From Coalescing Binary Black Holes to Quantum Amplitudes, and Back

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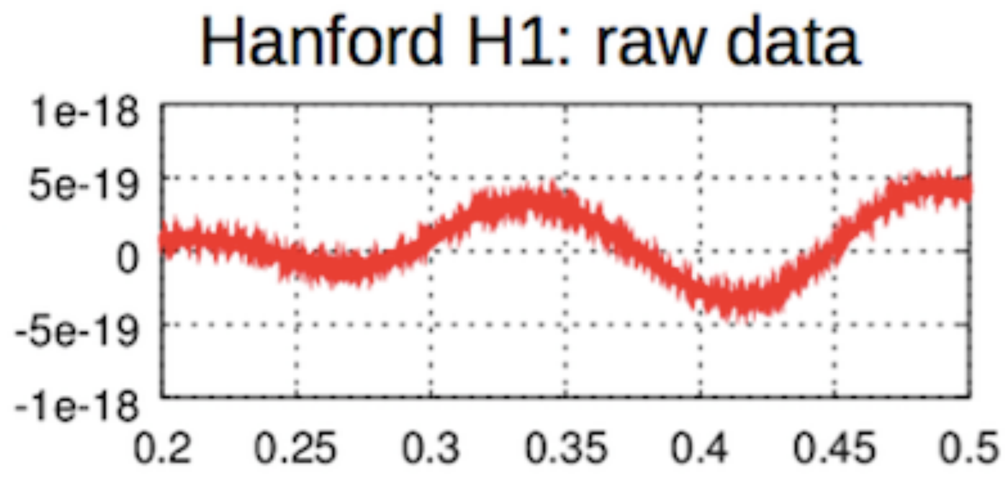
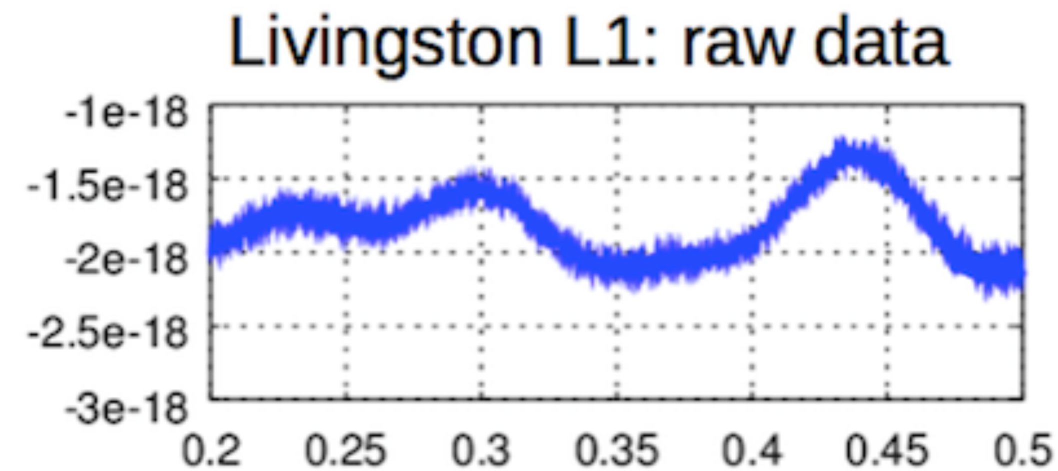
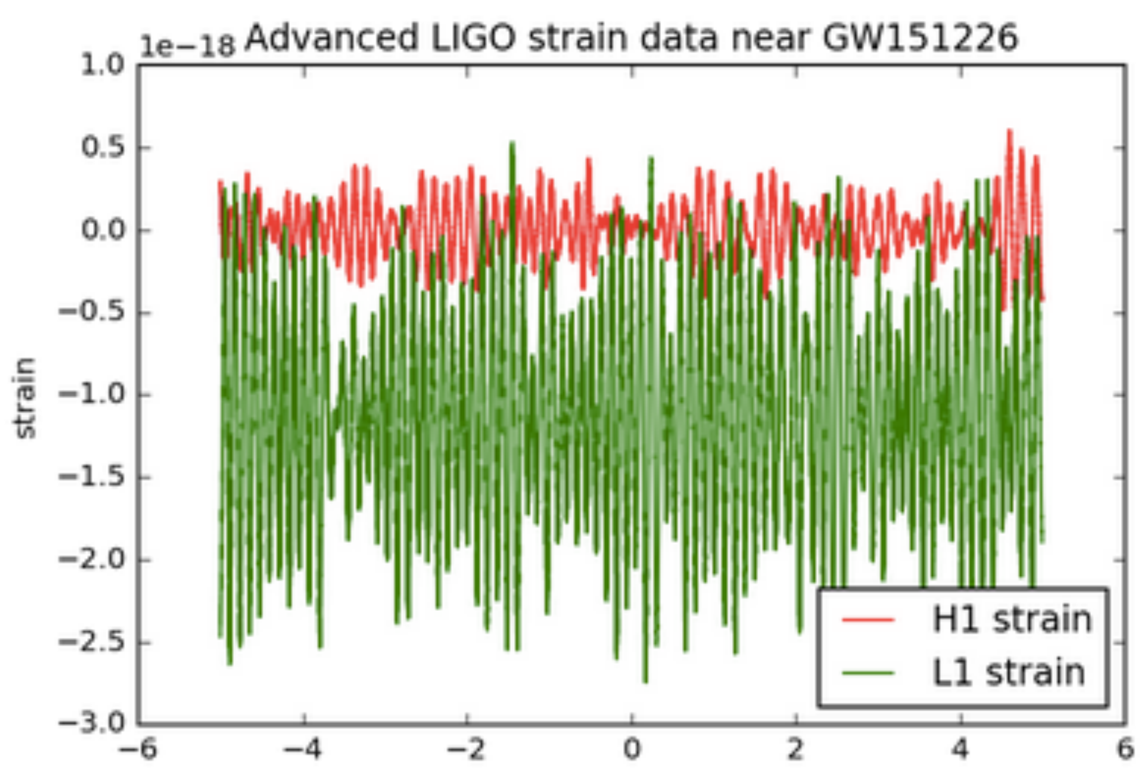
LIGO Raw Data for First Binary Black Hole Events

GW150914 from LIGO open data

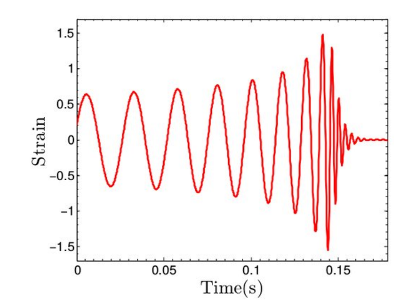




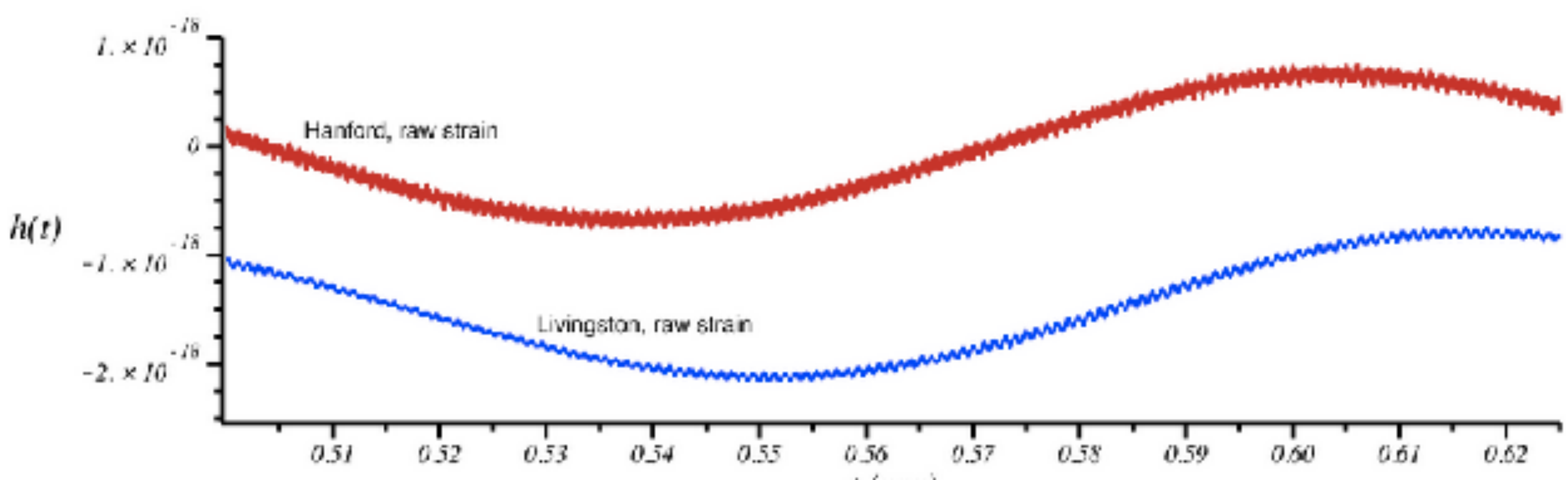
GW151226 from LIGO open data



$$\left[\frac{\delta L}{L}\right]_{\mathrm{obs}} \sim 10^{-18}$$
$$\gg h_{GW} \lesssim 10^{-21}$$



GW170104 from LIGO open data

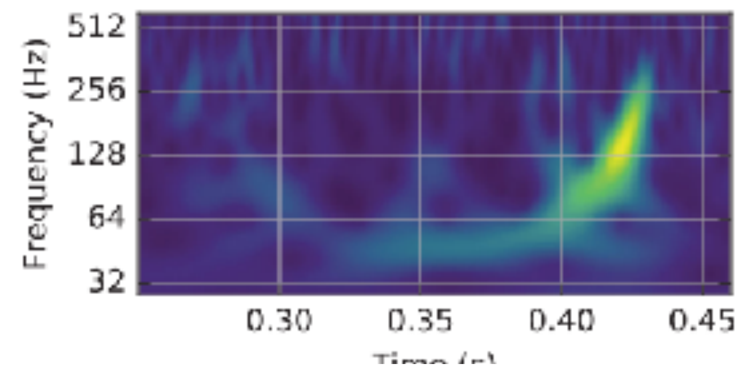


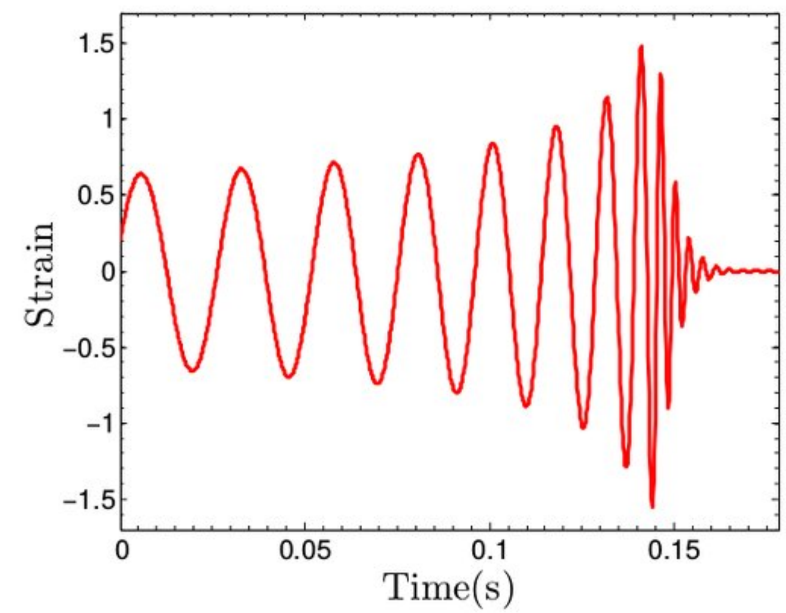
LIGO-Virgo data analysis

Various levels of search and analysis of signals:

Online trigger searches:

CoherentWaveBurst Time-frequency
(Wilson, Meyer, Daubechies-Jaffard-Journe, Klimenko et al.)
Omicron-LALInference sine-Gaussians
Gabor-type wavelet analysis **(Gabor,...,Lynch et al.)**
Matched-filter:
PyCBC (f-domain), gstLAL (t-domain)





Offline data analysis:

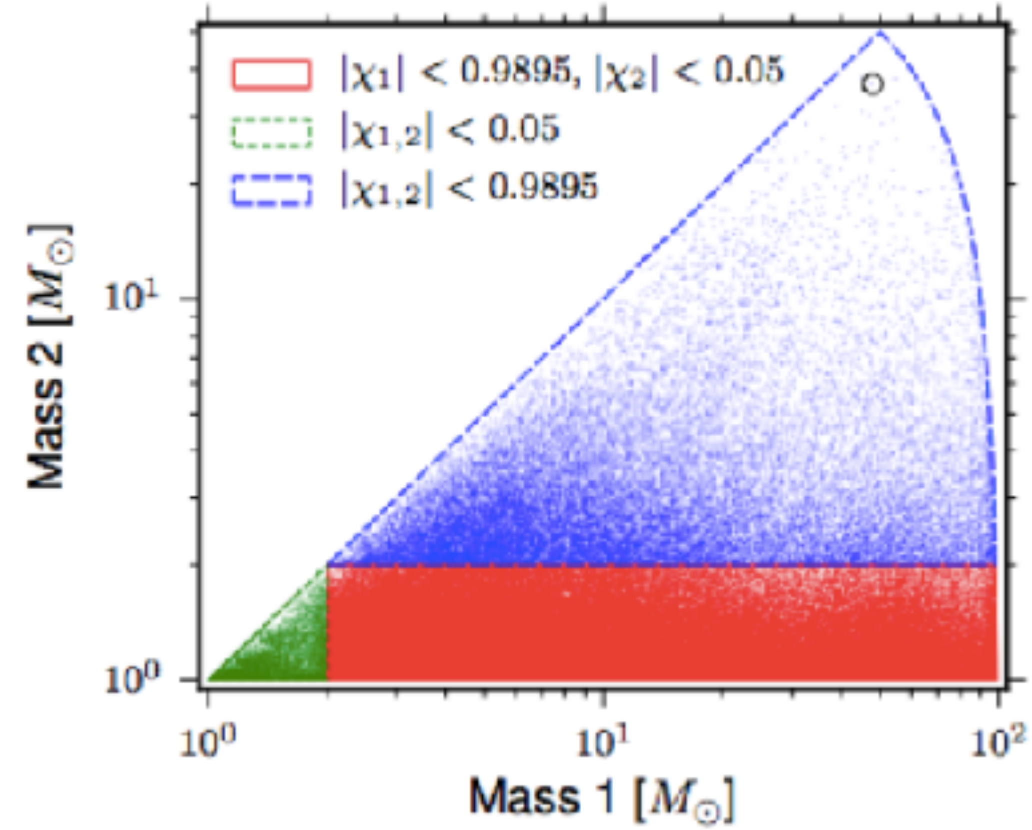
Generic transient searches
Binary coalescence searches

Here: focus on matched-filter definition
(crucial for high SNR, significance assessment, and parameter estimation)

Matched Filtering

$$\langle output|h_{\mathrm{template}}\rangle = \int \frac{df}{S_n(f)} o(f) h^*_{\mathrm{template}}(f)$$

LIGO’s bank of search templates
O1: 200 000 EOB + 50 000 PN
O2: 325 000 EOB + 75 000 PN
(Taracchini et al.'14, Bohé et al.’17)



PN

$v \ll c$

$R \gg GM/c^2$

EOB

LISA’s templates
via EOB[SF] ?

PM

$R \gg GM/c^2$

NR

$v \sim c$

$R \sim GM/c^2$

QFT
perturbation theory

STRING
perturbation theory

SF

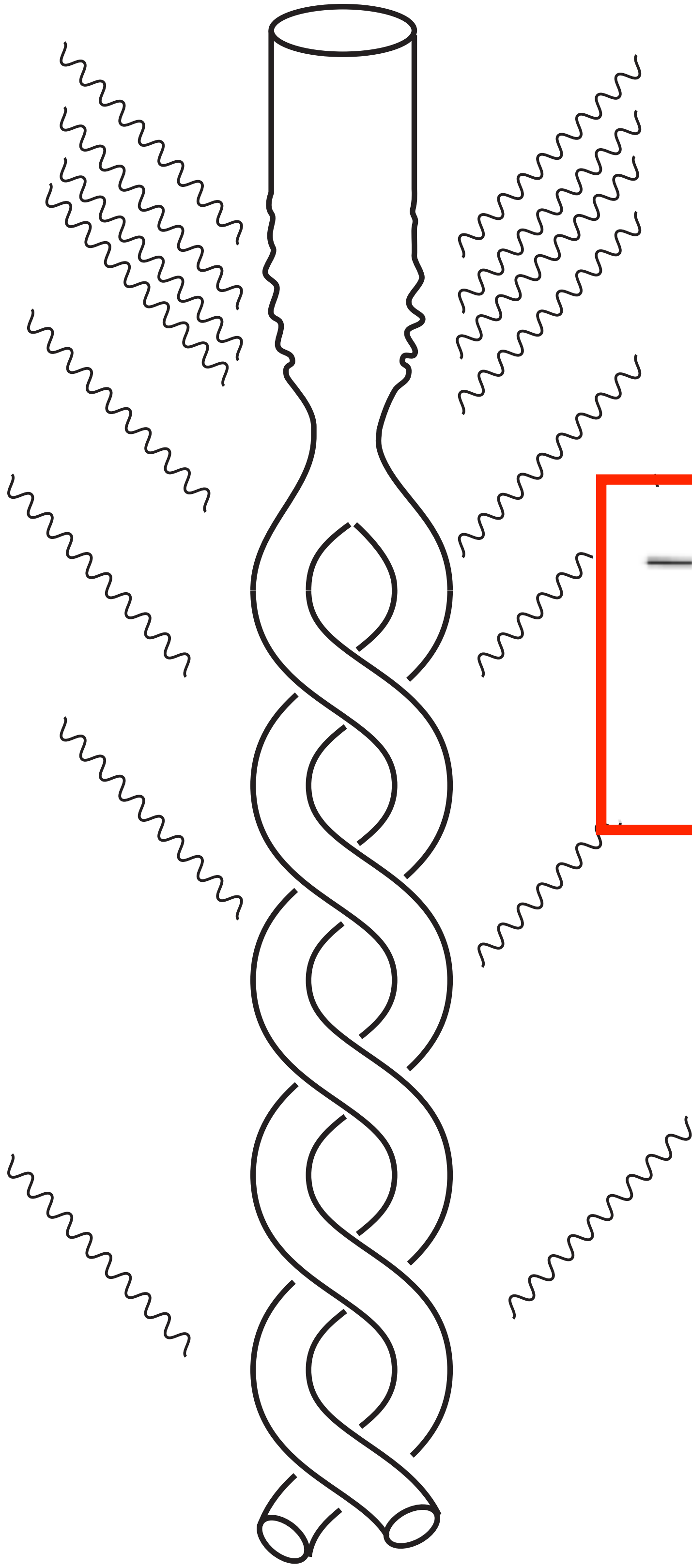
$m_1 \ll m_2$

Quantum Scattering Amplitudes

EOB= Effective-One-Body
Buonanno-Damour 1999, 2000;
Damour-Jaranowski-Schaefer 2000;
Damour 2001

