Field theories from IBP reduction of string integrands

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Based on 1812.03369 with Song He and Yong Zhang

Tree-level string amplitudes

Double copy construction:

$\otimes_{KLT}$	SYM	$(DF)^2 + \text{YM}$	$(DF)^2 + \text{YM} + \phi^3$
Z-theory	open superstring	open bosonic string	compactified open bosonic string
sv(type-I)	closed superstring	heterotic (gravity)	heterotic (gauge/gravity)
sv(open bosonic)	heterotic (gravity)	closed bosonic string	compactified closed bosonic string

[Azevedo, Chiodaroli, Johansson and Schlotterer, 1803.05452]

Open (closed) string = field theory $\otimes_{KLT}$ disk (sphere) integral

$$\mathcal{M}_n^{\text{string}} = \mathcal{M}_n^{\text{FT}} \otimes_{KLT} [Z \text{ or sv(open)}]$$

Introduction

Open string:

tree-level color-ordered massless amplitudes

$$\mathcal{M}_n^{\text{string}}(\rho) = \int_\rho \frac{d^n z}{\text{Vol SL}(2,\mathbb{R})} \prod_{i<j} |z_{ij}|^{s_{ij}} \mathcal{I}_n(z) \equiv \int_\rho d\mu_n^{\text{string}} \mathcal{I}_n(z)$$

$z_{ij} \equiv z_i - z_j$, $s_{ij} \equiv \alpha' k_i \cdot k_j$, integration region $z_{\rho_1} < z_{\rho_2} < \cdots < z_{\rho_n}$

Z theory:

$\mathcal{I}_n(z) = \text{PT}(\sigma)$

$$Z_\rho(\sigma) = \int_\rho d\mu_n^{\text{string}} \frac{1}{z_{\sigma_1\sigma_2} z_{\sigma_2\sigma_3} \cdots z_{\sigma_n\sigma_1}} \equiv \int_\rho d\mu_n^{\text{string}} \text{PT}(\sigma)$$

[Brödel, Schlotterer, Stieberger, 1304.7267]

CHY formalism:

z's are localized on the solutions to the scattering equations (SE)

$$\mathcal{M}_n^{\text{FT}}(\rho) = \int \frac{d^n z}{\text{Vol SL}(2,\mathbb{C})} \prod_{i=1}^{n}{}' \delta\Big(\sum_{j\neq i} \frac{s_{ij}}{z_{ij}}\Big) \text{PT}(\rho) \mathcal{I}_n^{\text{CHY}}(z) \equiv \int d\mu_n^{\text{CHY}} \text{PT}(\rho) \mathcal{I}_n^{\text{CHY}}(z)$$

[Cachazo, He, Yuan, 1307.2199]

Questions

- Is there a relation between the string and the CHY integrand in the double copy $\mathcal{M}_n^{\text{string}} = \mathcal{M}_n^{\text{FT}} \otimes_{KLT} Z$?

- How general is $\mathcal{M}_n^{\text{string}} = \mathcal{M}_n^{\text{FT}} \otimes_{KLT} Z$?

String and CHY integrand

- Z theory: $Z_\rho(\sigma) = \int d\mu_n^{\text{string}}\, \text{PT}(\sigma)$
- Bi-adjoint ϕ^3: $m(\rho|\sigma) = \int d\mu_n^{\text{CHY}} \text{PT}(\rho)\, \text{PT}(\sigma)$
- $(n-3)!$-dimensional minimal basis: $S = m^{-1}$ [Cachazo, He, Yuan, 1309.0885]
- Bern-Carasco-Johansson (BCJ) relation: [Bern, Carasco, Johansson, 0805.3993]

$$\text{PT}(\pi) \cong \sum_{\alpha,\beta\in S_{n-3}} m(\pi|\alpha) S[\alpha|\beta] \text{PT}(\beta)\,, \quad \pi \in S_{n-2}$$

  1. For Z theory: holds as an integration-by-parts (IBP) relation [Brödel, Schlotterer, Stieberger, 1304.7267]
  2. For field theories: holds on the support of SE [Cachazo, 1206.5970]

