Closed string amplitudes from single-valued correlation functions

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"There is no intellectual exercise that is not ultimately pointless" (J.L. Borges)

The knowledge of string theory perturbation is important for understanding many aspects of quantum and classical gravity

- The ultraviolet behaviour of $\mathcal{N}=8$ (maximal) supergravity in various dimensions up three loop order has been derived from a low-energy limit of superstring theory amplitude [Green, Schwarz, Brink; Green, Russo, Vanhove; Pioline]
- Match of the discontinuities of $\mathcal{N}=4$ amplitude with the low-energy limit of the flat-space $AdS_5 \times S^5$ string theory amplitude [Alday, Bissi, Perlmutter]
- The double copy formula has its origin in string theory. This has important application to the physics of gravitational waves. [many talks at this workshop]

Low energy expansion of string amplitudes

[Stieberger] noticed that the low-energy expansion of closed string tree-level amplitudes involve only single-valued multiple zeta values,

$$\zeta(s_1,\dots,s_r) = \sum_{n_r>\cdots>n_1\geqslant 1} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}}$$

Four gravitons tree-level closed string amplitude

$$\begin{aligned} A_{4g} = \Big(3 &+ 2\zeta(3)\alpha'^3\partial^6 + \zeta(5)\,\alpha'^5\partial^{10} + \frac{2\zeta(3)^2}{3}\,\alpha'^6\partial^{12} \\ &+ \frac{\zeta(7)}{2}\,\alpha'^7\partial^{14} + \frac{2}{3}\zeta(5)\zeta(3)\alpha'^8\partial^{16} + O(\alpha')^9\Big)\,\frac{\mathcal{R}^4}{\sigma_3} \end{aligned}$$

Whereas open string amplitudes have all zeta values in its low-energy expansion

$$A_4 = \Big(1 - \alpha'^2\zeta(2)\partial^4 - \alpha'^3\zeta(3)\partial^6 - \frac{\zeta(4)}{4}\alpha'^4\partial^8 + O(\alpha')^5\Big)\,\frac{F^4}{st}$$

Brown's single-valued multiple zeta values

Single-valued MZV are the value at $z=1$ of single-valued multiple polylogarithms $\mathrm{Li}^{sv}_{k_1,\dots,k_r}(z)$ on $\mathbb{C}\backslash\{0,1\}$:
$\zeta_{sv}(k_1,\dots,k_r) := \mathrm{Li}^{sv}_{k_1,\dots,k_r}(1)$

$$\zeta_{sv}(2n)=0; \qquad \zeta_{sv}(2n+1)=2\zeta(2n+1); \qquad n\in\mathbb{N}$$

At weight 11 a $\mathbb{Q}$-basis has dimension 9

$$\zeta(3,5,3),\ \zeta(3,5)\zeta(3),\ \zeta(3)^2\zeta(5),\ \zeta(11),$$
$$\zeta(2)\zeta(3)^3,\ \zeta(2)^4\zeta(3),\ \zeta(2)^3\zeta(5),\ \zeta(2)^2\zeta(7),\ \zeta(2)\zeta(9).$$

the basis of single-valued MZVs has dimension 3 [Brown; Schnetz]

$$\zeta_{sv}(3,5,3)=2\zeta(3,5,3)-2\zeta(3)\zeta(3,5)-10\zeta(3)^2\zeta(5),$$
$$\zeta_{sv}(3)^2\zeta_{sv}(5),\ \zeta_{sv}(11)$$

[Schlotterer, Stieberger; Stieberger, Taylor]
conjectured that

$$M^{\mathrm{closed}}=\mathrm{sv}A^{\mathrm{open}}$$

Proven recently [Schlotterer, Schnetz;Brown, Dupont]

Tree-level closed string amplitudes

Any closed string tree-level amplitudes (bosonic, type II superstring, heterotic string) can be decomposed on partial amplitudes

$$\begin{aligned} M_{N+3}(\mathbf{s},\boldsymbol{\epsilon}) &= \left\langle V_1(0)\prod_{r=2}^{N+1}\int d^2 w_i \mathcal{V}(w_r,\bar{w}_r) V_{N+2}(1) V_{N+3}(\infty)\right\rangle \\ &= \sum_r c_r(\mathbf{s},\boldsymbol{\epsilon}) M_{N+3}(\mathbf{s},\mathbf{n}^r,\bar{\mathbf{n}}^r) \end{aligned}$$

$c_r(\mathbf{s},\boldsymbol{\epsilon})$ rational functions of kinematic invariant, polarisation tensors, and colour factors