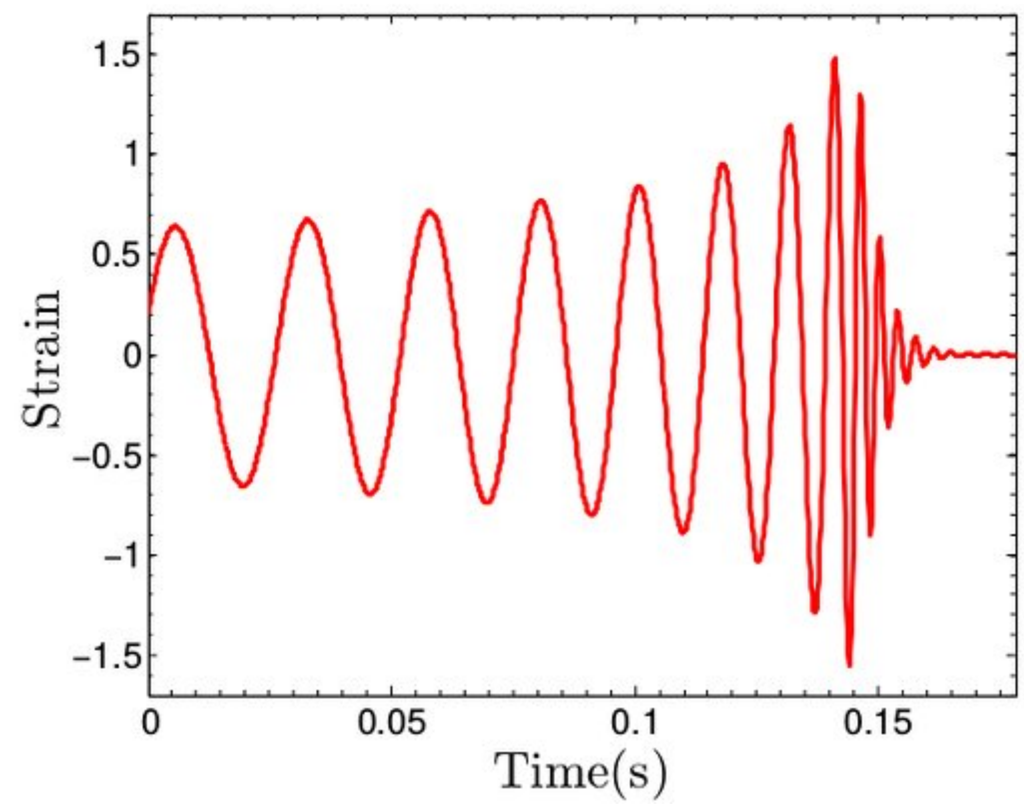
$$ds^2 = g_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu$$

$$\boxed{R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$



$$\boxed{R_{\mu\nu} = 0}$$

$$\boxed{\begin{aligned} -g^{\mu\nu} g_{\alpha\beta,\,\mu\nu} + g^{\mu\nu} g^{\rho\sigma} (g_{\alpha\mu,\,\rho} g_{\beta\nu,\,\sigma} - g_{\alpha\mu,\,\rho} g_{\beta\sigma,\,\nu} \\ + g_{\alpha\mu,\,\rho} g_{\nu\sigma,\,\beta} + g_{\beta\mu,\,\rho} g_{\nu\sigma,\,\alpha} - \frac{1}{2} g_{\mu\rho,\,\alpha} g_{\nu\sigma,\,\beta}) = 0 \end{aligned}}$$



Basic Physico-Mathematical Tools Having Allowed one to Predict GW signals from Coalescing Binary Black Holes

Matched Asymptotic Expansion approach to the motion of strongly self-gravitating bodies

Post-Minkowskian (PM) approximation theory to the motion of binary systems -> binary pulsars

$$\frac{dP}{dt} = -\frac{192\pi}{5c^5}\left(\frac{2\pi C}{P}\right)^{\frac{5}{3}} \mu M^{\frac{2}{3}} \frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{\frac{7}{2}}}$$

Post-Newtonian (PN) approximation theory to the motion of binary systems

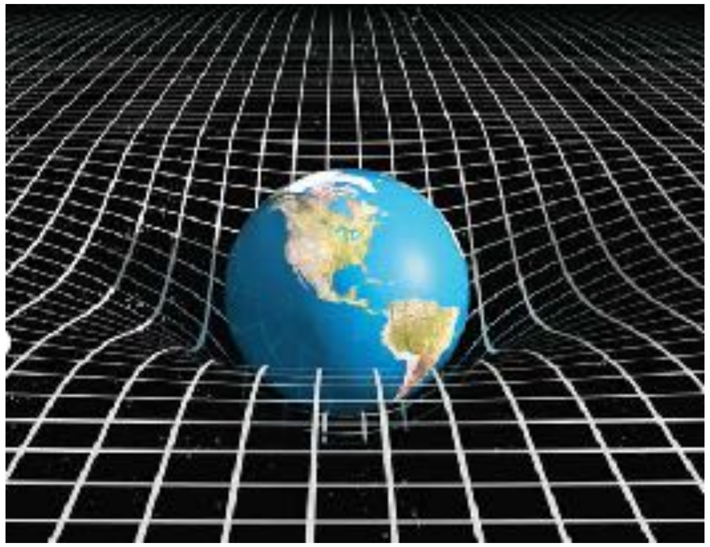
PN-matched Multipolar post-Minkowskian (MPM) approximation theory to the GW emission of binary systems

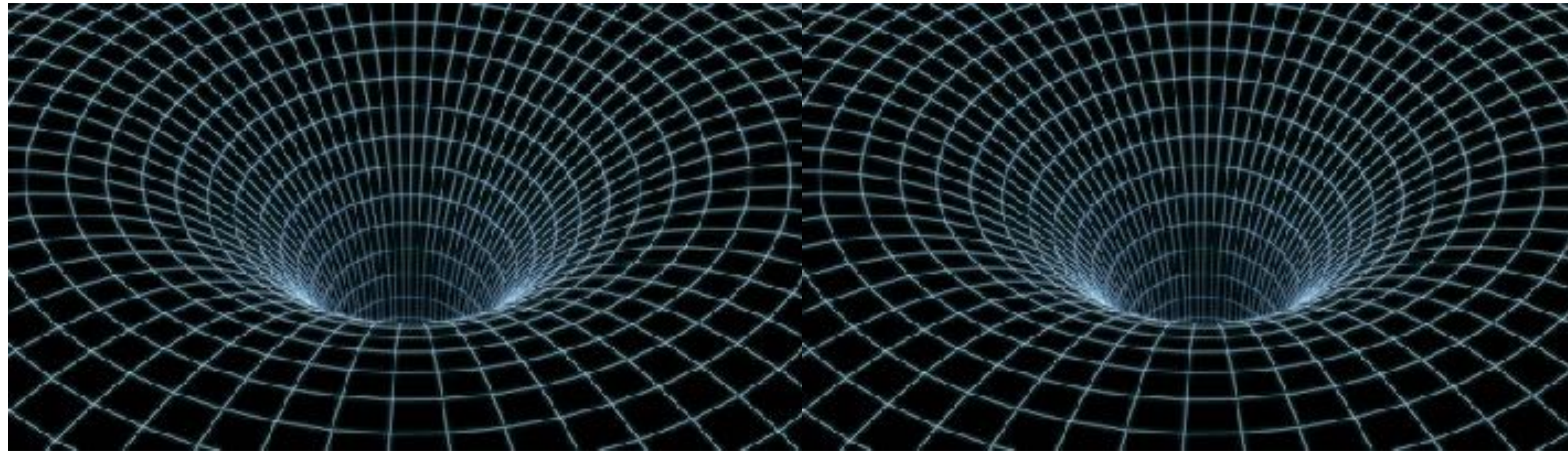
Effective One-Body (EOB) Approach to coalescing binary black holes (and binary neutron stars)

Hyperbolic Systems for Einstein's equations

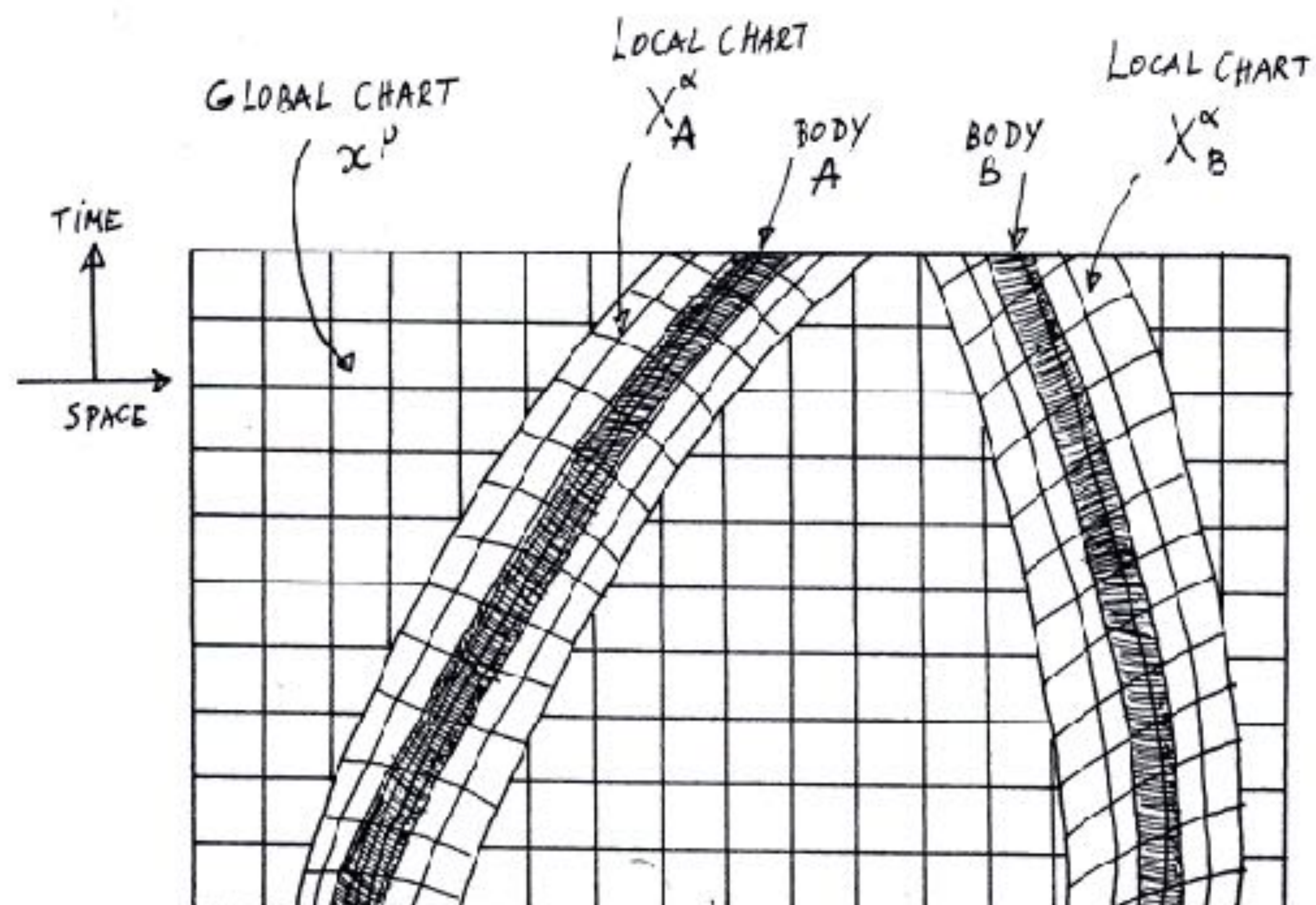
Numerical Relativity (NR) simulations of coalescing binary black holes (and binary neutron stars)

Challenge: Motion of Strongly Self-gravitating Bodies (NS, BH)





Multi-chart approach to motion of strong-self-gravity bodies, and matched asymptotic expansions [EIH '38], Manasse '63, Demianski-Grishchuk '74, D'Eath'75, Kates '80, Damour '82



Useful even for weakly self-gravitating bodies, i.e."relativistic celestial mechanics", Brumberg-Kopeikin '89, Damour-Soffel-Xu '91-94

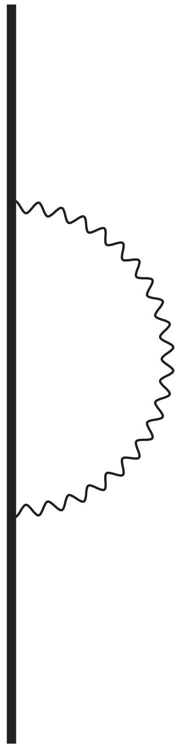
Combine two expansions in two charts:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + G h^{(1)}_{\mu\nu}(x) + G^2 h^{(2)}_{\mu\nu}(x) + \dots \quad G_{\alpha\beta}(x) = G^{(0)}_{\alpha\beta}(x) + G^{(1)}_{\alpha\beta}(x) + \cdots$$

Practical Techniques for Computing the Motion of Compact Bodies (NS or BH)

Skeletonization : $T_{\mu\nu} \longrightarrow$ point-masses (Mathisson ’31)

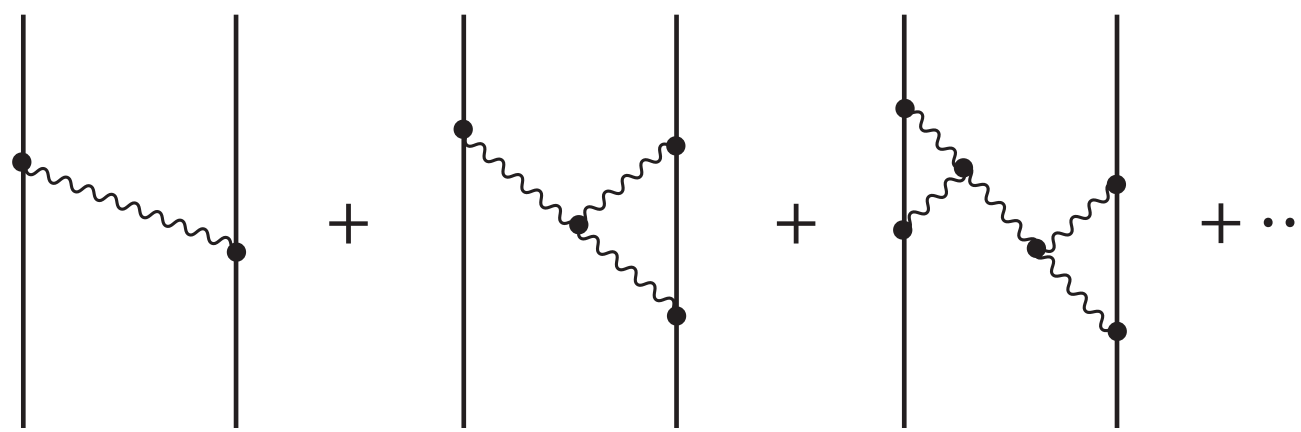
delta-functions in GR : Infeld ’54, Infeld-Plebanski ’60



justified by Matched Asymptotic Expansions (« Effacing Principle » Damour ’83 possible internal-structure dependence in strong self-gravity objects (NSs, BHs) only arise at 5PN= 5-loop level)

UV divergences linked to self-field effects (loops on external lines) [Dirac, 1938]

QFT’s analytic (Riesz ’49) or dimensional regularization (Bollini-Giambiagi ’72, t’Hooft-Veltman ’72) imported in GR (Damour ’80, Damour-Jaranowski-Schäfer ’01, …)



Feynman-like diagrams and

« Effective Field Theory » techniques

Bertotti-Plebanski ’60,

Damour-Esposito-Farèse ’96,

Goldberger-Rothstein ’06, Porto ‘06, Gilmore-Ross’ 08, Levi ’10,

Foffa-Sturani ’11 ‘13, Levi-Steinhoff ‘14, ‘15

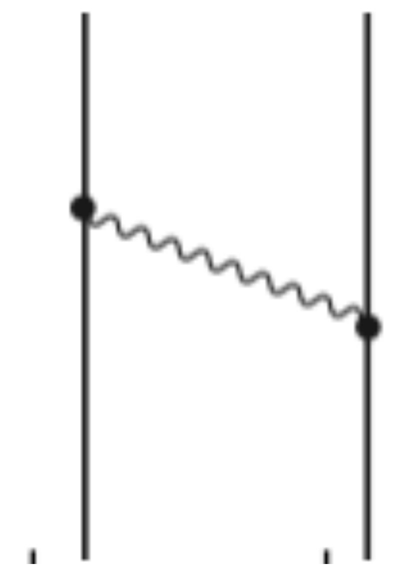
Reduced Worldline Action in Electrodynamics (Fokker 1929)

$$S_{\rm tot}[x_a^\mu, A_\mu] = -\sum_a \int m_a ds_a + \sum_a \int e_a dx_a^\mu A_\mu(x_a) - \int d^D x \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + S_{\rm gf}$$

« Integrate out » the field A_mu in the total (particle+field) action

$$\boxed{\begin{aligned} S_{\rm eff}^{\rm class}[x_a(s_a)] = &-\sum_a m_a \int \mathrm{d}s_a \\ &+ \frac{1}{2} \sum_{a,b} e_a e_b \iint \mathrm{d}x_a^\mu \, \mathrm{d}x_{b\mu} \, \delta\left((x_a - x_b)^2\right) . \end{aligned}}$$

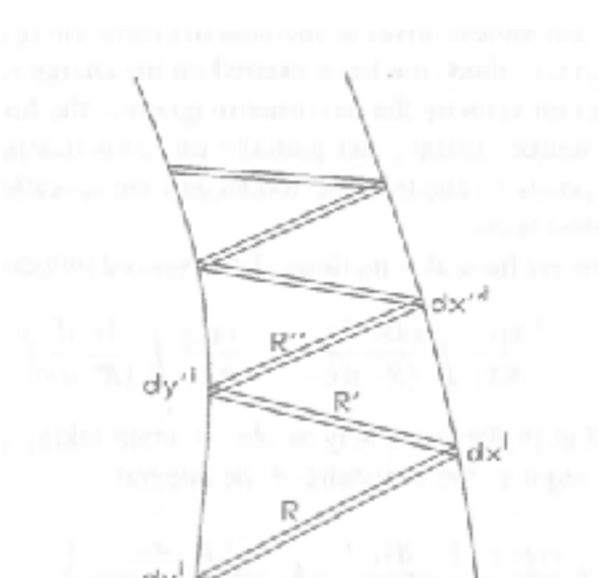
One-photon-exchange diagram

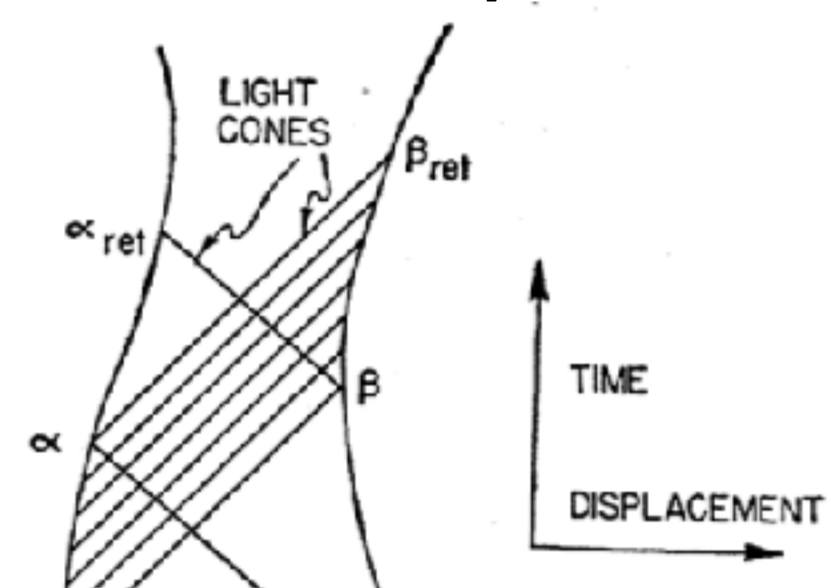


time-symmetric Green function G.