- $\phi^3 \otimes_{\text{KLT}}$ as a "resolution of identity": $Z = \phi^3 \otimes_{\text{KLT}} Z$

General feature of string double copy

As long as $\mathcal{I}_n$ is a weight-two rational function of z_{ij}, we can always use IBP relations to reduce it to

$$\mathcal{I}_n \overset{\text{IBP}}{\cong} \mathcal{I}'_n = \sum_{\pi \in S_{n-2}} N_\pi \, \text{PT}(\pi)$$

[Mafra, Schlotterer, Stieberger, 1104.5224, 1106.2645]
[Huang, Schlotterer, Wen, 1602.01674] [Mizera, 1711.00469, Amplitude 2018] [He, **FT**, Zhang 1812.03369]

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Can be realized by

- A single master integration-by-parts relation
- Partial fraction identities

Benefit:

- A streamlined algorithm
- Land on a redundant basis → N_π is local
- Lead to a compact CHY integrand → closed-form $(DF)^2 + \text{YM} + \phi^3$ integrand

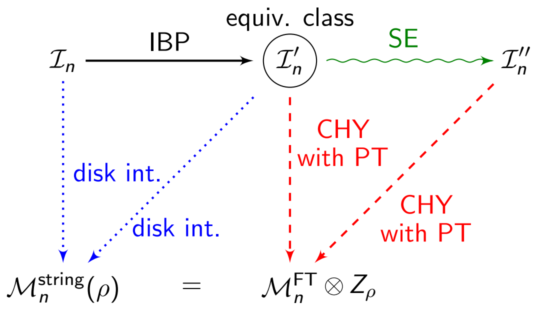
General feature of string double copy

- $\mathcal{I}'_n$ is also the CHY integrand of $\mathcal{M}_n^{\text{FT}}$ that appears in the double copy:

$$\begin{aligned}\mathcal{M}_n^{\text{string}}(\rho) = \int_\rho d\mu_n^{\text{string}}\, \mathcal{I}'_n &= \sum_{\pi\in S_{n-2}} N_\pi\, m(\pi|\alpha) S[\alpha|\beta] Z_\rho(\beta) \\ &= \mathcal{M}_n^{\text{FT}} \otimes_{\text{KLT}} Z_\rho \\ \mathcal{M}_n^{\text{FT}}(\alpha) = \sum_{\pi\in S_{n-2}} N_\pi\, m(\pi|\alpha) &= \sum_{\pi\in S_{n-2}} \int d\mu_n^{\text{CHY}} N_\pi\, \text{PT}(\pi)\text{PT}(\alpha) \\ &= \int d\mu_n^{\text{CHY}} \text{PT}(\alpha)\, \mathcal{I}'_n\end{aligned}$$

- We can usually bring $\mathcal{I}'_n$ into a more compact form $\mathcal{I}''_n$ by SE.
- N_π's are the DDM basis BCJ numerators of the field theory.

String double copy



Example: type-I $=$ SYM $\otimes_{\text{KLT}} Z$ [Mafra, Schlotterer, Stieberger, 1104.5224, 1106.2645]

$$\mathcal{I}_n = (\text{type-I correlator}) \xrightarrow{\text{IBP}} \mathcal{I}'_n = \sum_{\pi \in S_{n-2}} N_\pi \, \text{PT}(1, \pi, n) \overset{\text{SE}}{\rightleftharpoons} \mathcal{I}''_n = \text{Pf}'(\boldsymbol{\Psi}_n)$$

(The inverse direction is easier to realize [Du, **FT**, 1703.05717])

Logarithmic functions

- Parke-Taylor form is a logarithmic form: [Mizera, 1706.08527]

$$\mathrm{PT}(1,2\cdots n)\,\frac{d^n z}{\mathrm{Vol}\ \mathrm{SL}(2)} = (-1)^n \bigwedge_{i=2}^{n-2} d\log \frac{z_{i-1,i}}{z_{i,i+1}}$$

