

$$A_1 = A_2$$

We found agreement at $O(G^3 v^4)$, *i.e.* 4PN.

conclusions

- There is a concrete and scaleable path to connect amplitudes results to LIGO.

- We computed the scattering amplitude at 3PM using newly developed integration methods which are quite efficient.

- Merging unitarity and EFT methods, we obtain the 3PM potential and it satisfies multiple nontrivial checks.

map of perturbative theory

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	7PN		
1PM	$\big(1$	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8$	$+ v^{10}$	$+ v^{12}$	$+ v^{14}$	$+ \cdots \big)$	G
2PM		$\big(1$	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8$	$+ v^{10}$	$+ v^{12}$	$+ \cdots \big)$	G^2
3PM			$\big(1$	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8$	$+ v^{10}$	$+ \cdots \big)$	G^3
4PM				$\big(1$	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8$	$+ \cdots \big)$	G^4
5PM					$\big(1$	$+ v^2$	$+ v^4$	$+ v^6$	$+ \cdots \big)$	G^5

$$\vdots$$

map of perturbative theory

0PN 1PN 2PN 3PN 4PN 5PN 6PN 7PN

1PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + v^{14} + \cdots \right) G$$

2PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots \right) G^2$$

3PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \cdots \right) G^3$$

4PM $$\left(1 + v^2 + v^4 + v^6 + v^8 + \cdots \right) G^4$$

5PM $$\left(1 + v^2 + v^4 + v^6 + \cdots \right) G^5$$

$$\vdots$$

map of perturbative theory

0PN 1PN 2PN 3PN 4PN 5PN 6PN 7PN

1PM $\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + v^{14} + \cdots\right) G$

2PM $\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots\right) G^2$

3PM $\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \cdots\right) G^3$

4PM $\left(1 + v^2 + v^4 + v^6 + v^8 + \cdots\right) G^4$

5PM $\left(1 + v^2 + v^4 + v^6 + \cdots\right) G^5$

$\vdots$

looks doable

thank you!