$$G(x) = \delta(-\eta_{\mu\nu} x^\mu x^\nu) = \frac{1}{2r} \left(\delta(t-r) + \delta(t+r)\right) \; ; \; \Box G(x) = -4\pi \delta^4(x)$$

The effective action S_eff(x_a) was heavily used in the (second) Wheeler-Feynman paper (1949) together with similar diagrams to those used by Fokker





Reduced Action in Gravity and its Diagrammatic Expansion

either $S_{\text{eff}}^{\text{class}}[x_a(s_a)] = [S_{\text{pm}} + S_{\text{EH}} + S_{\text{gf}}]_{g_{\mu\nu}(x) \to g_{\mu\nu}^{\text{gf}}[x_a(s_a)]}$ Damour-Esposito-Farese ‘96

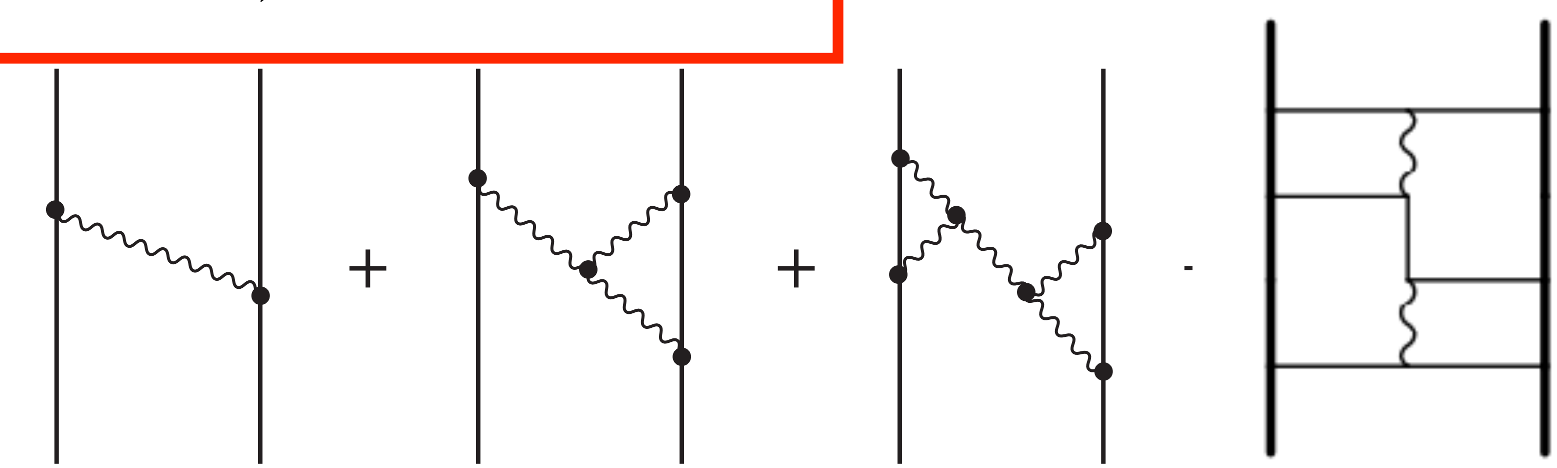
or $e^{\frac{i}{\hbar} S_{\text{eff}}^{\text{quant}}} = \int D g_{\mu\nu}\, e^{\frac{i}{\hbar}(S_{\text{pm}} + S_{\text{EH}} + S_{\text{gf}})}$. Goldberger-Rothstein ’06

Needs gauge-fixed* action and time-symmetric Green function G.*E.g. Arnowitt-Deser-Misner Hamiltonian formalism or harmonic coordinates.Perturbatively solving (in dimension D=4 - eps) Einstein’s equations to get the equations of motion and the action for the conservative dynamics

$$g = \eta + h$$
$$S(h,T) = \int \left(\frac{1}{2} h \Box h + \partial\partial hhh + \ldots + (h + hh + \ldots) T \right)$$
$$\Box h = -T + \ldots \to h = GT + \ldots$$
$$S_{\text{red}}(T) = \frac{1}{2} TGT + V_3(GT, GT, GT) + \ldots$$

Slow Motion (PN) expansion: in powers of 1/c^2:

1PN= (v/c)^2; 2PN= (v/c)^4, etc
nPN=(v/c)^(2n)



O(G)= Newtonian + (v/c)^n corrections | O(G^2)=1PN =1 loop | O(G^3)=2PN =2 loop | O(G^5)=4PN =4 loop

It has been explicitly shown that S_eff was UV finite (in ADM gauge, and in dim.reg.) at 3 loops (Damour-Jaranowski-Schäfer ’01) and 4 loops (Damour-Jaranowski-Schäfer ’14, Jaranowski-Schäfer ’15)

There appear IR divergences at 4PN (4 loop) linked to non-locality (Blanchet-Damour ’88).

Post-Newtonian Equations of Motion [2-body, wo spins]

- 1PN (including v^2/c^2) [Lorentz-Droste '17], Einstein-Infeld-Hoffmann '38
- 2PN (inc. v^4/c^4) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
  Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc. v^5/c^5) Damour-Deruelle '81, Damour '82, Schäfer '85,
  Kopeikin '85
- 3 PN (inc. v^6/c^6) Jaranowski-Schäfer '98, Blanchet-Faye '00,
  Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
  Blanchet-Damour-Esposito-Farèse' 04, Foffa-Sturani '11
- 3.5 PN (inc. v^7/c^7) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
  Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09
- 4PN (inc. v^8/c^8) Jaranowski-Schäfer '13, Foffa-Sturani '13,'16
  Bini-Damour '13, Damour-Jaranowski-Schäfer '14, Bernard et al'16

New feature : non-locality in time (Blanchet-Damour'88)

Inclusion of spin-dependent effects: Barker-O' Connell'75, Faye-Blanchet-Buonanno'06, Damour-Jaranowski-Schaefer'08, Porto-Rothstein '06, Levi '10, Steinhoff-Hergt-Schaefer '10, Steinhoff'11, Levi-Steinhoff'15-18

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_{\mathrm{N}}(\mathbf{x}_a,\mathbf{p}_a)=\frac{\mathbf{p}_1^2}{2m_1}-\frac{1}{2}\frac{Gm_1m_2}{r_{12}}+(1\leftrightarrow 2)$$

$$c^2H_{\mathrm{1PN}}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2}+14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}+(1\leftrightarrow 2),$$

$$c^4H_{\mathrm{2PN}}(\mathbf{x}_a,\mathbf{p}_a)=\frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}-\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.$$
$$\left.-6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}-\frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)$$
$$+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2}+19\frac{\mathbf{p}_2^2}{m_2^2}\right)-\frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2)+6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}+(1\leftrightarrow 2),$$

2-body Taylor-expanded 3PN Hamiltonian [DJS 01]

$$
\begin{aligned}
c^6 H_{3\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = {} & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{G m_1 m_2}{r_{12}}\left(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4 m_2^2}\right.\\
& - 10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4 m_2^2}\\
& + 2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2\mathbf{p}_2^2 - 10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2^3}\\
& + \frac{(\mathbf{p}_1^2\mathbf{p}_2^2 - 2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2^3} + 15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3}\\
& \left. - 18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3} + 5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3}\right) + \frac{G^2 m_1 m_2}{r_{12}^2}\left(\frac{1}{16}(m_1 - 27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\right.\\
& - \frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} - \frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\
& - \frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2}\\
& + \frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\\
& \left. - \frac{1}{48}(220m_1 + 193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right) + \frac{G^3 m_1 m_2}{r_{12}^3}\left(-\frac{1}{48}\left(425m_1^2 + \left(473 - \frac{3}{4}\pi^2\right)m_1 m_2 + 150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\
& + \frac{1}{16}\left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \frac{1}{16}\left(20m_1^2 - \left(43 + \frac{3}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\
& \left. + \frac{1}{16}\left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right)\\
& + \frac{1}{8}\frac{G^4 m_1 m_2^3}{r_{12}^4}\left(\left(\frac{227}{3} - \frac{21}{4}\pi^2\right)m_1 + m_2\right) + (1 \leftrightarrow 2).
\end{aligned}
$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$c^8 H^{\rm loc}_{\rm 4PN}(\mathbf{x}_a,\mathbf{p}_a)=\frac{7(\mathbf{p}_1^2)^5}{256 m_1^9}+\frac{G m_1 m_2}{r_{12}}H_{48}(\mathbf{x}_a,\mathbf{p}_a)+\frac{G^2 m_1 m_2}{r_{12}^2}m_1 H_{46}(\mathbf{x}_a,\mathbf{p}_a)$$
$$+\frac{G^3 m_1 m_2}{r_{12}^3}\left(m_1^2 H_{441}(\mathbf{x}_a,\mathbf{p}_a)+m_1 m_2 H_{442}(\mathbf{x}_a,\mathbf{p}_a)\right)$$
$$+\frac{G^4 m_1 m_2}{r_{12}^4}\left(m_1^3 H_{421}(\mathbf{x}_a,\mathbf{p}_a)+m_1^2 m_2 H_{422}(\mathbf{x}_a,\mathbf{p}_a)\right)$$
$$+\frac{G^5 m_1 m_2}{r_{12}^5}H_{40}(\mathbf{x}_a,\mathbf{p}_a)+(1\leftrightarrow 2). \qquad \text{(A3)}$$

$$\begin{aligned}
H_{48}(\mathbf{x}_a,\mathbf{p}_a)={}&\frac{45(\mathbf{p}_1^2)^4}{128 m_1^8}-\frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64 m_1^6 m_2^2}+\frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64 m_1^6 m_2^2}-\frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16 m_1^6 m_2^2}\\
&-\frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32 m_1^6 m_2^2}+\frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64 m_1^6 m_2^2}-\frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64 m_1^6 m_2^2}-\frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256 m_1^5 m_2^3}\\
&+\frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128 m_1^5 m_2^3}+\frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256 m_1^5 m_2^3}-\frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3}\\
&-\frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2^3}-\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3}+\frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3}\\
&+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3}-\frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3}+\frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3}+\frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3}\\
&-\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128 m_1^5 m_2^3}-\frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256 m_1^5 m_2^3}-\frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3}\\
&+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128 m_1^5 m_2^3}-\frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3}-\frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64 m_1^4 m_2^4}+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64 m_1^4 m_2^4}\\
&-\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4 m_1^4 m_2^4}+\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^4 m_2^4}-\frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64 m_1^4 m_2^4}+\frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64 m_1^4 m_2^4}\\
&-\frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4}-\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4 m_1^4 m_2^4}+\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16 m_1^4 m_2^4}+\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16 m_1^4 m_2^4}\\
&-\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4}+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64 m_1^4 m_2^4}-\frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32 m_1^4 m_2^4}-\frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128 m_1^4 m_2^4}, \qquad \text{(A4a)}
\end{aligned}$$

$$\begin{aligned}
H_{46}(\mathbf{x}_a,\mathbf{p}_a)={}&\frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160 m_1^6}-\frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192 m_1^6}+\frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16 m_1^6}-\frac{63(\mathbf{p}_1^2)^3}{64 m_1^6}-\frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128 m_1^5 m_2}\\
&+\frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16 m_1^5 m_2}-\frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128 m_1^5 m_2}+\frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2}-\frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2}\\
&+\frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2}+\frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2}+\frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480 m_1^4 m_2^2}-\frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840 m_1^4 m_2^2}\\
&-\frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320 m_1^4 m_2^2}+\frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480 m_1^4 m_2^2}+\frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2}\\
&-\frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840 m_1^4 m_2^2}+\frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920 m_1^4 m_2^2}-\frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2}+\frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2}-\frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8 m_1^3 m_2^3}\\
&+\frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192 m_1^3 m_2^3}-\frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3}-\frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3}\\
&+\frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192 m_1^3 m_2^3}+\frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96 m_1^3 m_2^3}+\frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96 m_1^3 m_2^3}-\frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32 m_1^3 m_2^3}\\
&+\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3}-\frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3}-\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4 m_1^2 m_2^4}+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4 m_1^2 m_2^4}\\
&-\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2 m_1^2 m_2^4}+\frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^2 m_2^4}+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6 m_1^2 m_2^4}+\frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48 m_1^2 m_2^4}\\
&-\frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24 m_1^2 m_2^4}-\frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96 m_1^2 m_2^4}+\frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96 m_1^2 m_2^4}-\frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48 m_1^2 m_2^4}+\frac{13(\mathbf{p}_2^2)^3}{8 m_2^6}, \qquad \text{(A4b)}
\end{aligned}$$

$$\begin{aligned}
H_{441}(\mathbf{x}_a,\mathbf{p}_a)={}&\frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384 m_1^4}-\frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960 m_1^4}-\frac{6695(\mathbf{p}_1^2)^2}{1152 m_1^4}-\frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640 m_1^3 m_2}\\
&+\frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920 m_1^3 m_2}+\frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2}-\frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800 m_1^3 m_2}\\
&-\frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960 m_1^2 m_2^2}+\frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800 m_1^2 m_2^2}-\frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400 m_1^2 m_2^2}\\
&+\frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400 m_1^2 m_2^2}+\frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600 m_1^2 m_2^2}-\frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800 m_1^2 m_2^2}+\frac{105(\mathbf{p}_2^2)^2}{32 m_2^4}, \qquad \text{(A4c)}
\end{aligned}$$

$$\begin{aligned}
H_{442}(\mathbf{x}_a,\mathbf{p}_a)={}&\left(\frac{2749\pi^2}{8192}-\frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4}+\left(\frac{63347}{1600}-\frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4}+\left(\frac{375\pi^2}{8192}-\frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4}\\
&+\left(\frac{10631\pi^2}{8192}-\frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}+\left(\frac{13723\pi^2}{16384}-\frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2}\\
&+\left(\frac{1411429}{19200}-\frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2 m_2^2}+\left(\frac{248991}{6400}-\frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2 m_2^2}\\
&-\left(\frac{30383}{960}+\frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}+\left(\frac{1243717}{14400}-\frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2}\\
&+\left(\frac{2369}{60}+\frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2}+\left(\frac{43101\pi^2}{16384}-\frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3 m_2}\\
&+\left(\frac{56955\pi^2}{16384}-\frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2}, \qquad \text{(A4d)}
\end{aligned}$$

$$H_{421}(\mathbf{x}_a,\mathbf{p}_a)=\frac{64861\mathbf{p}_1^2}{4800 m_1^2}-\frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8 m_1 m_2}+\frac{105\mathbf{p}_2^2}{32 m_2^2}-\frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600 m_1^2}-\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2 m_1 m_2}, \qquad \text{(A4e)}$$

$$\begin{aligned}
H_{422}(\mathbf{x}_a,\mathbf{p}_a)={}&\left(\frac{1937033}{57600}-\frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2}+\left(\frac{176033\pi^2}{24576}-\frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2}+\left(\frac{282361}{19200}-\frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2}\\
&+\left(\frac{698723}{19200}+\frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}-\left(\frac{63641\pi^2}{24576}-\frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\\
&+\left(\frac{3200179}{57600}-\frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}, \qquad \text{(A4f)}
\end{aligned}$$

$$H_{40}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{m_1^4}{16}+\left(\frac{6237\pi^2}{1024}-\frac{169799}{2400}\right)m_1^3 m_2+\left(\frac{44825\pi^2}{6144}-\frac{609427}{7200}\right)m_1^2 m_2^2. \qquad \text{(A4g)}$$

$$H^{\rm nonloc}_{\rm 4PN}(t)=-\frac{1}{5}\frac{G^2 M}{c^8}I^{(3)}_{ij}(t)\times \mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|}I^{(3)}_{ij}(t+v).$$

Perturbative Theory of the Generation of Gravitational Radiation

Einstein '16, '18 (+ Landau-Lifshitz 41, and Fock '55) : h_+, h_x and quadrupole formula

Relativistic, multipolar extensions of LO quadrupole radiation :

Sachs-Bergmann '58, Sachs '61, Mathews '62, Peters-Mathews '63, Pirani '64

Campbell-Morgan '71,

Campbell et al '75,

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

nonlinear effects:

Bonnor-Rotenberg '66,

Epstein-Wagoner-Will '75-76

Thorne '80, .., Will et al 00

$$x = \left(\frac{v}{c}\right)^2 = \left(\frac{G(m_1+m_2)\Omega}{c^3}\right)^{\frac{2}{3}} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{2}{3}}$$

MPM Formalism:

Blanchet-Damour '86,

Damour-Iyer '91,

Blanchet '95 '98

Combines multipole exp. ,

Post Minkowkian exp.,

analytic continuation,

and PN matching

$$\boxed{\begin{aligned}\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Big\{ 1 &+ \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} \\
&+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
&+ \Big[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16\,x) \\
&\quad + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\Big]x^3 \\
&+ \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Big\}.\end{aligned}}$$

Analytical GW Templates for BBH Coalescences ?