- It ties into the geometric picture of scattering amplitudes

  [Arkani-Hamed, Bai, Lam, 1703.04541] [Arkani-Hamed, Bai, He, Yan, 1711.09102]

- $\mathrm{PT}(1,2\cdots n)$ is a logarithmic function on $\mathcal{M}_{0,n}$
- Generic logarithmic function is a linear combination of PT factors:

$$\mathcal{I}_n' = \sum_{\pi\in S_{n-2}} N_\pi\,\mathrm{PT}(1,\pi,n)$$

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$$\mathcal{I}'_n = \sum_{\pi\in S_{n-2}} N_\pi\,\mathrm{PT}(1,\pi,n)$$

- Statement:

$$\mathcal{I}_n \overset{\mathrm{IBP}}{\cong} \text{logarithmic function } \mathcal{I}'_n$$

Proof of the claim

As long as $\mathcal{I}_n$ is a weight-two rational function of z_{ij}, we can always use IBP relations to reduce it to

$$\mathcal{I}_n \overset{\text{IBP}}{\cong} \mathcal{I}'_n = \sum_{\pi \in S_{n-2}} N_\pi \, \text{PT}(\pi)$$

- Show by induction that it holds for a class of rational functions:

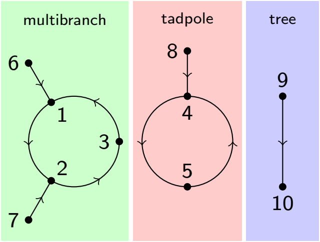
  After gauge fixing a certain $z_a \to \infty$, no z_{ij} in the numerators

- Single/multi-trace heterotic string integrands belong to this category
- Strategy: break all the PT subcycles in $\mathcal{I}_n$ by IBP relations
- Generic weight-two rational functions can be reduced to this form by absorbing certain cross ratios into the Koba-Nielsen factor

  (*i.e.* a string-like dual model [Baadsgaard, Bjerrum-Bohr, Bourjaily, Damgaard, 1610.04228])

IBP reduction

Under the gauge fixng $z_n \to \infty$, the part of $\mathcal{I}_n$ that needs IBP reduction has the following generic form:



$(z_{11} \to \infty)$

$$\mathrm{PT}(1,2,3)\,\mathrm{PT}(4,5)\,\frac{1}{z_{61}\,z_{72}}\,\frac{1}{z_{84}}\,\frac{1}{z_{9,10}}$$

$$\frac{1}{z_{ij}} \to \; i \bullet\!\!\longrightarrow\!\!\bullet\, j$$

IBP reduction: tadpole graphs

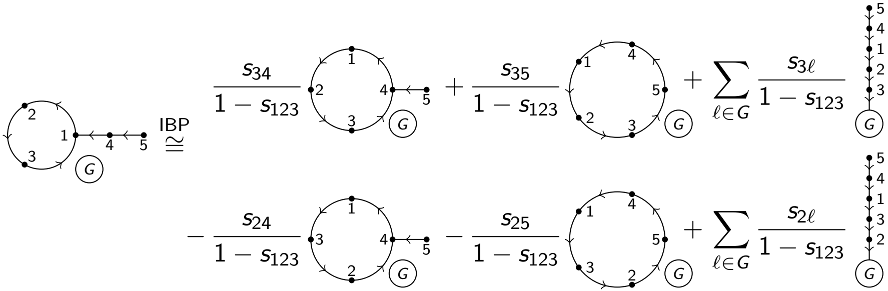
Tadpoles can be broken by the IBP relation: (assuming multiplied by a function independent of z_2 to z_m)

$$\mathrm{PT}(12\cdots m) \overset{\mathrm{IBP}}{\cong} \left(\sum_{\ell=2}^{m} \sum_{j=m+1}^{n-1} \sum_{\rho \in X \sqcup\!\sqcup Y^T} \frac{(-1)^{|Y|}}{1 - s_{12\cdots m}} \times \frac{s_{\ell j}}{z_{1\rho_1} z_{\rho_1 \rho_2} \cdots z_{\rho_{|\rho|}\ell} z_{\ell j}} \right),$$