The partial amplitudes are generic building blocks to any closed string amplitudes

Closed string amplitudes from correlators

$$M_{N+3}(\mathbf{s}, \mathbf{n}, \bar{\mathbf{n}}) = \mathcal{G}_N(1,1)$$

is $z = 1$ value of CFT correlators ($\tilde{V}$ special polarisation strings vop)

$$\mathcal{G}_N(z,\bar{z}) :=$$
$$\left\langle \widetilde{V}_1(0) \prod_{r=2}^{N+1} \int d^2 z_i \widetilde{\mathcal{V}}(w_r, \bar{w}_r) \widetilde{V}_{N+2}(1) \widetilde{V}_{N+3}(\infty) : e^{ik_* \cdot X(z,\bar{z})} : \right\rangle$$
$$= \int_{\mathbb{C}^N} \prod_{1 \leqslant i < j \leqslant N} (w_i - w_j)^{g_{ij}} (\bar{w}_i - \bar{w}_j)^{\bar{g}_{ij}} \prod_{i=1}^{N} d^2 w_i$$
$$\prod_{i=1}^{N} w_i^{a_i} (w_i - 1)^{b_i} (w_i - z)^{c_i} \bar{w}_i^{\bar{a}_i} (\bar{w}_i - 1)^{\bar{b}_i} (\bar{w}_i - \bar{z})^{\bar{c}_i}$$

It is important that difference between the holomorphic and anti-holomorphic exponent are integer (spins)

$$\mathbf{a} - \bar{\mathbf{a}}, \mathbf{b} - \bar{\mathbf{b}}, \mathbf{c} - \bar{\mathbf{c}}, \mathbf{g} - \bar{\mathbf{g}} \in \mathbb{Z}$$

Holomorphic factorisation

CFT correlator decompose on conformal blocks

$$\mathcal{G}_N(z,\bar z) = \sum_{r,s=1}^{(N+1)!} G_{r,s} I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) I_s(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar z)$$

The conformal block are the ordered integrals

$$I_{(\sigma,\rho)}(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) = \int_{\Delta(\sigma,\rho)} \prod_{j=1}^{N} dw_j$$

$$\prod_{m,n} |w_m - w_n|^{g_{mn}} \prod_m w_m^{a_m} (w_m - 1)^{b_m} (w_m - z)^{c_m},$$

integrated along the real line

$$\Delta(\sigma,\rho) := \{0 \leqslant w_{\rho(1)} \leqslant \cdots \leqslant w_{\rho(s)} \leqslant z \leqslant 1 \leqslant w_{\sigma(1)} \leqslant \cdots \leqslant w_{\sigma(r)}\}$$

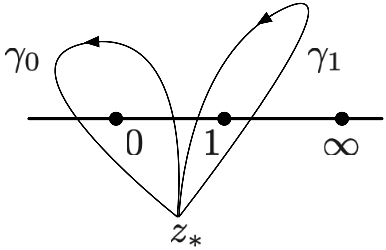
Their value at $z = 1$ are open string amplitudes

Monodromies

As CFT correlator $\mathcal{G}_N(z,\bar{z})$ is single-valued in $\mathbb{C}$

$$\mathcal{G}_N(z,\bar{z}) = \sum_{r,s=1}^{(N+1)!} G_{r,s} I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) I_s(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$$

The integrals $I_r(\cdots;z)$ have monodromies



$$I_r(\cdots;z) \xrightarrow{\gamma_0} \sum_s (g_0)_r{}^s I_s(\cdots;z)$$

$$I_r(\cdots;z) \xrightarrow{\gamma_1} \sum_s (g_1)_r{}^s I_s(\cdots;z)$$

The monodromy matrices g_0 and g_1 are the *same* for $I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z)$ and $I_r(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$ because $\mathbf{a}-\bar{\mathbf{a}} \in \mathbb{Z}^N$, $\mathbf{b}-\bar{\mathbf{b}} \in \mathbb{Z}^N$, $\mathbf{c}-\bar{\mathbf{c}} \in \mathbb{Z}^N$, $\mathbf{g}-\bar{\mathbf{g}} \in \mathbb{Z}^{\frac{N(N+1)}{2}}$

Monodromies around $z = 0$

$I_r(\cdots; z)$ have diagonal monodromies around $z = 0$

$$\mathcal{G}_N(z, \bar{z}) = \sum_{r,s=1}^{(N+1)!} G_{r,s} I_r(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{g}; z) I_s(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}; \bar{\mathbf{g}}; \bar{z})$$