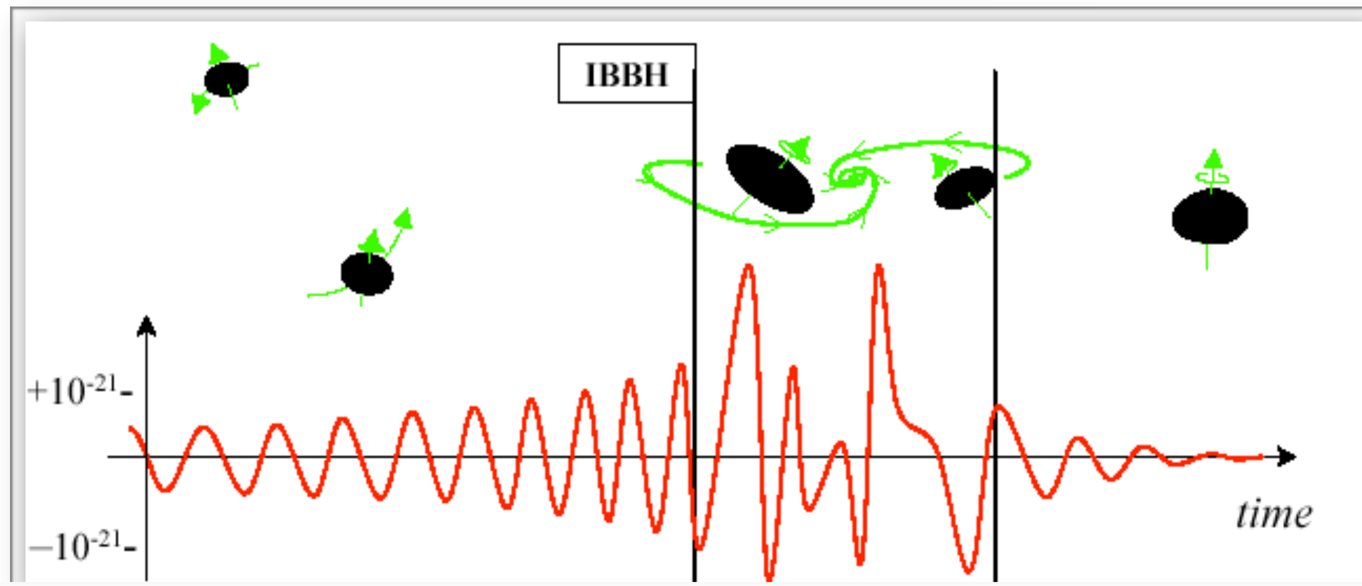
PN corrections to Einstein's quadrupole frequency « chirping »
from PN-improved balance equation dE(f)/dt = - F(f)

$$\frac{d\phi}{d\ln f} = \frac{\omega^2}{d\omega/dt} = Q_\omega^N \widehat{Q}_\omega \qquad \frac{v}{c} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{1}{3}}$$

$$Q_\omega^N = \frac{5\,c^5}{48\,\nu\,v^5}\,;\ \widehat{Q}_\omega = 1 + c_2\left(\frac{v}{c}\right)^2 + c_3\left(\frac{v}{c}\right)^3 + \cdots \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

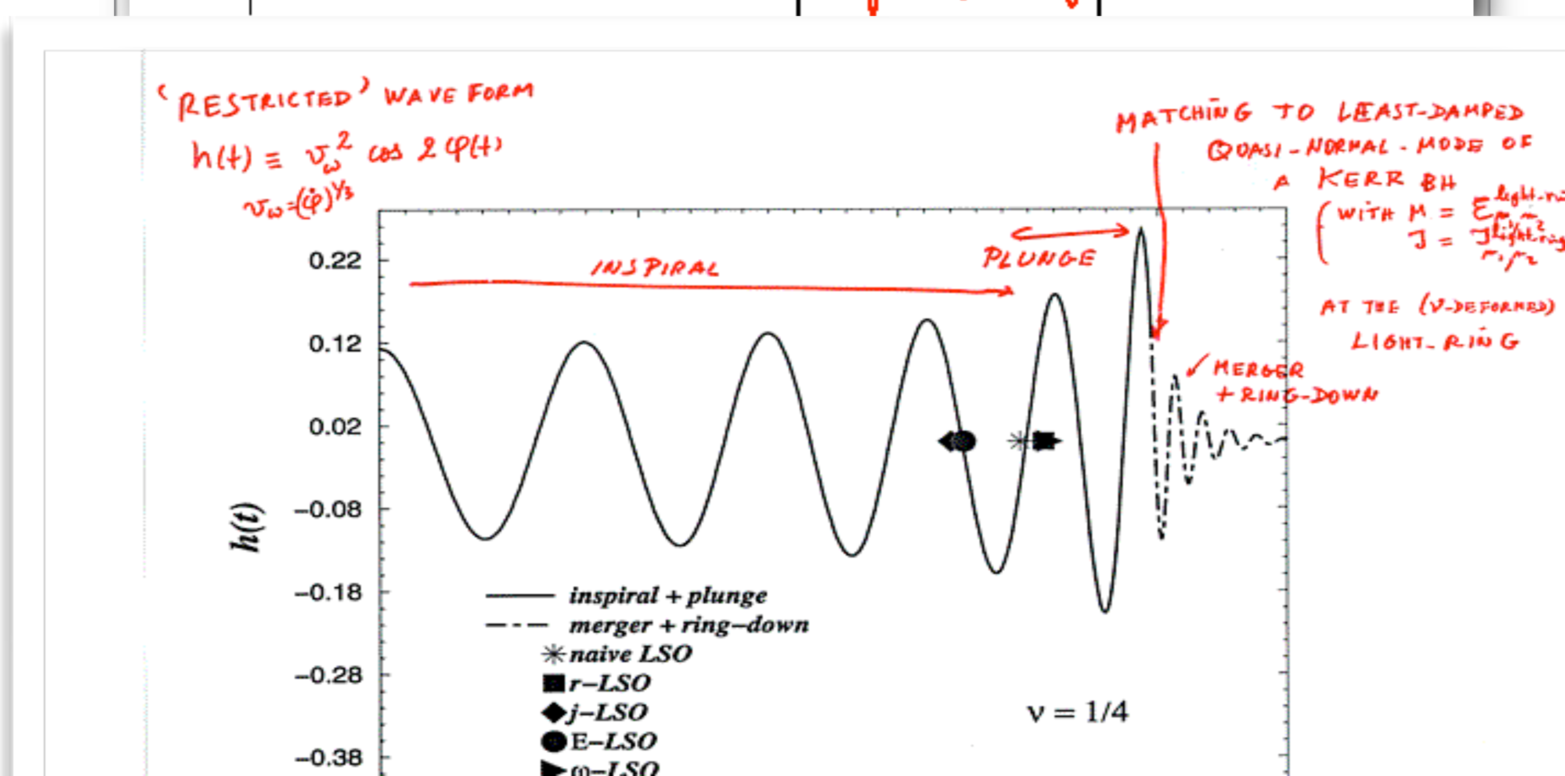
Cutler et al. '93:
« slow convergence of PN »

Brady-Creighton-Thorne'98:
« inability of current computational techniques to evolve a BBH through its last ~10 orbits of inspiral » and to compute the merger



Damour-Iyer-Sathyaprakash'98:
use resummation methods for E and F

Buonanno-Damour '99-00:
novel, resummed approach:
Effective-One-Body
analytical formalism

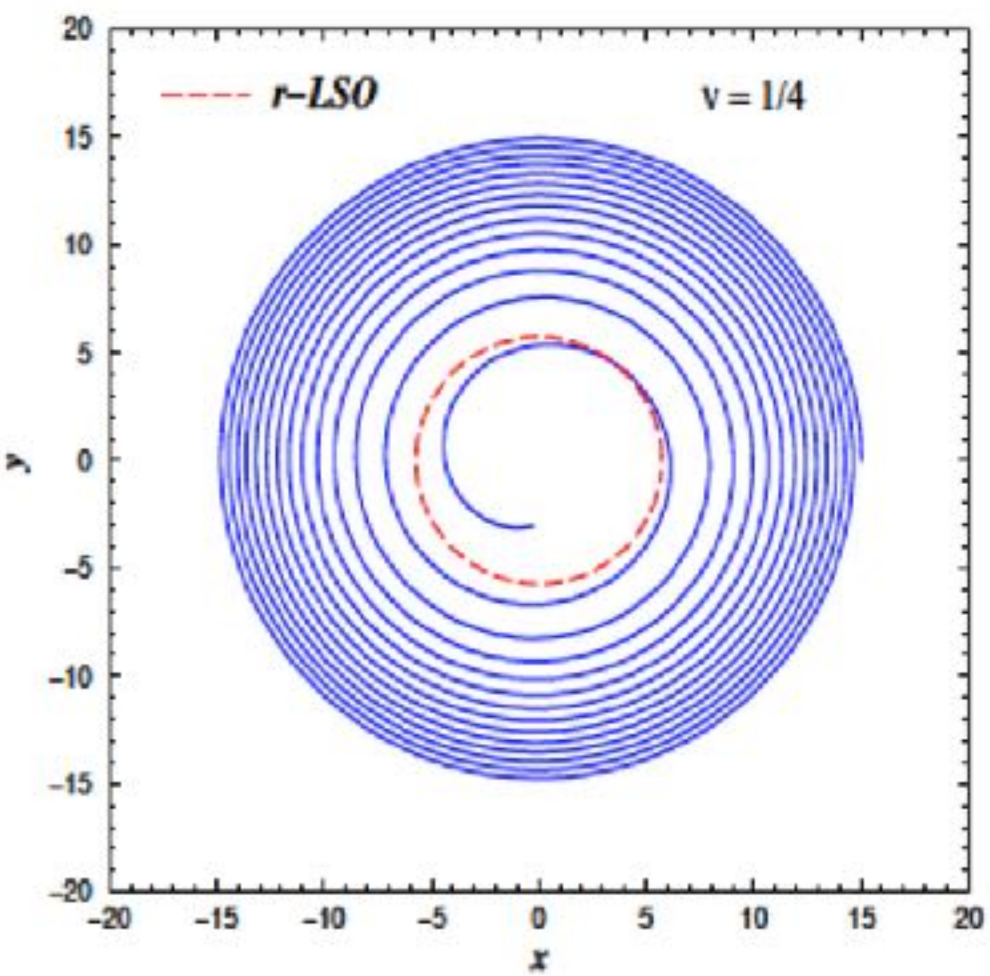


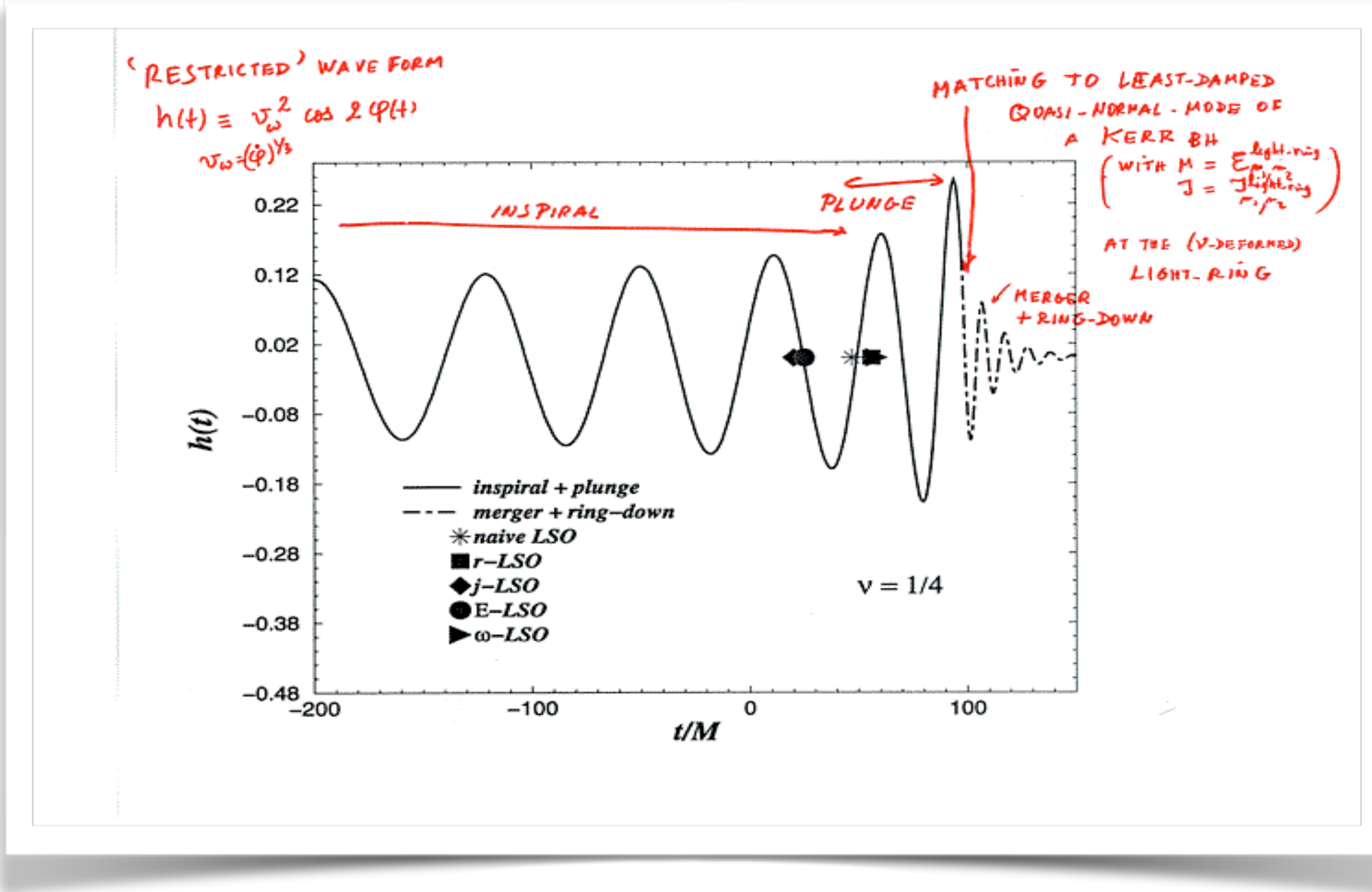
Effective One Body (EOB) Method)

Buonanno-Damour 1999, 2000; Damour-Jaranowski-Schaefer 2000; Damour 2001; Damour-Nagar 2007; Damour-Iyer-Nagar 2009

Resummation of both the Hamiltonian, the waveform and radiation-reaction
—> description of the coalescence + addition of ringdown (Vishveshwara 70, Davis-Ruffini-Tiomno 72)

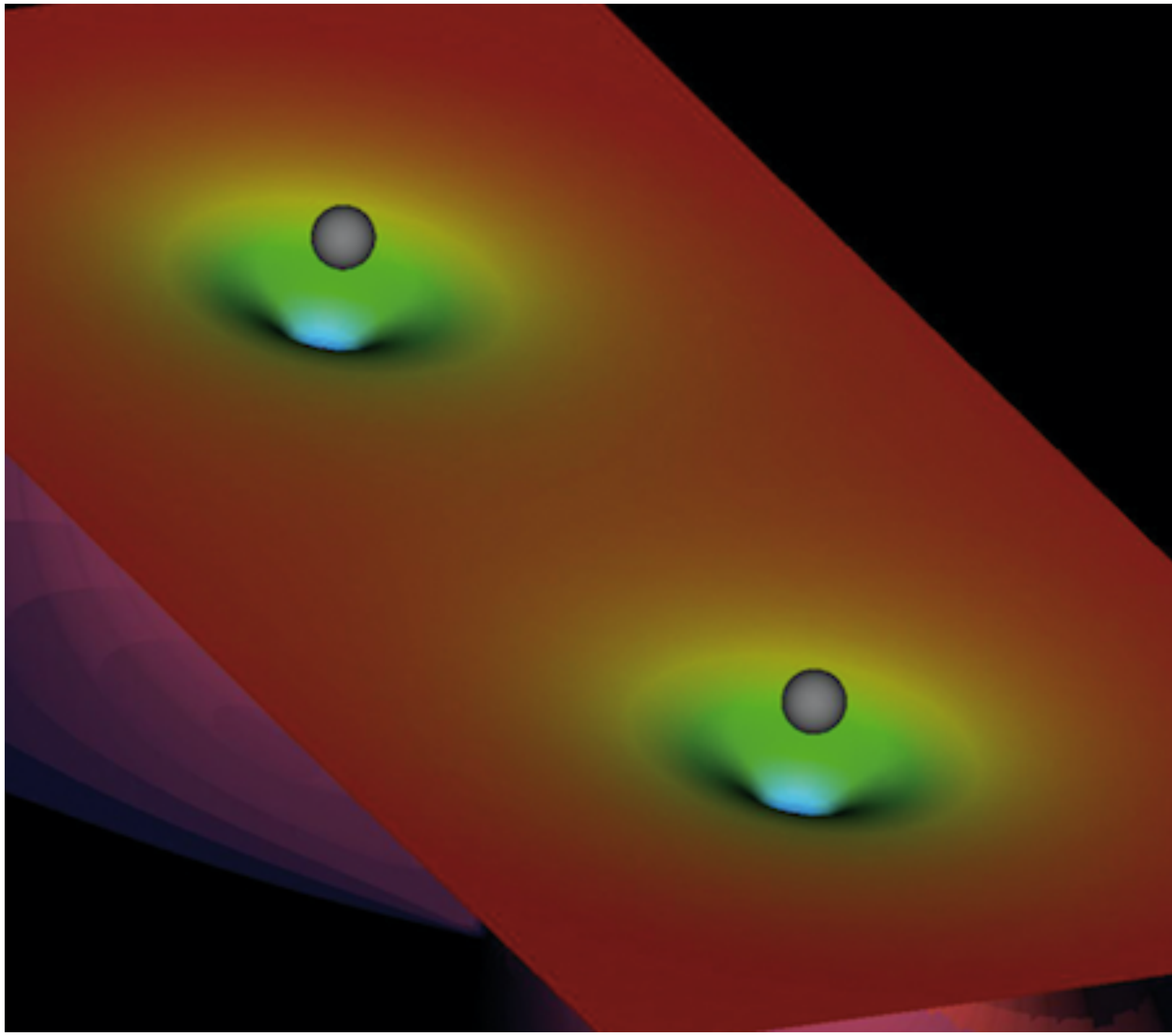
Buonanno-Damour 2000

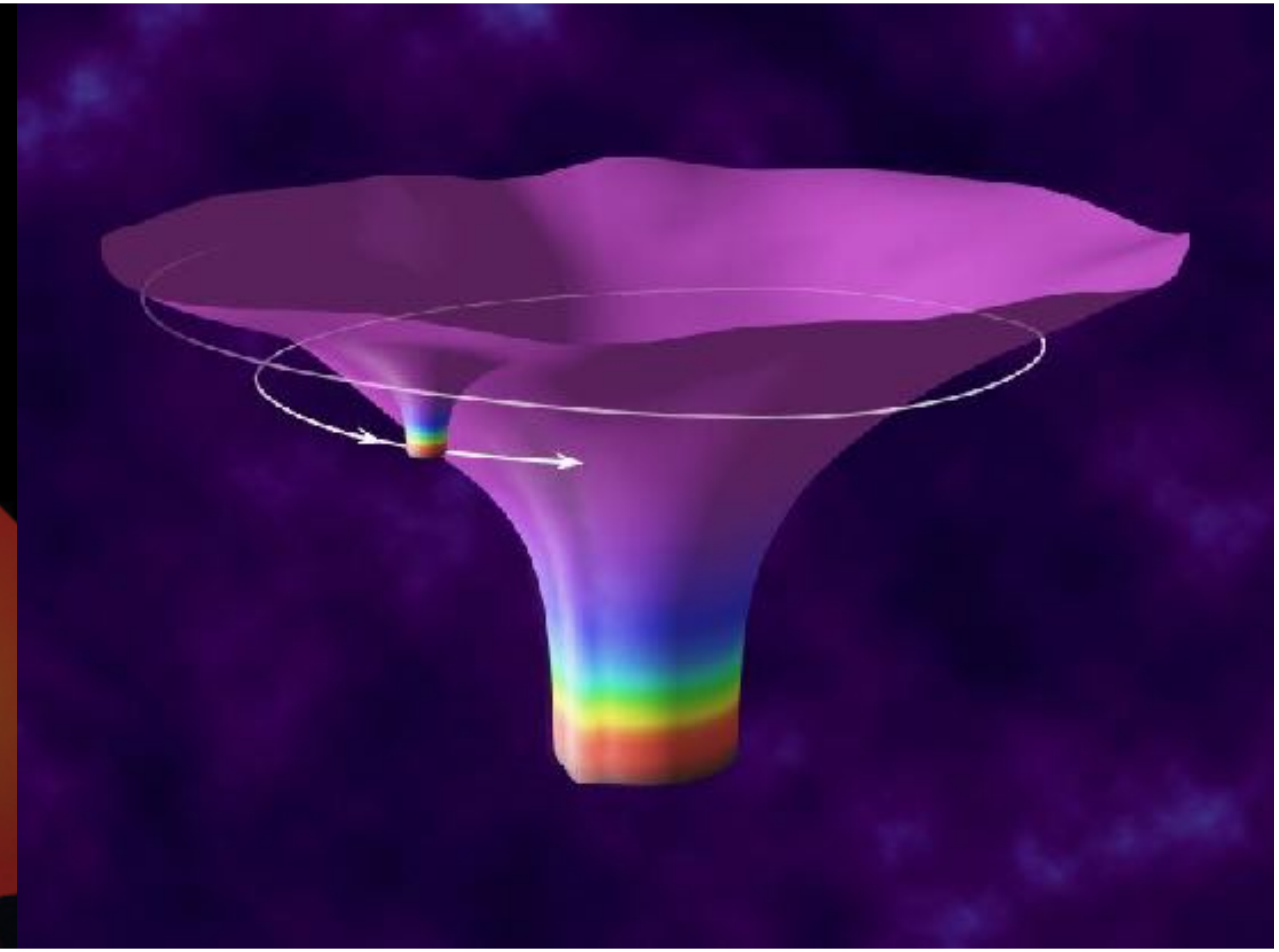




Predictions as early as 2000 :
continued transition, non adiabaticity, first complete waveform, final spin (OK within 10%), final mass