[Schlotterer, 1608.00130]

where X and Y are obtained by matching $(1, 2\cdots m) = (1, X, \ell, Y)$

Example:



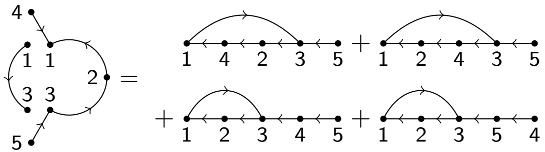
IBP reduction: multibranch graphs

- Reduce trees into lines: [Gao, He, Zhang, 1708.08701]

$$\mathcal{C}[T] = \sum_{\rho \in \mathcal{O}(T)} \mathcal{L}(1, \rho)$$

where $\mathcal{O}(T) = \{\rho \in S_{n-2} |\ i \to \cdots \to j \text{ in } T \Rightarrow i < j \text{ in } \rho\}$

- Turn multibranches into tadpoles algebraically:



- Repeating the above two steps ⇒ a sum of labeled trees ⇒ logarithmic function
- $\mathrm{PT}(1, \sigma, n)$: labeled line $\mathcal{L}(1, \sigma)$ starting at 1
- Use SE to turn the logarithmic $\mathcal{I}'_n$ into a compact non-log CHY integrand $\mathcal{I}''_n$

Lagrangian of $(DF)^2 + \text{YM} + \phi^3$

- $(DF)^2$ Lagrangian: [Johansson, Nohle, 1707.02965]

$$\mathcal{L}_{(DF)^2} = \frac{1}{2}(D_\mu F^{\mu\nu,a})^2 - \frac{g}{3}F^3 + \frac{1}{2}(D_\mu \varphi^\alpha)^2 + \frac{g}{2}C^{\alpha ab}\varphi^\alpha F^a_{\mu\nu}F^{\mu\nu,b} + \frac{g}{6}d^{\alpha\beta\gamma}\varphi^\alpha\varphi^\beta\varphi^\gamma$$

- Mass deformation: $\mathcal{L}_{(DF)^2+\text{YM}} = \mathcal{L}_{(DF)^2} + \frac{1}{2\alpha'}(F^a_{\mu\nu})^2 + \frac{1}{\alpha'}\varphi^\alpha\varphi^\alpha$
- $(DF)^2 + \text{YM} + \phi^3$:

$$\mathcal{L}_{(DF)^2+\text{YM}+\phi^3} = \mathcal{L}_{(DF)^2+\text{YM}} + \frac{1}{2}\left(D_\mu \phi^{aA}\right)^2 + \frac{g}{2}C^{\alpha ab}\varphi^\alpha\phi^{aA}\phi^{bA} + \frac{g\lambda}{3!}f^{abc}\hat{f}^{ABC}\phi^{aA}\phi^{bB}\phi^{cC}\,,$$

- $(DF)^2 + \text{YM} + \phi^3 \xrightarrow{\alpha'\to\infty} (DF)^2 + \phi^3$
- $(DF)^2 + \text{YM} + \phi^3 \xrightarrow{\alpha'=0}$ YM-scalar
- $(DF)^2 \otimes_{KLT}$ YM $=$ Berkovits-Witten conformal gravity: $\mathcal{F} = i\bar{\tau} = e^{-\phi} + i\chi$

$$e^{-1}\mathcal{L} = -\frac{\mathcal{F}}{2}\left[\frac{1}{2}(W^+_{\mu\nu\rho\sigma})^2 - \bar{P}^\mu\tilde{\nabla}_\mu\tilde{\nabla}_\nu P^\nu - 2(R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R)\bar{P}^\mu P^\nu + P^2\bar{P}^2 + \frac{1}{3}(P\cdot\bar{P})^2\right] + \text{h.c.}$$

[Johansson, Mogull, **FT**, 1806.05124] [Butter, Ciceri, de Wit, Sahoo, 1609.09083]