This imposes that the matrix G_{rs} has the bloc diagonal form

$$G_N = \begin{pmatrix} G_N^{(1)} & 0 & 0 \\ 0 & G_N^{(2)} & 0 \\ 0 & 0 & G_N^{(3)} \end{pmatrix}$$

- $G_N^{(i)}$ with $i = 1, 3$ are real square matrices of size $N!$
- $G_N^{(2)}$ are diagonal matrix of size $(N-1)\,N!$

Monodromies around $z = 1$

The monodromies of $I_r(\cdots ; z)$ around $z = 1$ are not diagonal but $J_r(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{g}; z) := I_r(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{g}; 1 - z)$ have diagonal monodromies around $z = 1$

$$\mathcal{G}_N(z, \bar{z}) = \sum_{r,s=1}^{(N+1)!} \hat{G}_{r,s} J_r(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{g}; z) J_s(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}; \bar{\mathbf{g}}; \bar{z})$$

therefore

$$\hat{G}_N = \begin{pmatrix} \hat{G}_N^{(1)} & 0 & 0 \\ 0 & \hat{G}_N^{(2)} & 0 \\ 0 & 0 & \hat{G}_N^{(3)} \end{pmatrix}$$

- $\hat{G}_N^{(i)}$ with $i = 1, 3$ are real square matrices of size $N!$
- $\hat{G}_N^{(2)}$ are diagonal matrix of size $(N - 1)\, N!$

Monodromies constraints

The two sets of integral are related by linear relations derived using the contour deformation method of [Bjerrum-Bohr, Damgaard, Vanhove]

$$I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) = \sum_{r=1}^{(N+1)!} S(\mathbf{A},\mathbf{B},\mathbf{C};\mathbf{G})_r{}^s J_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z)$$

We need to solve the linear system

$$S(\mathbf{A},\mathbf{B},\mathbf{C};\mathbf{G}) \begin{pmatrix} G_N^{(1)} & 0 & 0 \\ 0 & G_N^{(2)} & 0 \\ 0 & 0 & G_N^{(3)} \end{pmatrix} S(\mathbf{A},\mathbf{B},\mathbf{C};\mathbf{G})$$

must have the above block diagonal form of $\hat{G}_N$

Monodromies constraints

- The linear system has *unique solution* up to an scale
- Matching the closed string partial amplitude *determines* the scale factor, *therefore there is no ambiguities*
- The coefficients of the matrices G_N and $\hat{G}_N$ are rational functions $\sin(\pi\alpha' x)$ where x are linear combination of kinematic invariants. This is a non-local version of the momentum kernel
- The small α' expansion of the $I_r(\cdots;z)$ and $J_r(\cdots;z)$ are on multiple polylogarithm with coefficients polynomials of MZV's and $2\pi i$.
- The proof is constructive as it is for CFT minimal models

  [Dotsenko, Fateev]

Matching closed string amplitudes

At $z = 1$ we get the colour-ordered open string amplitudes

$$J_{(\sigma,\emptyset)}(\mathfrak{a},\mathfrak{b},\mathfrak{c};\mathfrak{g};1) = A_{N+3}(\sigma(1,\ldots,N+1),1,N+2,N+3;\mathfrak{n})$$
$$J_{(\sigma,\rho)}(\mathfrak{a},\mathfrak{b},\mathfrak{c};\mathfrak{g};1) = 0$$

$$M_{N+3}(\mathfrak{s},\mathfrak{n},\bar{\mathfrak{n}}) = \sum_{\sigma,\rho\in\mathfrak{S}_N} \hat{G}_{\sigma,\rho} \times A_{N+3}(\sigma(2,\ldots,N+1),1,N+2,N+3;\mathfrak{n}) \times \bar{A}_{N+3}(\rho(2,\ldots,N+1),1,N+2,N+3;\bar{\mathfrak{n}}),$$

The α' has only single-valued multiple zeta values as the valuation at $z = 1$ of combination of single-valued multiple polylogarithms

Remarks

- It is not necessary that the total amplitude is given by the special value at $z = 1$ of a single-valued correlation function. It is enough that each partial amplitude arises this way
- a given order in the α'-expansion can mix single-valued multiple zeta values of different way (due to tachyonic pole in the kinematic coefficients $c_r(\mathbf{s}, \boldsymbol{\epsilon})$ for heterotic-string amplitudes)
- We mathematically prove that each of these convergent integrals, i.e. the coefficients of the small α'-expansion, is a single-valued multiple-zeta value. This is done by repeatedly applying Lemma D.1 of part II of the paper