EOB: resumming the energetics of bound states of a two-body system (m_1,m_2,S_1,S_2) in terms of the energetics of the bound states of a particle of mass mu and spin S* moving in some effective metric g(M,S)





Effective metric for non-spinning bodies: a nu-deformation of Schwarzschild

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \nu = \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$\boxed{ds^2_{\rm eff} = -A(r;\nu)\, dt^2 + B(r;\nu)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}$$

TWO-BODY/EOB "CORRESPONDENCE":

THINK QUANTUM-MECHANICALLY (J.A. WHEELER)

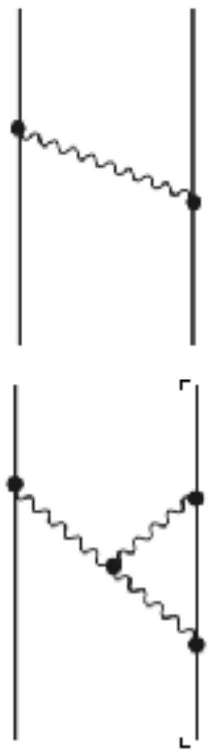
Real 2-body system
(in the c.o.m. frame)
(m_1, m_2)

1:1 map

An effective particle
in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$g_{\mu\nu}^{\text{eff}}$$



+ PN
expansion
in
v^2/c^2
and
G/c^2

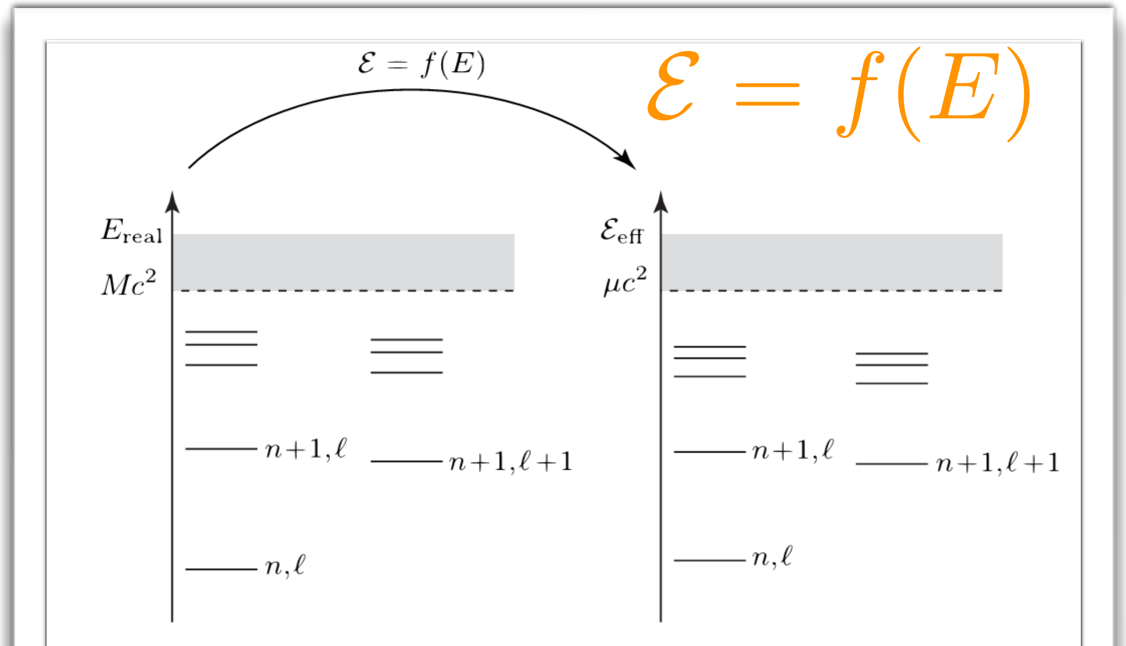


Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. n denotes the 'principal quantum

$$\mathcal{E} = f(E)$$

$$\mu^2 + g_{\text{eff}}^{\mu\nu} \frac{\partial S_{\text{eff}}}{\partial x^\mu} \frac{\partial S_{\text{eff}}}{\partial x^\nu} + \mathcal{O}(p^4) = 0$$

Bohr-Sommerfeld's
Quantization Conditions
(action-angle variables &
Delaunay Hamiltonian)

$$\begin{aligned} J &= \ell\hbar = \frac{1}{2\pi}\oint p_\varphi d\varphi \\ N &= n\hbar = I_r + J \\ I_r &= \frac{1}{2\pi}\oint p_r dr \end{aligned}$$

$$H^{\text{classical}}(q,p) \longrightarrow H^{\text{classical}}(I_a) \longrightarrow \boxed{E^{\text{quantum}}(I_a = n_a h) = f^{-1}[\mathcal{E}_{\text{eff}}^{\text{quantum}}(I_a^{\text{eff}} = n_a h)]}$$

Explicit 3PN EOB dynamics (Damour-Jaranowski-Schaefer '01)

A **simple**, but crucial transformation between the real energy and the effective one:

$$\boxed{\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}.}$$

A **simple post-geodesic** effective mass-shell:

$$\boxed{g_{\rm eff}^{\mu\nu}\, P'_\mu\, P'_\nu + \mu^2\, c^2 + Q(P'_\mu) = 0\,,}$$

$$\boxed{ds_{\rm eff}^2 = -A(R;\nu)dt^2 + B(R;\nu)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)}$$

$$M = m_1 + m_2\,; \qquad \mu = \frac{m_1 m_2}{m_1+m_2}\,, \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2} = \frac{\mu}{M}$$

$$u \equiv \frac{GM}{R\,c^2}$$

$$A^{\rm 3PN}(u) = 1 - 2u + 2\nu\, u^3 + \left(\frac{94}{3} - \frac{41}{32}\,\pi^2\right)\nu u^4,$$

$$\overline{D}^{\rm 3PN}(u) = 1 + 6\nu\, u^2 + (52\nu - 6\nu^2)\, u^3,$$

$$\widehat{Q}^{\rm 3PN} \equiv \frac{Q}{\mu^2 c^2} = (8\nu - 6\nu^2)\, u^2\, \frac{p_r^4}{c^4}.$$

2-body Taylor-expanded N + 1PN + 2PN+ 3PN Hamiltonian

$$H_{\mathrm{N}}(\mathbf{x}_a,\mathbf{p}_a)=\frac{\mathbf{p}_1^2}{2m_1}-\frac{1}{2}\frac{Gm_1m_2}{r_{12}}+(1\leftrightarrow 2)$$

$$\begin{aligned}c^2H_{\mathrm{1PN}}(\mathbf{x}_a,\mathbf{p}_a)=&-\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2}+14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)\\&+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}+(1\leftrightarrow 2),\end{aligned}$$

$$\begin{aligned}c^4H_{\mathrm{2PN}}(\mathbf{x}_a,\mathbf{p}_a)=&\frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}-\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.\\&\left.-6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}-\frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)\\&+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2}+19\frac{\mathbf{p}_2^2}{m_2^2}\right)-\frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2)+6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)\\&-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}+(1\leftrightarrow 2),\end{aligned}$$

$$\begin{aligned}c^6H_{\mathrm{3PN}}(\mathbf{x}_a,\mathbf{p}_a)=&-\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7}+\frac{1}{32}\frac{Gm_1m_2}{r_{12}}\left(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6}+4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2}+6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}\right.\\&-10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2}+24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2}\\&+2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}+\frac{(7\mathbf{p}_1^2\mathbf{p}_2^2-10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3}\\&+\frac{(\mathbf{p}_1^2\mathbf{p}_2^2-2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}+15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}\\&\left.-18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}+5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\right)+\frac{G^2m_1m_2}{r_{12}^2}\left(\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\right.\\&-\frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}+\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3}+\frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\&-\frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2}-\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}\\&+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}+\frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\\&\left.-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)+\frac{G^3m_1m_2}{r_{12}^3}\left(-\frac{1}{48}\left(425m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\&+\frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{1}{16}\left(20m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\&\left.+\frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)\\&+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left(\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right)+(1\leftrightarrow 2).\end{aligned}$$

Spinning EOB effective Hamiltonian

$$H_{\text{eff}} = H_{\text{orb}} + H_{\text{so}} \quad \rightarrow H_{\text{EOB}} = Mc^2\sqrt{1+2\nu\left(\frac{H_{\text{eff}}}{\mu c^2}-1\right)}$$

$$\hat{H}^{\text{eff}}_{\text{orb}} = \sqrt{A\left(1 + B_p \boldsymbol{p}^2 + B_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 - \frac{1}{1+\frac{(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}{r^2}}\frac{(r^2+2r+(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2)}{\mathcal{R}^4+\Delta(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}((\boldsymbol{n}\times\boldsymbol{p})\cdot\boldsymbol{\chi}_0)^2 + Q_4\right)}.$$

$$H_{\text{so}} = G_S \boldsymbol{L}\cdot\boldsymbol{S} + G_{S^*}\boldsymbol{L}\cdot\boldsymbol{S}^*,$$

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}\,;\ \mathbf{S}_* = \frac{m_2}{m_1}\mathbf{S_1} + \frac{m_1}{m_2}\mathbf{S_2}\,,$$

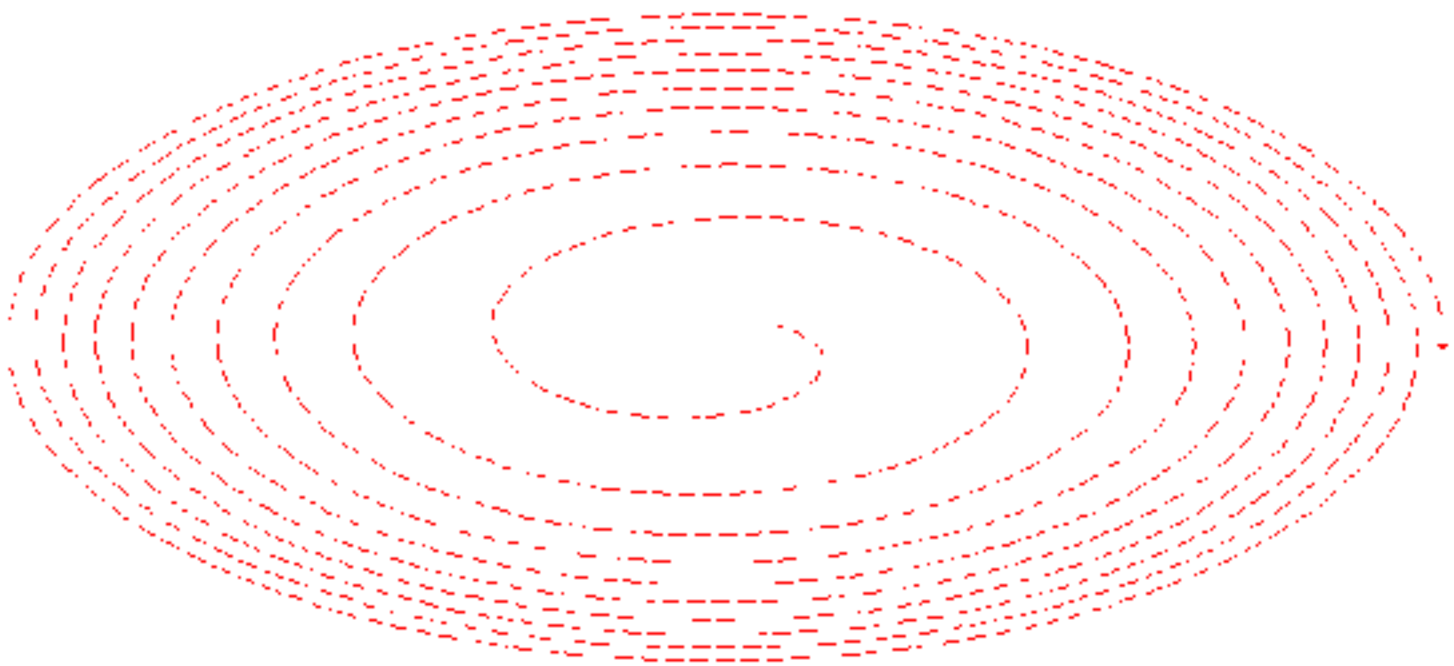
Gyrogravitomagnetic ratios (when neglecting spin^2 effects)

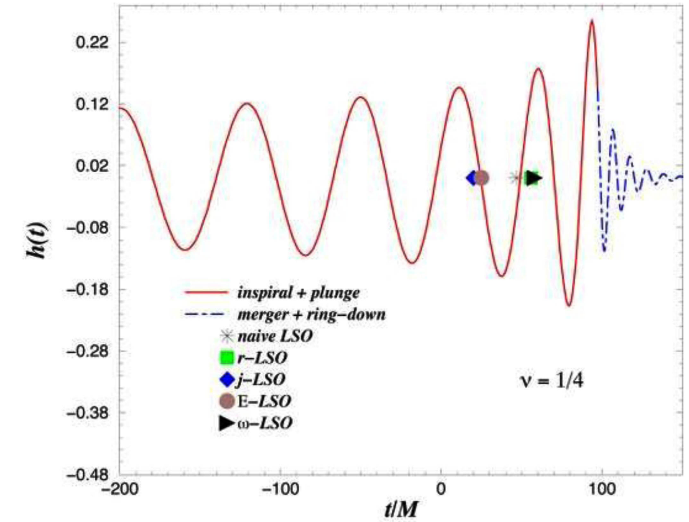
$$r^3 G_S^{\text{PN}} = 2 - \frac{5}{8}\nu u - \frac{27}{8}\nu p_r^2 + \nu\left(-\frac{51}{4}u^2 - \frac{21}{2}u p_r^2 + \frac{5}{8}p_r^4\right) + \nu^2\left(-\frac{1}{8}u^2 + \frac{23}{8}u p_r^2 + \frac{35}{8}p_r^4\right)$$

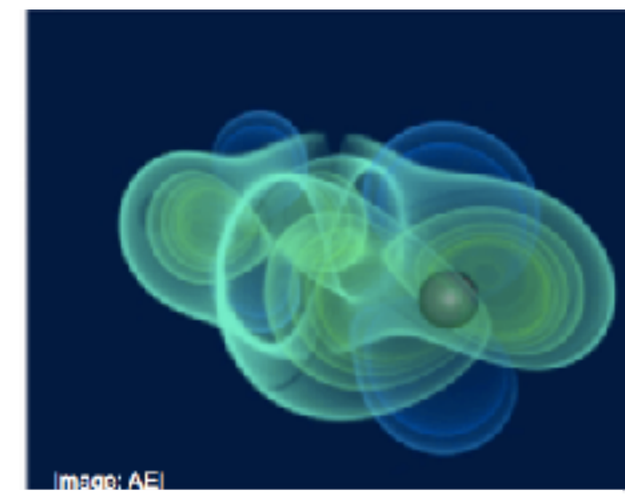
$$r^3 G_{S_*}^{\text{PN}} = \frac{3}{2} - \frac{9}{8}u - \frac{15}{8}p_r^2 + \nu\left(-\frac{3}{4}u - \frac{9}{4}p_r^2\right) - \frac{27}{16}u^2 + \frac{69}{16}u p_r^2 + \frac{35}{16}p_r^4 + \nu\left(-\frac{39}{4}u^2 - \frac{9}{4}u p_r^2 + \frac{5}{2}p_r^4\right)$$
$$+\nu^2\left(-\frac{3}{16}u^2 + \frac{57}{16}u p_r^2 + \frac{45}{16}p_r^4\right)$$

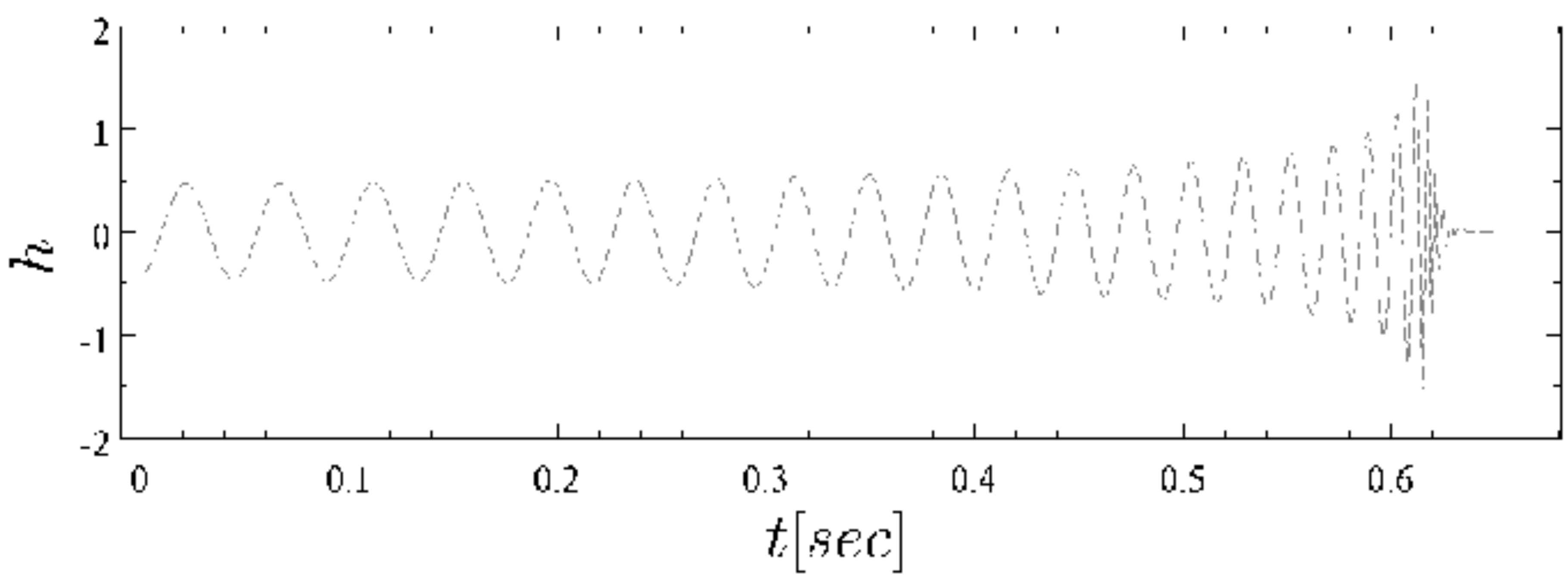
Effective One Body (EOB) Method

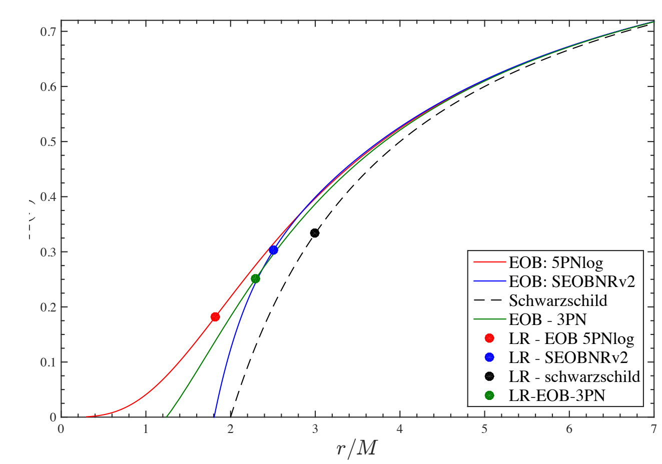
Buonanno-Damour 1999, 2000 ; Damour-Jaranowski-Schäfer 2000, Damour 2001