Single-trace three-gluon integrand for $(DF)^2 + \mathrm{YM} + \phi^3$

- CHY half-integrand: $\mathcal{I}''_n(123;\rho) = \mathrm{PT}(\rho)\,\mathcal{P}_3$
- $\mathcal{P}_3$ has a cycle expansion form:

$$\mathcal{P}_3 = C_1 C_2 C_3 + C_1 \Psi_{(23)} + C_2 \Psi_{(31)} + C_3 \Psi_{(12)} + \Psi_{(123)} + \Psi_{(132)}$$

$$\Psi_{(a)} = C_a = \sum_{b \neq a} \frac{\epsilon_a \cdot k_b}{z_{ab}}\,, \qquad \Psi_{(ab)} = -T_{ab}\,\mathrm{PT}(ab)$$

$$\Psi_{(abc)} = -\frac{T_{abc}\,\mathrm{PT}(abc)}{2}$$

- T's are α'-deformed Lorentz traces of $f_{\mu\nu} = k_\mu \epsilon_\nu - k_\nu \epsilon_\mu$:

$$T_{ab} = \frac{\mathrm{tr}(ab)}{1 - s_{ab}} \equiv \frac{1}{2}\frac{\mathrm{tr}(f_a f_b)}{1 - s_{ab}}$$

$$T_{abc} = \frac{1}{1 - s_{abc}}\left[\mathrm{tr}(f_a f_b f_c) + \frac{\alpha'}{2}\left(\frac{\mathrm{tr}(f_a f_b) k_b \cdot f_c \cdot k_a}{1 - s_{ab}} + \mathrm{cyc}(abc)\right)\right]$$

- $\alpha' = 0$ limit: $\mathcal{P}_3 \to \mathrm{Pf}(\boldsymbol{\Psi}_3)$

Single-trace CHY integrand for $(DF)^2 + \text{YM} + \phi^3$

- $\mathcal{I}''_n = \mathcal{I}'_n = \sum_\pi N_\pi \, \text{PT}(\pi)$ on the support of SE:

$$\mathcal{I}''_n(12\cdots m;\rho) = \text{PT}(\rho)\mathcal{P}_m$$

- $\mathcal{P}_m$ is given by the cycle expansion: (same structure as Pf $\boldsymbol{\Psi}_m$, [Lam, Yao, 1602.06419])

$$\mathcal{P}_m = \sum_{(I)(J)\cdots(K)\in S_m} \Psi_{(I)}\Psi_{(J)}\cdots\Psi_{(K)}$$

$$\Psi_{(I)} = -\frac{T_I \text{PT}(I)}{2} \quad \text{for} \quad |I| \geqslant 3$$

- T's are α'-deformed Lorentz traces:

$$T_{ab} \equiv \frac{\text{tr}(ab)}{1-s_{ab}} \equiv \frac{1}{2}\frac{\text{tr}(f_a f_b)}{1-s_{ab}}\,, \qquad T_I = \frac{1}{1-s_I}\left[\text{tr}(I) + \sum_{\text{CP}} G^{I_1}_{i_2} G^{I_2}_{i_3}\cdots G^{I_p}_{i_1}\right]$$

[He, **FT**, Zhang, 1812.03369]

α'-deformed trace T_I

$$T_I = \frac{1}{1-s_I}\left[\mathrm{tr}(I) + \sum_{\mathrm{CP}} G^{I_1}_{i_2} G^{I_2}_{i_3} \cdots G^{I_p}_{i_1}\right]$$

- $\mathrm{tr}(I) \equiv \mathrm{tr}(f_{a_1} f_{a_2} \cdots f_{a_{|I|}})$; the first element of I_ℓ is i_ℓ.
- $\sum_{\mathrm{CP}}$ sums over all the cyclic partitions $\{I_1, I_2 \cdots I_p\}$ of I, with $|I_\ell| \geqslant 2$:

$$\begin{aligned} I = (1,2,3,4) \xrightarrow{\mathrm{CP}} & \{(1,2),(3,4)\},\ \{(2,3),(4,1)\} \\ & \{(1,2,3,4)\},\ \{(2,3,4,1)\},\ \{(3,4,1,2)\},\ \{(4,1,2,3)\} \end{aligned}$$