Four point case

The partial amplitudes of four points tree-level closed string amplitudes are (n_{ij}, $\bar{n}_{ij}$ in $\mathbb{Z}$)

$$M_4(\mathbf{s},\mathbf{n},\bar{\mathbf{n}}) = \int_{\mathbb{C}} d^2w |w|^{2\alpha' k_1\cdot k_2} |1-w|^{2\alpha' k_2\cdot k_3} \times w^{n_{12}} \bar{w}^{\bar{n}_{12}} (1-w)^{n_{23}} (1-\bar{w})^{\bar{n}_{23}} .$$

and the single-valued correlator

$$\mathcal{G}_1(z,\bar{z}) := \int_{\mathbb{C}} w^{a_1}(w-1)^{b_1}(w-z)^{c_1} \bar{w}^{\bar{a}_1}(\bar{w}-1)^{\bar{b}_1}(\bar{w}-\bar{z})^{\bar{c}_1} d^2w$$

such that $\mathcal{G}_1(1,1) = M_4(\mathbf{s},\mathbf{n},\bar{\mathbf{n}})$

Four point case: holomorphic factorisation

$$\mathcal{G}_1(z,\bar z) = \begin{pmatrix} J_{((1),\emptyset)}(\bar a,\bar b,\bar c;\bar z) & J_{(\emptyset,(1))}(\bar a,\bar b,\bar c;\bar z)\end{pmatrix} \hat G_1 \begin{pmatrix} J_{((1),\emptyset)}(a,b,c;z) \\ J_{(\emptyset,(1))}(a,b,c;z)\end{pmatrix}$$

with

$$\hat G_1 = \begin{pmatrix} -\frac{\sin(\pi(A_1+B_1+C_1))\sin(\pi A_1)}{\sin(\pi(B_1+C_1))} & 0 \\ 0 & -\frac{\sin(\pi C_1)\sin(\pi B_1)}{\sin(\pi(B_1+C_1))}\end{pmatrix}$$

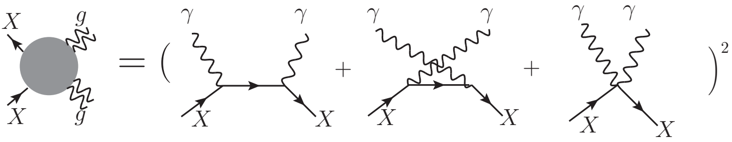
The J_r integrals map at $z=1$ to the open string amplitudes

$$J_{((1),\emptyset)}(a,b,c;1) = A_4(2,1,3,4;\mathfrak{n});$$
$$J_{(\emptyset,(1))}(a,b,c;1) = 0\,.$$

The value at $z=1$ gives $M_4(\mathfrak{s},\mathfrak{n},\bar{\mathfrak{n}}) = \mathcal{G}_1(1,1)$ gives the non-local version of the KLT relations given in [Bjerrum-Bohr, Damgaard, Vanhove]

$$M_4(\mathfrak{s},\mathfrak{n},\bar{\mathfrak{n}}) = \frac{\sin(2\pi\alpha' k_1\cdot k_2)\sin(2\pi\alpha' k_2\cdot k_4)}{\sin(2\pi\alpha' k_2\cdot k_3)}|A_4(2,1,3,4;\mathfrak{n})|^2$$

Gravitational Compton scattering



The gravity Compton scattering is a product of two abelian QED Compton amplitudes [Bjerrum-Bohr, Donoghue, Vanhove]

$$\mathfrak{M}(X^s g \to X^s g) = G_N \frac{(p_1 \cdot k_1)(p_1 \cdot k_2)}{k_1 \cdot k_2} \mathcal{A}_s(1324)\tilde{\mathcal{A}}_0(1324)$$

This expression was particular useful when evaluation the post-Minkowskian correction from one-loop amplitude

Conclusion

- The construction clarifies the role of the momentum kernel in the single-valued projection $\mathcal{S}_{\alpha'}$ is one block of $\mathrm{G}_N S(\mathbf{A}, \mathbf{B}, \mathbf{C}; \mathbf{G})$
- Notice that the α' expansion does not need to have uniform weight: tree-level heterotic string from the tachyonic pole, or genus two type II expansion [Green, Vanhove]
- Closed string amplitude are special value single-valued CFT correlators, and open strings are multivalued conformal block extended to higher genus
- The low-energy expansion of genus one closed string amplitudes has single-valued modular graph functions [D'Hoker, Green, Gurdogan, Vanhove; Zerbini; Brown; Gerken, Kleinschmidt, Schlotterer]
- Single-valued modular graph functions in degeneration limits of genus-two amplitudes [D'Hoker, Green, Pioline]