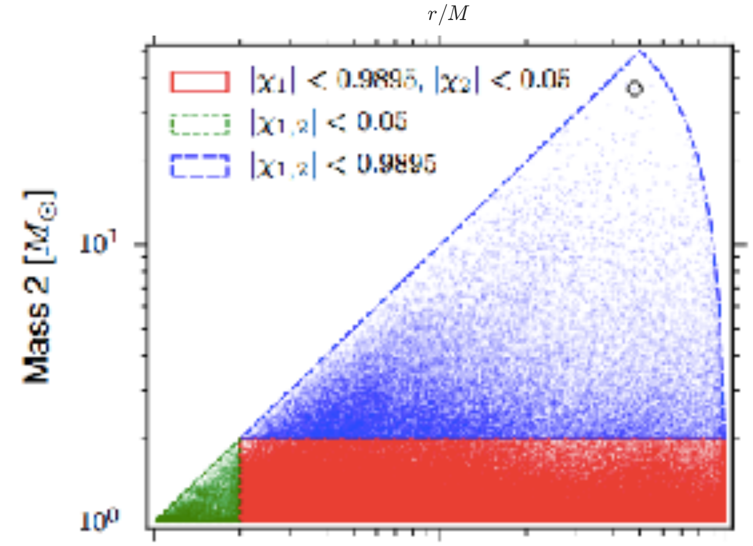






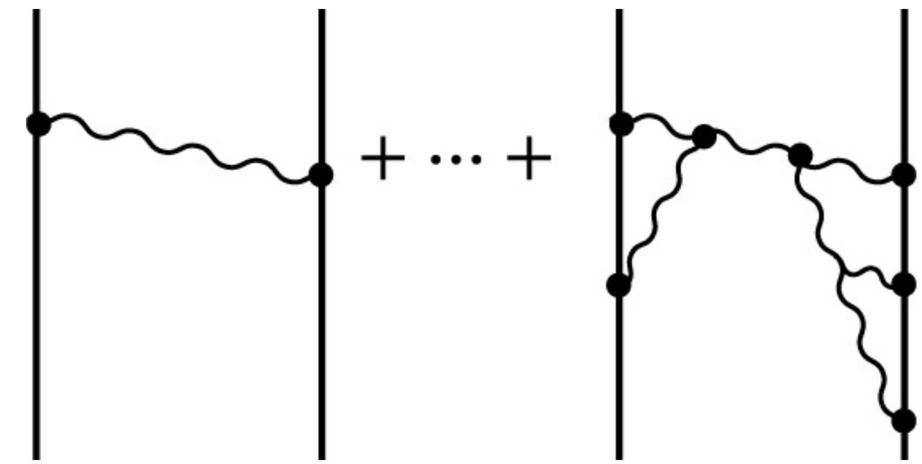






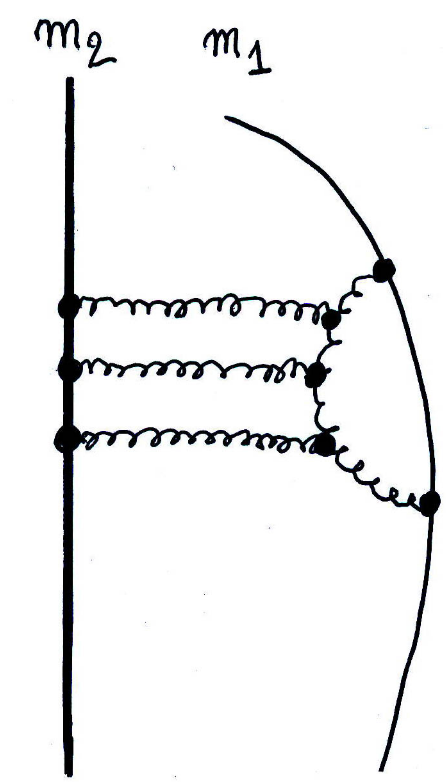
200 000 Numerical-Relativity-completed EOB waveforms

EOB AND GSF



Comparable-mass case: $m_1 \sim m_2$

Gravitational Self-Force Theory : $m_1 << m_2$
based on BH perturbation theory:
Regge-Wheeler-Zerilli-Teukolsky



• Analytical high-PN results : Blanchet-Detweiler-LeTiec-Whiting '10, Damour '10, Blanchet et al '10, LeTiec et al '12, Bini-Damour '13-15, Kavanagh-Ottewill-Wardell '15 Bini-Damour-Geralico'16, Hopper-Kavanagh-Ottewill'16

• (gauge-invariant) Numerical results : Detweiler '08, Barack-Sago '09, Blanchet-Detweiler-LeTiec-Whiting '10, Barack-Damour-Sago '10, Shah-Friedman-Keidl '12, Dolan et al '14, Nolan et al '15, Akcay-van de Meent '16

• Analytical PN results from high-precision (hundreds to thousands of digits !) numerical results : Shah-Friedman-Whiting '14, Johnson-McDaniel-Shah-Whiting '15

GSF : ANALYTICAL HIGH-ORDER_PN RESULTS (22 LOOPS)

Bini-Damour 15

$$\boxed{A(u;\nu) = 1 - 2u + \nu a(u) + O(\nu^2)}$$

Kavanagh et al 15

$$
\begin{aligned}
a_{10}^{c} = & \frac{18605478842060273}{7079830758000}\ln(2) - \frac{1619008}{405}\zeta(3) - \frac{21339873214728097}{1011404394000}\gamma \\
& + \frac{27101981341}{100663296}\pi^6 - \frac{6236861670873}{125565440}\ln(3) + \frac{360126}{49}\ln(2)\ln(3) + \frac{180063}{49}\ln(3)^2 \\
& - \frac{121494974752}{9823275}\ln(2)^2 - \frac{24229836023352153}{549755813888}\pi^4 + \frac{1115369140625}{124540416}\ln(5) + \frac{96889}{2779} \\
& + \frac{75437014370623318623299}{186907532011200000} - \frac{60648244288}{9823275}\ln(2)\gamma + \frac{200706848}{280665}\gamma^2 \\
& + \frac{11980569677139}{2306867200}\pi^2 + \frac{360126}{49}\gamma\ln(3), \\
a_{10}^{\ln} = & -\frac{21275143333512097}{2022808788000} + \frac{200706848}{280665}\gamma - \frac{30324122144}{9823275}\ln(2) + \frac{180063}{49}\ln(3), \\
a_{10}^{\ln^2} = & \frac{50176712}{280665}, \\
a_{10.5}^{c} = & -\frac{185665618769828101}{24473489040000}\pi + \frac{377443508}{77175}\ln(2)\pi + \frac{2414166668}{1157625}\pi\gamma - \frac{5846788}{11025}\pi^3 - 24 \\
a_{10.5}^{\ln} = & \frac{1207083334}{1157625}\pi.
\end{aligned}
$$

$$
\begin{aligned}
c_{15} = & -\frac{206954345058376961934037672 4}{32547744208650608437 5}\zeta(3) + \frac{651950262982450079 36}{22370298575625}\gamma\zeta(3) - \frac{5049442304}{25725}\gamma^2\zeta(3) + \frac{1262360576}{15435}\pi^2\zeta(3) \\
& + \frac{171722752}{441}\zeta(3)^2 + \frac{1613866959570176}{496621125}\zeta(5) - \frac{343445504}{441}\gamma\zeta(5) - \frac{146997248}{105}\zeta(7) + \frac{56314978304}{385875}\zeta(3)\log^2(2) \\
& - \frac{106445664}{343}\zeta(3)\log^2(3) + \frac{1516709982448497976 96}{22370298575625}\zeta(3)\log(2) - \frac{190336581632}{1157625}\gamma\zeta(3)\log(2) \\
& + \frac{28863591064624341}{4909804900}\zeta(3)\log(3) - \frac{212891328}{343}\gamma\zeta(3)\log(3) - \frac{212891328}{343}\zeta(3)\log(2)\log(3) - \frac{77186767578125}{19876428}\zeta(3)\log \\
& - \frac{2039263232}{3675}\zeta(5)\log(2) - \frac{49128768}{49}\zeta(5)\log(3) + \frac{29826742751501839701973659217528941950139153944429084 9}{658761222254465322614246840503191731953125 0} \\
& - \frac{680766176845363776831328694806032908750141 9}{70431094812480372256260772954406250 0}\gamma + \frac{159834694441260324783100628982938 8}{5261717150386770335918906 25}\gamma^2 - \frac{1007647146215971027644}{335890033113009375} \\
& + \frac{461219496448}{72930375}\gamma^4 - \frac{2833827508207759158785506345027630379006576290724319 7}{999703155845143418115744045792755712000000}\pi^2 + \frac{25191178655399275691104}{67178006622601875}\gamma\pi^2 \\
& - \frac{230609748224}{14586075}\gamma^2\pi^2 + \frac{105480323357757226894713787760391180776248036241}{30424535483131602802509905532026880000 0}\pi^4 + \frac{1262360576}{385875}\gamma\pi^4 \\
& - \frac{620847283961296697269145713114 3}{26693015135410024611840 0}\pi^6 + \frac{357317878192092911828132 9}{151996487423754240}\pi^8 - \frac{10136323685888}{72930375}\log^4(2) + \frac{38438712}{2401}\log^4(3) \\
& - \frac{177896086126482679647872}{549638236003106 25}\log^3(2) - \frac{896860131061 76}{364651875}\gamma\log^3(2) + \frac{153754848}{2401}\log^3(2)\log(3) \\
& - \frac{131463845322790269123}{245735735245000}\log^3(3) + \frac{153754848}{2401}\gamma\log^3(3) + \frac{153754848}{2401}\log(2)\log^3(3) + \frac{11933074267578125}{51161925672}\log^3(5) \\
& + \frac{38782586741666289745954206352002 04}{18942181741392373209308062 5}\log^2(2) - \frac{344085637991460169215116 8}{10076700993390281 25}\gamma\log^2(2) - \frac{16582891400192}{121550625}\gamma^2\log^2(2) \\
& + \frac{4145722850048}{72930375}\pi^2\log^2(2) - \frac{523697163373483905609}{245735735245000}\log^2(2)\log(3) + \frac{461264544}{2401}\gamma\log^2(2)\log(3) \\
& + \frac{45454535766189065888302299261759}{65697282267898830348800 00}\log^2(3) - \frac{394391535968370807369}{245735735245000}\gamma\log^2(3) + \frac{230632272}{2401}\gamma^2\log^2(3) \\
& - \frac{96096780}{2401}\pi^2\log^2(3) - \frac{437493411770075173449}{245735735245000}\log(2)\log^2(3) + \frac{461264544}{2401}\gamma\log(2)\log^2(3) \\
& + \frac{230632272}{2401}\log^2(2)\log^2(3) + \frac{11933074267578125}{17053975224}\log^2(2)\log(5) - \frac{25058426969931459437054980468 75}{40213632089533222243123 2}\log^2(5) \\
& + \frac{11933074267578125}{17053975224}\gamma\log^2(5) + \frac{11933074267578125}{17053975224}\log(2)\log^2(5) + \frac{47929508316470415142010251}{56464635170211840000}\log^2(7) \\
& - \frac{18163606721689522042153774768525369973449465 9}{63387985331232335030634695658965625 00}\log(2) + \frac{742036621552191085437995316530101 36}{47355454353480933023270156 25}\gamma\log(2) \\
& - \frac{14821693265224925154993 92}{10076700993390281 25}\gamma^2\log(2) - \frac{4905667647488}{364651875}\gamma^3\log(2) + \frac{371228115490667668451168}{604602059603416875}\pi^2\log(2) \\
& + \frac{1226416911872}{72930375}\gamma\pi^2\log(2) + \frac{23792072704}{17364375}\pi^4\log(2) - \frac{414115837539718030238709512493585574772 7}{108266631596274488880198656000000}\log(3) \\
& + \frac{94593580011315754543320552762 39}{69155033966209295104000 0}\gamma\log(3) - \frac{394391535968370807369}{245735735245000}\gamma^2\log(3) + \frac{153754848}{2401}\gamma^3\log(3) \\
& + \frac{131463845322790269123}{196588588196000}\pi^2\log(3) - \frac{192193560}{2401}\gamma\pi^2\log(3) + \frac{8870472}{1715}\pi^4\log(3) \\
& + \frac{21441150106021138984596292714838 1}{13139456453579766069760000}\log(2)\log(3) - \frac{437493411770075173449}{122867867622500}\gamma\log(2)\log(3) \\
& + \frac{461264544}{2401}\gamma^2\log(2)\log(3) - \frac{192193560}{2401}\pi^2\log(2)\log(3) + \frac{978612948501709853277095576118865234375}{179427491919561270211323849031 68}\log(5) \\
& - \frac{25058426969931459437054980468 75}{20106816044766611121561 6}\gamma\log(5) + \frac{11933074267578125}{17053975224}\gamma^2\log(5) - \frac{59665371337890625}{204647702688}\pi^2\log(5) \\
& - \frac{25058426969931459437054980468 75}{20106816044766611121561 6}\log(2)\log(5) + \frac{11933074267578125}{8526987612}\gamma\log(2)\log(5) \\
& - \frac{585800617379230891566511301391464808 1}{32391919320751280297779200000 0}\log(7) + \frac{47929508316470415142010251}{28232317585105920000}\gamma\log(7) \\
& + \frac{47929508316470415142010251}{28232317585105920000}\log(2)\log(7) + \frac{740024994425816010121 1}{65676344832000000}\log(11),
\end{aligned}
$$

Numerical GSF computation of the O(nu) A(u;nu)-potential: with singularity at u=1/3 (Akcay et al 2012)

$$\boxed{a(u) \simeq 0.25(1-3u)^{-1/2}}$$

GSF can also bring scattering information

(Damour 2010, Barack et al, in prepar)

Classical Gravitational Scattering and the EOB description of the GR 2-body dynamics

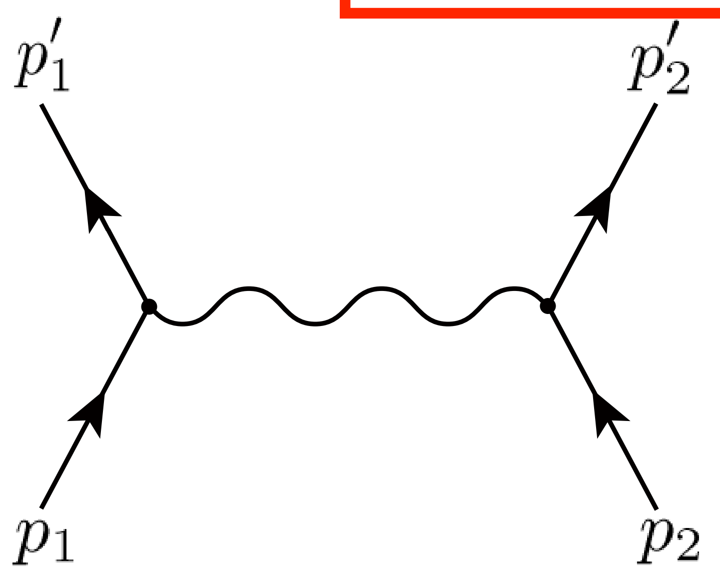
Original EOB dictionary based on bound states.

$$\boxed{f(E_{\mathrm{real}}(I_a)) = E_{\mathrm{eff}}(I_a)}$$

New (equivalent) dictionary for scattering states:
applicable to the PM approximation (no restriction on v/c).
[Damour2016]

$$\boxed{\chi_{\mathrm{eff}}(\mathcal{E}_{\mathrm{eff}}, J) = \chi_{\mathrm{real}}(\mathcal{E}_{\mathrm{real}}, J);}$$

G^1



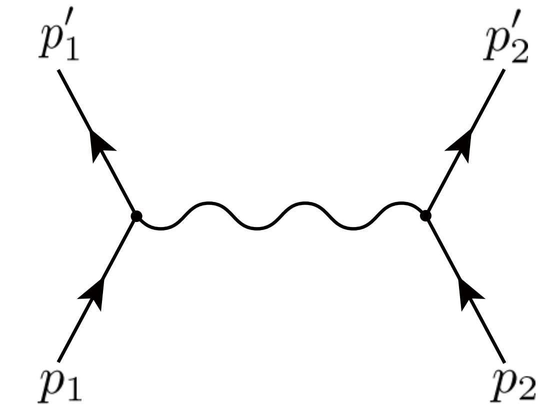
$$\boxed{\frac{1}{2}\chi^{\mathrm{real}}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2 p_2^2}}.}$$

$$\frac{1}{2}\chi_{\mathrm{class}}(E, J) = \frac{1}{j}\chi_1(\hat{E}_{\mathrm{eff}}, \nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\mathrm{eff}}, \nu) + O(G^3)$$

New results already at the 1PM order (linear in G)

Derivation of EOB energy map to all orders in v/c:

$$\boxed{\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}.}$$



to order G^1, the relativistic dynamics of a two-body system (of masses m_1 , m_2) is equivalent to the relativistic dynamics of an effective test particle of mass $\mu = m_1 m_2/(m_1 + m_2)$ moving in a Schwarzschild metric of mass $M = m_1 + m_2$, i.e. the rather complicated 1PM Hamiltonian of Ledvinka-Schaefer-Bicak2010: with

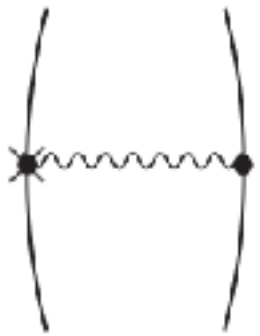
$$\begin{aligned}
H_{\rm lin} = {} & \sum_a \overline{m}_a + \frac{1}{4} G \sum_{a,b\neq a} \frac{1}{r_{ab}} \left(7\, \mathbf{p}_a\cdot\mathbf{p}_b + (\mathbf{p}_a\cdot\mathbf{n}_{ab})(\mathbf{p}_b\cdot\mathbf{n}_{ab})\right) - \frac{1}{2} G \sum_{a,b\neq a} \frac{\overline{m}_a \overline{m}_b}{r_{ab}} \\
& \times \left(1 + \frac{p_a^2}{\overline{m}_a^2} + \frac{p_b^2}{\overline{m}_b^2}\right) - \frac{1}{4} G \sum_{a,b\neq a} \frac{1}{r_{ab}} \frac{(\overline{m}_a \overline{m}_b)^{-1}}{(y_{ba}+1)^2 y_{ba}} \Bigg[2\Big(2(\mathbf{p}_a\cdot\mathbf{p}_b)^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 \\
& - 2(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b)\mathbf{p}_b^2 + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 \mathbf{p}_b^4 - (\mathbf{p}_a\cdot\mathbf{p}_b)^2 \mathbf{p}_b^2 \Big) \frac{1}{\overline{m}_b^2} + 2\Big[-\mathbf{p}_a^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 \\
& + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + 2(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b) + (\mathbf{p}_a\cdot\mathbf{p}_b)^2 - (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 \mathbf{p}_b^2 \Big] \\
& + \Big[-3\mathbf{p}_a^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + 8(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b) \\
& + \mathbf{p}_a^2 \mathbf{p}_b^2 - 3(\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 \mathbf{p}_b^2 \Big] y_{ba} \Bigg], \qquad y_{ba} = \frac{1}{\overline{m}_b} \sqrt{m_b^2 + (\mathbf{n}_{ba}\cdot\mathbf{p}_b)^2}\,.
\end{aligned} \tag{6}$$

$$\overline{m}_a = (m_a^2 + \mathbf{p}_a^2)^{\frac{1}{2}}$$

is fully described by the EOB energy map applied to

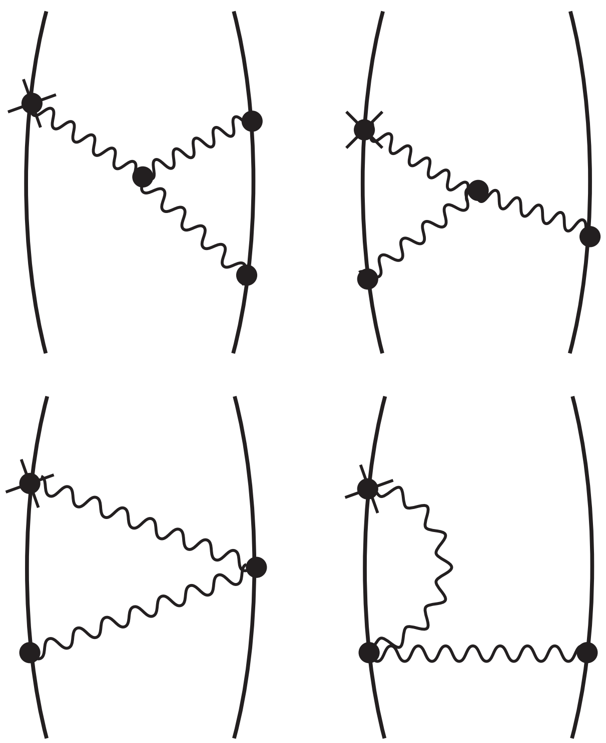
$$\boxed{ds_{\rm lin}^2 = -(1 - 2\frac{GM}{r})dt^2 + (1 + 2\frac{GM}{r})dr^2 + r^2 d\Omega^2}$$

Classical Gravitational Scattering and the GR 2-body problem beyond the linear-in-G level



At each order in G: EOB transcription of the gauge-invariant scattering function

$$\boxed{\frac{1}{2}\chi = \Phi(E_{\mathrm{real}}, J; m_1, m_2, G)}$$



as a simple energy-dependent modification of a Schwarzschild-metric mass-shell condition

$$\boxed{0 = g_{\mathrm{eff}}^{\mu\nu} P_\mu P_\nu + \mu^2 + Q}$$

Schwarzschild metric M=m1+m2

$$\boxed{Q = \left(\frac{GM}{R}\right)^2 q_2(E) + \left(\frac{GM}{R}\right)^3 q_3(E) + O(G^4)}$$

2PM **3PM**

$$\boxed{\frac{1}{2}\chi_{\text{class}}(E,J) = \frac{1}{j}\chi_1(\hat{E}_{\text{eff}},\nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\text{eff}},\nu) + O(G^3)}$$

O(G^1) —> Schwarzschild modulo energy map

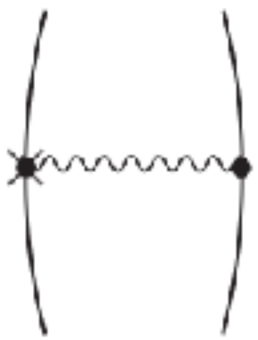
O(G^2) —>q_2(E)

$$\frac{1}{2}\chi^{\text{real}}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2 p_2^2}}.$$

Portilla '80, Damour'16

$$\chi_2(\hat{\mathcal{E}}_{\text{eff}},\nu) = \frac{3\pi}{8}\frac{5\hat{\mathcal{E}}^2_{\text{eff}} - 1}{\sqrt{1+2\nu(\hat{\mathcal{E}}_{\text{eff}}-1)}}.$$

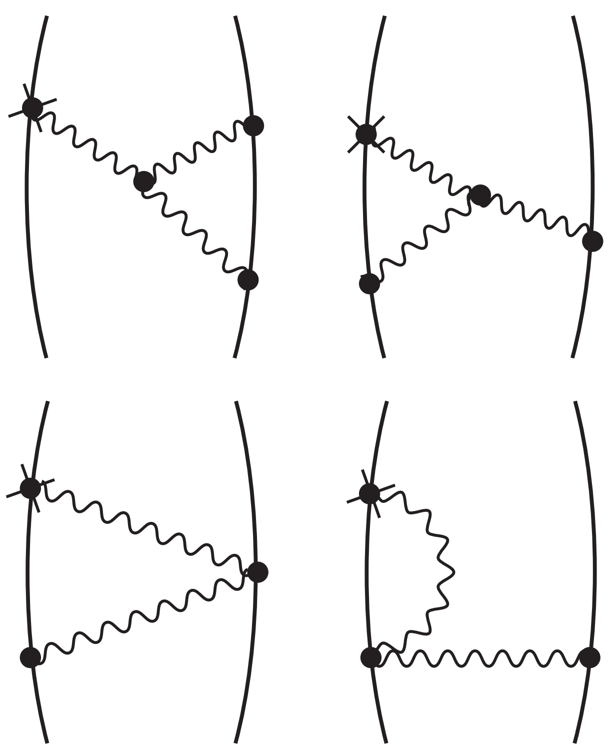
G^1



$$\boxed{\mathcal{E}_{\text{eff}} = \frac{(\mathcal{E}_{\text{real}})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}}$$

Damour'18, using Westpfahl-Goller '79, Bel-Damour-Deruelle-Ibanez-Martin'81,Westpfahl '85

G^2



$$\boxed{q_2(\hat{H}_{\text{Schw}},\nu) = \frac{3}{2}(5\hat{H}^2_{\text{Schw}} - 1)\left[1 - \frac{1}{\sqrt{1+2\nu(\hat{H}_{\text{Schw}}-1)}}\right]}$$

$$\boxed{0 = g^{\mu\nu}_{\text{Schw}} P_\mu P_\nu + \mu^2 + q_2(E)(GM/R)^2}$$

Predicted Regge-like Behavior of High-Energy Unstable Circular Bound States

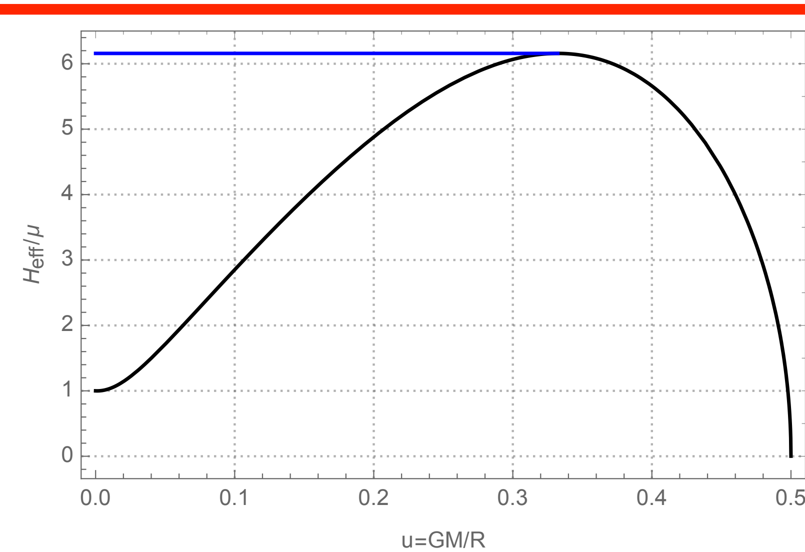
$$u \equiv \frac{GM}{R}$$

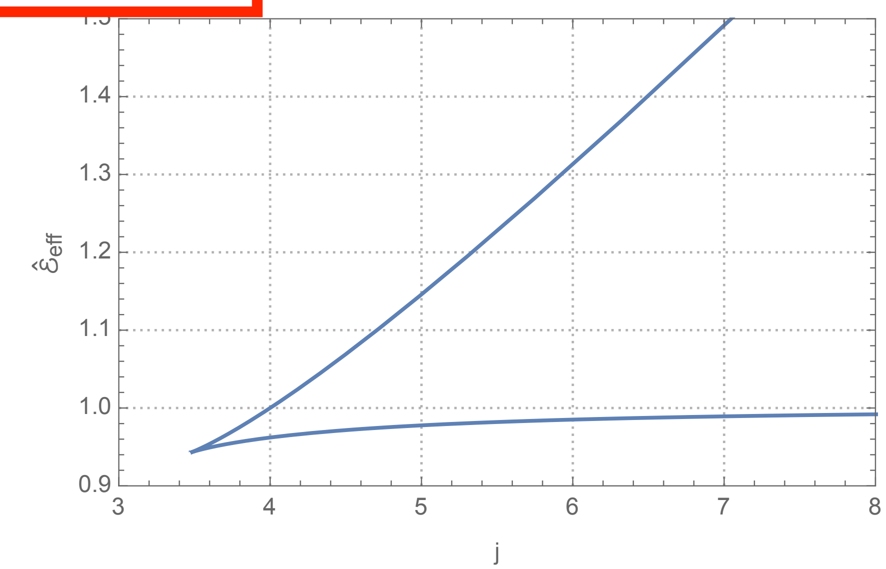
$$\widehat{H}^2_{\rm eff}(p_r, r, p_\varphi; \nu) = \widehat{H}^2_{\rm Schw} + \frac{3}{2}(1-2u)u^2 \left(5\,\widehat{H}^2_{\rm Schw} - 1\right)\left(1 - \frac{1}{\sqrt{1+2\nu(\widehat{H}_{\rm Schw}-1)}}\right)$$

$$H^2_{\rm Schw} = \left(1 - \frac{2GM}{R}\right)\left[\mu^2 + (1 - \frac{2GM}{R})P_R^2 + \frac{J^2}{R^2}\right]$$

High J —> H^2_eff ~ B(u) J^2
but

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}.$$





HE unstable circular bound states:
asymptotic constant Regge slope

string-like

$$s = E^2_{\rm real} \propto J$$

$$\frac{ds}{dJ} = \frac{2}{G}\frac{d\hat{\mathcal{E}}_{\rm eff}}{dj} \overset{\rm HE}{\approx} \frac{0.719\,964}{G} \qquad E^2_{\rm real} \overset{\rm HE}{=} C\,\frac{J}{G},$$

Self-Force Expansion , Light-Ring Behavior

Small mass-ratio expansion: \nu ->0

$$\widehat{H}^2_{\rm eff}(p_r, r, p_\varphi; \nu) = \widehat{H}^2_{\rm Schw} + \frac{3}{2}(1-2u)u^2\left(5\,\widehat{H}^2_{\rm Schw} - 1\right)\left(1 - \frac{1}{\sqrt{1+2\nu(\widehat{H}_{\rm Schw}-1)}}\right)$$

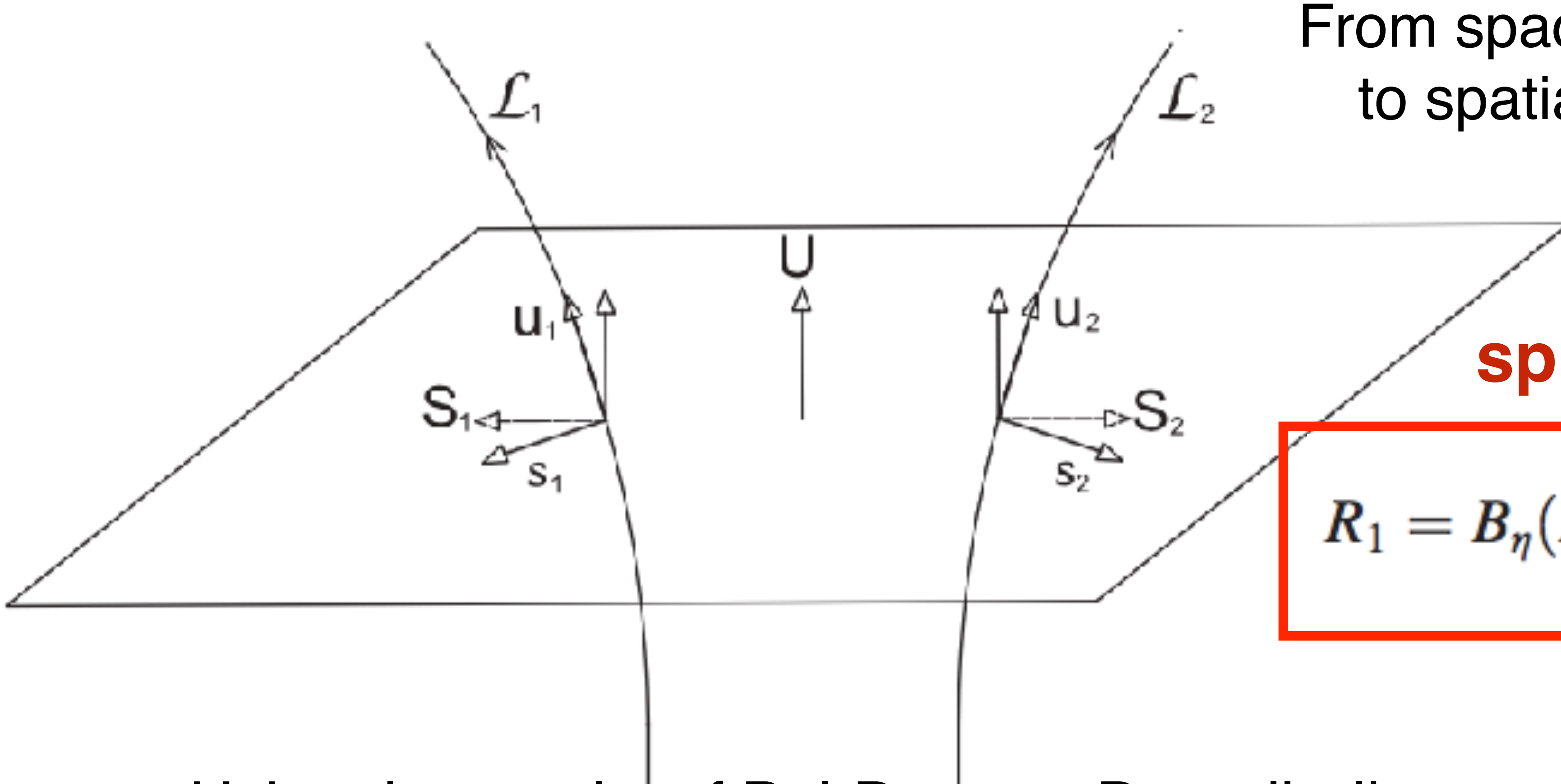
$$1 - \frac{1}{\sqrt{1+2\nu(\hat{H}_{\rm Schw}-1)}} = \nu(\hat{H}_{\rm Schw}-1) - \frac{3}{2}\nu^2(\hat{H}_{\rm Schw}-1)^2 + \frac{5}{2}\nu^3(\hat{H}_{\rm Schw}-1)^3 + \cdots$$

$$\hat{H}^2_{\rm eff} = \hat{H}^2_{\rm Schw} + \frac{3}{2}\nu(1-2u)u^2(5\hat{H}_{\rm Schw}{}^2 - 1)(\hat{H}_{\rm Schw}-1) \times \left[1 - \frac{3}{2}\nu(\hat{H}_{\rm Schw}-1) + \frac{5}{2}\nu^2(\hat{H}_{\rm Schw}-1)^2 + \cdots\right]. \quad (8.7$$

Singular Light-Ring Behavior of Self-Force expansion in DJS gauge (Akcay-Barack-Damour-Sago'12)

$$\bar{A}^{\rm SF}(\bar{u};\nu) = 1 - 2\bar{u} + \nu a_{1\rm SF}(\bar{u}) + \nu^2 a_{2\rm SF}(\bar{u}) + O(\nu^3). \quad a_{1\rm SF}(\bar{u}) \underset{\bar{u}\to\frac{1}{3}}{\sim} \frac{1}{4}\zeta(1-3\bar{u})^{-1/2}, \quad \text{with} \quad \zeta \approx 1.$$

1PM and 2PM-accurate spin-orbit couplings



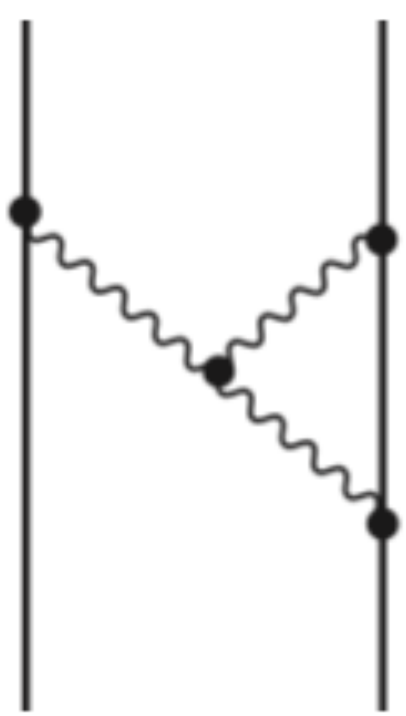
From spacetime spin four-vectors s_1,s_2 to spatial spin three-vectors S_1, S_2

Bini-Damour '17,'18 (see also Vines'18,Vines-Steinhoff-Buonanno'18)

spin-scattering holonomy

$$R_1 = B_\eta(\Lambda_1 u_1^- \to U^{\rm as})\Lambda_1[B_\eta(u_1^- \to U^{\rm as})]^{-1}.$$

Using the results of Bel-Damour-Deruelle-Ibanez-Martin'81 for the 2PM metric, one gets the 2PM-accurate value of the integrated spin rotation (spin holonomy)



$$\begin{aligned}\theta_1 \;=\; & -\frac{2}{hj\sqrt{\gamma^2-1}}\left[\gamma X_2 + (2\gamma^2-1)(X_1-h)\right] \\ & +\frac{\pi}{4h^2j^2}\left[-3(5\gamma^2-1)(X_1-h) - 6\gamma X_2 \right. \\ & \left. +\gamma(5\gamma^2-3)X_1X_2\right] . \qquad (5\end{aligned}$$

EOB transcription of the 2PM-accurate spin-rotation

energy spin-gauge
instead of DJS gauge

$$\boxed{\begin{aligned} g_S &= g_S^{1\mathrm{PM}}(H_{\mathrm{eff}}) + g_S^{2\mathrm{PM}}(H_{\mathrm{eff}})\, u + O(u^2)\,, \\ g_{S*} &= g_{S*}^{1\mathrm{PM}}(H_{\mathrm{eff}}) + g_{S*}^{2\mathrm{PM}}(H_{\mathrm{eff}})\, u + O(u^2)\,, \end{aligned}}$$

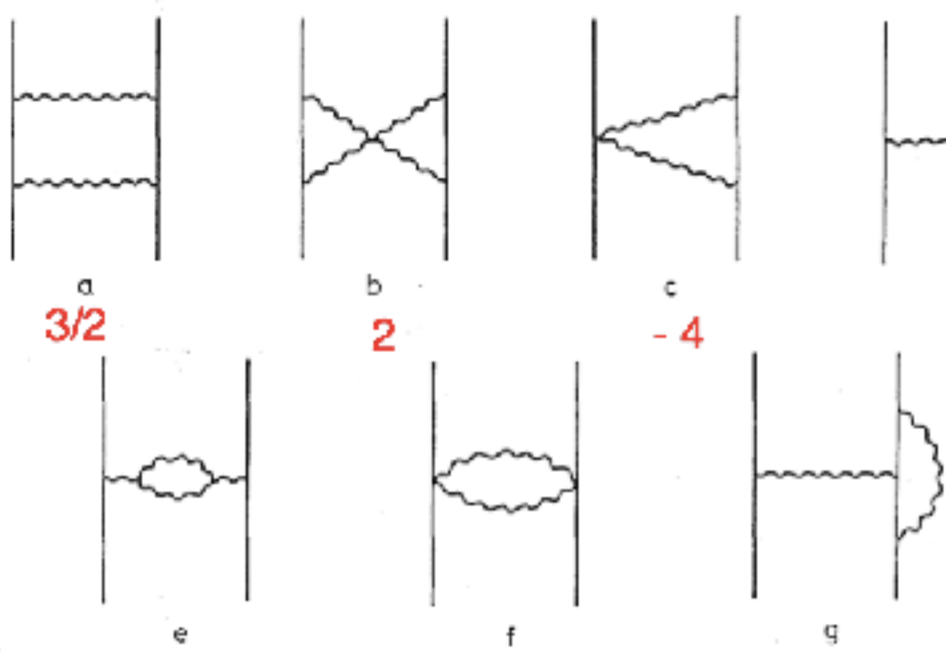
$$\theta_1^{\mathrm{EOB}} = G \int \frac{\mathbf{L}}{R^3}\left(g_S + \frac{m_2}{m_1} g_{S*}\right)\frac{B}{A} E_{\mathrm{eff}} \frac{dR}{P_R}\,,$$

$$\begin{aligned} g_S^{1\mathrm{PM}}(\gamma,\nu) &= \frac{(2\gamma+1)(2\gamma+h)-1}{h(h+1)\gamma(\gamma+1)} \\ &= \frac{1}{h(h+1)}\left[4 + \frac{h-1}{\gamma+1} + \frac{h-1}{\gamma}\right] \\ g_{S*}^{1\mathrm{PM}}(\gamma,\nu) &= \frac{2\gamma+1}{h\gamma(\gamma+1)} \\ &= \frac{1}{h}\left[\frac{1}{\gamma+1} + \frac{1}{\gamma}\right]. \end{aligned}$$

$$\boxed{\gamma = \hat{H}_{\mathrm{eff}} \qquad h = \sqrt{1+2\nu(\gamma-1)}}$$

$$\boxed{\begin{aligned} g_S^{2\mathrm{PM}}(\gamma,\nu) &= -\frac{\nu}{\gamma(\gamma+1)^2 h^2 (h+1)^2}\left[2(2\gamma+1)(5\gamma^2-3)h + (\gamma+1)(35\gamma^3-15\gamma^2-15\gamma+3)\right] \\ &= \frac{\nu}{h^2(h+1)^2}\left[-5(7\gamma+4h-10) + \frac{8(3h-4)}{\gamma+1} - \frac{4h}{(\gamma+1)^2} + \frac{3(2h-1)}{\gamma}\right] \\ g_{S*}^{2\mathrm{PM}}(\gamma,\nu) &= -\frac{1}{2\gamma(\gamma+1)^2 h^2 (h+1)}\left[(5\gamma^2+6\gamma+3)(h+1) + 4\nu(1+2\gamma)(5\gamma^2-3)\right] \\ &= \frac{1}{h^2(h+1)}\left[-20\nu + \frac{24\nu-h-1}{\gamma+1} + \frac{h+1-4\nu}{(\gamma+1)^2} - \frac{3}{2}\frac{h+1-4\nu}{\gamma}\right] \\ &= \frac{1}{h^2(h+1)}\left[-\frac{20\gamma\nu}{\gamma+1} + (h+1-4\nu)\left(\frac{1}{(\gamma+1)^2} - \frac{1}{\gamma+1} - \frac{3}{2}\frac{1}{\gamma}\right)\right]. \end{aligned}}$$

Quantum Scattering Amplitudes and 2-body Dynamics



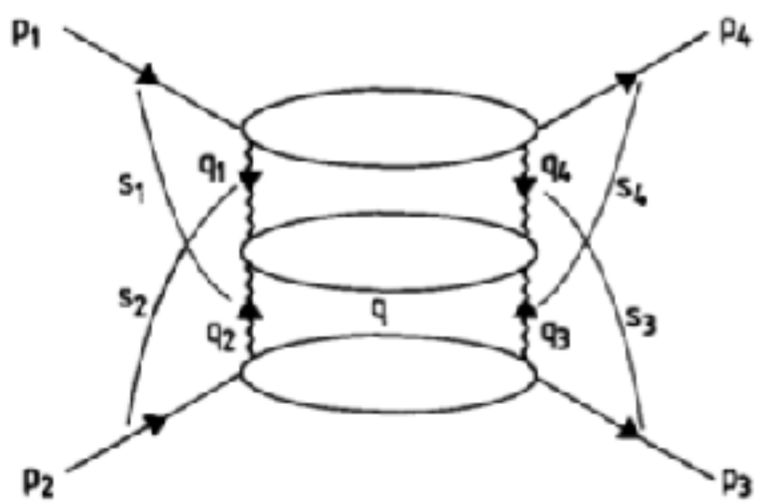


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

- Quantum Scattering Amplitudes —> Potential

one-graviton exchange :
Corinaldesi '56 '71,
Barker-Gupta-Haracz 66,
Barker-O'Connell 70, Hiida-Okamura72

Nonlinear: Iwasaki 71 [First post-Newtonian approx.],
Okamura-Ohta-Kimura-Hiida 73[2 PN]

Amati-Ciafaloni-Veneziano 1987-2008
Ultra-High-Energy (s >> M_Planck^2)
Four-graviton Scattering at 2 loops
Eikonal phase \delta in D=4
with one- and two-loop corrections
using the Regge-Gribov approach

$$\boxed{\delta = \frac{Gs}{\hbar}\left(\log\left(\frac{L_{IR}}{b}\right) + \frac{6\ell_s^2}{\pi b^2} + \frac{2G^2 s}{b^2}\left(1 + \frac{2i}{\pi}\log(\cdots)\right)\right)}$$

New technique: use EOB as a scattering-> Hamiltonian translation device

Progress in gravity amplitudes (Bern, Carrasco et al., Cachazo et al., Bjerrum-Bohr et al., Cachazo-Guevara,…) can be used (Damour '18) to improve the classical 2-body dynamics: need a quantum/classical dictionary.

High-energy limit of 2-body scattering and 2-body dynamics

Using the (eikonal) ultra-high-energy results of Amati-Ciafaloni-Veneziano: get HE information up to G^4

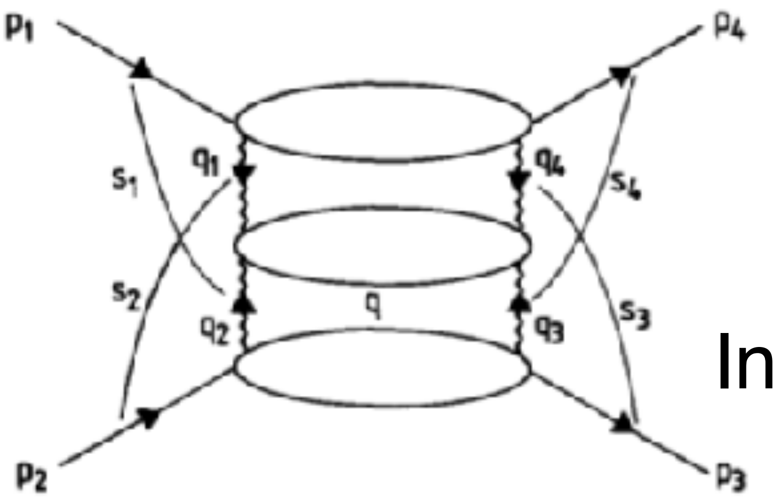


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

$$\sin\frac{1}{2}\chi^{\text{ACV}} \overset{\text{HE}}{=} 2\alpha + (2\alpha)^3 + O(\alpha^5),$$

with $\alpha_{\text{HE}} = \dfrac{GE_{\text{real}}}{b}$

In HE limit the EOB energy map is such that

$$\alpha = \frac{GME_{\text{eff}}}{J} = \frac{G}{2}\frac{(E_{\text{real}})^2 - m_1^2 - m_2^2}{J} \approx_{\text{HE}} \frac{GE_{\text{real}}}{b}$$

The masses disappear and the HE scattering is equivalent to a null geodesic in the « effective HE metric »

$$ds^2 = -A_{\text{HE}}(u)dT^2 + \frac{dR^2}{1-2u} + R^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$\boxed{A_{\text{HE}}(u) = (1-2u)\left(1 + \frac{15}{2}u^2 - 3u^3 + \frac{1749}{16}u^4 + O(u^5)\right)}$$

2PM 3PM 4PM

Translating quantum scattering amplitudes into classical dynamical information

How to translate a scattering amplitude into a classical Hamiltonian ?

$$\mathcal{M}(s,t) = \mathcal{M}^{(\frac{G}{\hbar})}(s,t) + \mathcal{M}^{(\frac{G^2}{\hbar^2})}(s,t) + \cdots,$$

$$\mathcal{M}^{(\frac{G}{\hbar})}(s,t) = 16\pi \frac{G}{\hbar} \frac{2(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{-t}.$$

Problem: The domain of validity of the Born expansion
is GE_1 E_2/(hbar v) << 1, while
the domain of validity of the classical scattering
is GE_1 E_2/(hbar v) >> 1!

It is an accident that the Born approximation of
a 1/r potential yields the exact cross section.

A way out: quantize the classical EOB Hamiltonian dynamics.

2PM, O(G^2) EOB classical mass-shell condition

$$p_\infty^2 = \hat{\mathcal{E}}_{\rm eff}^2 - 1,$$

$$w_1 = 2(2\hat{\mathcal{E}}_{\rm eff}^2 - 1),$$

$$w_2 = \frac{3}{2}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{h(\hat{\mathcal{E}}_{\rm eff})}.$$

$$\mathbf{p}^2 = p_\infty^2 + \bar{W}(\bar{u}) = p_\infty^2 + w_1\bar{u} + w_2\bar{u}^2 + O(\bar{u}^3),$$

Quantized version:

$$-\hat{\hbar}^2\Delta_{\mathbf{x}}\psi(\mathbf{x}) = \left[p_\infty^2 + \frac{w_1}{\bar{r}} + \frac{w_2}{\bar{r}^2} + O\left(\frac{1}{\bar{r}^3}\right)\right]\psi(\mathbf{x}).$$

Scattering amplitude for this potential scattering at the second Born approximation

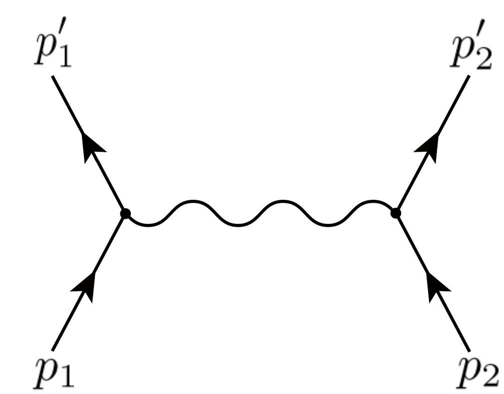
$$f_{\mathbf{k}_a}^{+\mathrm{B}1}(\mathbf{k}_b) = \frac{1}{\hat{\hbar}^2}\left[e^{\delta_C}\frac{w_1}{q^2} + \frac{\pi}{2}\frac{w_2}{q}\right],$$

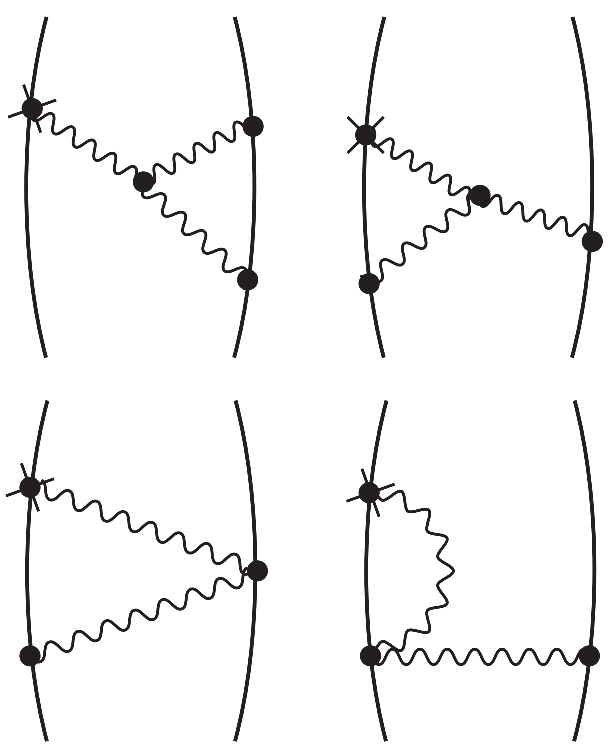
$$\delta_C = i\frac{w_1}{2k\hat{\hbar}^2}\ln\left(\sin^2\frac{\theta}{2}\right) + 2i\arg\Gamma\left(1 - i\frac{w_1}{2k\hat{\hbar}^2}\right).$$

Classical/quantum dictionary: prediction for one-loop result

$\mathcal{M}^{G^2}/\mathcal{M}^{G^1}$ with

$$\mathcal{M}^{(\frac{G}{\hbar})}(s,t) = 16\pi \frac{G}{\hbar}\frac{2(p_1\cdot p_2)^2 - p_1^2 p_2^2}{-t}.$$





+ Ladder

$$\boxed{\frac{f^+_{(1/q)}}{f^+_{(1/q^2)}} = \frac{3\pi}{8}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{2\hat{\mathcal{E}}_{\rm eff}^2 - 1}\frac{G(m_1+m_2)\sqrt{-t}}{\hbar} + O(G^2).}$$

OK with one-loop result of Guevara 1706.02314;
and Bjerrum-Bohr et al. 2018

2-loop amplitude gives the
3PM O(G^3) EOB Hamiltonian: q_3 u^3

From 2-loop amplitude to 3PM O(G^3) EOB Hamiltonian i.e. q_3 u^3

$$\hat{Q} = u^2 q_2(\hat{H}_{\rm eff}^{\rm Schw}) + u^3 q_3(\hat{H}_{\rm eff}^{\rm Schw}) + O(u^4).$$

$$\mathbf{p}^2 = p_\infty^2 + \bar{W}(\bar{u}) = p_\infty^2 + w_1 \bar{u} + w_2 \bar{u}^2 + w_3 \bar{u}^3 + \cdots$$

Potential scattering

$$\boxed{-\hat{\hbar}^2 \Delta_{\mathbf{x}} \psi(\bar{x}) = W\left(\frac{1}{\bar{r}}\right) \psi(\bar{x})}$$

$$\boxed{\bar{W}(\bar{u}) = w_1 \bar{u} + w_2 \bar{u}^2 + w_3 \bar{u}^3 + \cdots}$$

3PM

$$p_\infty^2 = \hat{\mathcal{E}}_{\rm eff}^2 - 1,$$

$$w_1 = 2(2\hat{\mathcal{E}}_{\rm eff}^2 - 1),$$

$$w_2 = \frac{3}{2}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{h(\hat{\mathcal{E}}_{\rm eff})}.$$

$$w_3(\hat{\mathcal{E}}_{\rm eff}) = -q_3(\hat{\mathcal{E}}_{\rm eff}) \;\&\; q_2(\hat{\mathcal{E}}_{\rm eff}) \;\&\; 1$$

Born expansion of scattering amplitude

(Coulomb + second-order in w_1 w_2)

$$f_{k_a}^{+}(k_b) = \frac{1}{\hat{\hbar}^2}\left[e^{\delta_C}\frac{w_1}{q^2} + \frac{\pi}{2}\frac{w_2}{q} + (w_3 \;\&\; w_1 w_2)\log q\right],$$

directly determines q_3(E_eff)

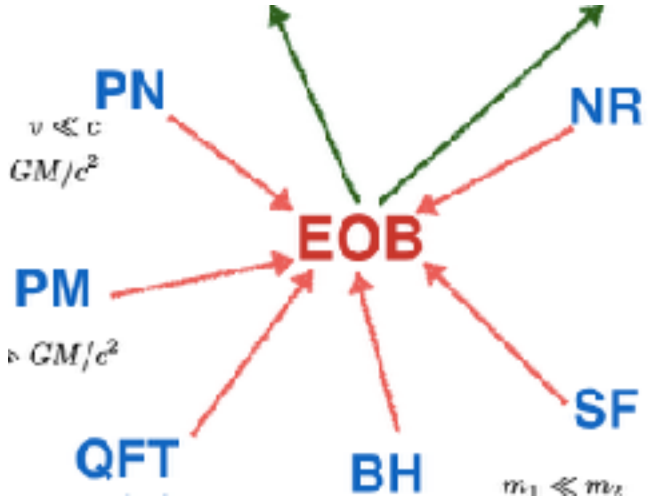
Checks: **4PN**

$$q_3^{\rm PN}(\hat{H}_{\rm Schw}, \nu) = 5\nu + \frac{1}{4}(108\nu - 23\nu^2)(\hat{H}_{\rm Schw}^2 - 1)$$ + 4PN

and EOB[ACV]

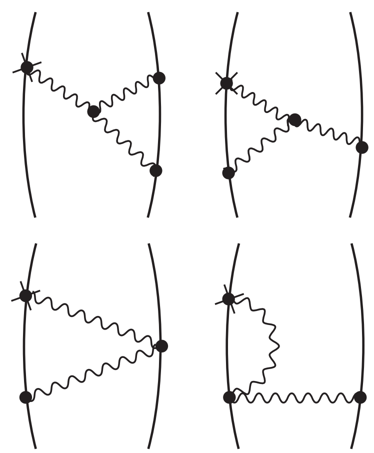
$$\boxed{q_3(\hat{\mathcal{E}}_{\rm eff}; \nu) \overset{\rm HE}{=} -3\,\hat{\mathcal{E}}_{\rm eff}^2}$$

Summary



EOB theory (Hamiltonian + Waveform) is directly used by LIGO-Virgo to define the many needed accurate templates

The EOB formulation of 2-body dynamics is a useful tool for transcribing classical and quantum scattering information into bound-state information.



The classical one-loop (G^ 2) scattering has been transcribed in EOB theory thereby giving new vistas on high-energy gravitational interactions.

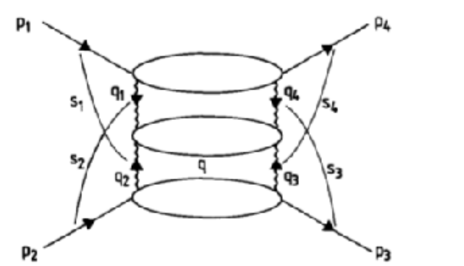
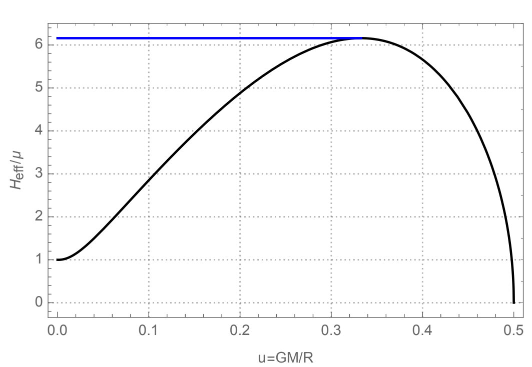


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

The Amati-Ciafaloni-Veneziano 2-loop HE result has been transcribed in EOB theory.

The HE gravitational EOB interaction predicts string-like (Regge-like) unstable « whirl » bound states: s ~ C J/G



A quantum/classical dictionary has been established.

Using it, the two-loop quantum scattering amplitude of gravitationally interacting scalar particles is easily translated into 3PM EOB Hamiltonian

EOB offers also a framework to transcribe spin effects from classical to quantum

$$-\hat{\hbar}^2 \Delta_{\mathbf{x}} \psi(\mathbf{x}) = \left[p_\infty^2 + \frac{w_1}{\bar{r}} + \frac{w_2}{\bar{r}^2} + O\left(\frac{1}{\bar{r}^3}\right) \right] \psi(\mathbf{x}).$$