- Given an ordered set $A = (a_1, a_2 \cdots a_r)$:

$$G^A_b = \sum_{q=2}^{s-\delta_{a_1 b}} T_{a_1 a_2 [a_3 [\cdots [a_{q-1} a_q] \cdots]} V^{a_q b}_{a_{q+1} \cdots a_r}\,,$$

- $V^{a_q b}_{a_{q+1}\cdots a_r} = \alpha' k_{a_q} \cdot f_{a_{q+1}} \cdots f_{a_r} \cdot k_b$ and $V^{ab} = s_{ab}$

Consistency checks

[Cachazo, He, Yuan, 1309.0885]

- $\alpha' = 0$: Yang-Mills-scalar

$$T_I \to \mathrm{tr}(I)\,, \qquad \mathcal{P}_m \to \mathrm{Pf}(\mathbf{\Psi}_m)$$

- $\alpha' \to \infty$: $(DF)^2 + \phi^3$ [Azevedo, Engelund, 1707.02192]

$$T_I \to 0 \text{ for } |I| \geqslant 2\,, \qquad \mathcal{P}_m = C_1 C_2 \cdots C_m$$

- Correct soft limit: $\mathcal{P}_m \to C_m \mathcal{P}_{m-1}$
- Correct scalar factorization (automatic due to the cycle expansion)
- Gluon factorization channel is free from double pole $1/p^4$
- $\mathcal{P}_m$ inherits the "corank-two" property of $\mathbf{\Psi}_m$

$$\mathcal{P}_m = 0 \quad \text{for } m\text{- and } (m+1)\text{-point massless on-shell kinematics}$$

- Numerical checks performed to very high multiplicities
  1. eleven points with IBP results
  2. eight points with Feynman diagram calculations

Multitrace integrands

- Double trace:

$$\mathcal{I}''_n(h';\sigma,\rho) = \left(\begin{array}{c} \text{the coefficient of} \\ \epsilon_{\sigma_1}\cdot k_{\sigma_2}\epsilon_{\sigma_2}\cdot k_{\sigma_3}\cdots\epsilon_{\sigma_{|\sigma|}}\cdot k_{\sigma_1} \\ \text{in } \mathcal{I}''_n(h;\rho) \end{array} \right), \quad \sigma \subset h$$

- Multitrace: repeat this process

$$\mathcal{I}''_n(h';\tau,\sigma,\rho) = \left(\begin{array}{c} \text{the coefficient of} \\ \epsilon_{\tau_1}\cdot k_{\tau_2}\epsilon_{\tau_2}\cdot k_{\tau_3}\cdots\epsilon_{\tau_{|\tau|}}\cdot k_{\tau_1} \\ \text{in } \mathcal{I}''_n(h';\sigma,\rho) \end{array} \right), \quad \tau \subset h'$$

- Closed-form formula for double-trace-one-graviton and triple-trace integrand
- Numerical checks up to eleven points and five traces

Conclusion and outlook

- A general algorithm to reduce a string integrand $\mathcal{I}_n$ to a logarithmic function $\mathcal{I}'_n$
- The function $\mathcal{I}'_n$ is the string integrand as well as the CHY integrand for

$$\mathcal{M}_n^{\text{string}} = \mathcal{M}_n^{\text{FT}} \otimes_{\text{KLT}} Z$$

- As an application, we have found a new family of CHY integrands for $(DF)^2 + \text{YM} + \phi^3$ theory

- $(DF)^2 + \text{YM}$ and ambitwistor?
- Relation between loop level string integrand and CHY integrand?

Thanks for your attention!

Back-up slides

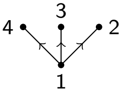
Labeled trees, PT factors and BCJ numerators

Equivalently, logarithmic functions can be expanded by labeled trees:

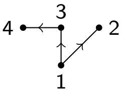
[Gao, He, Zhang, 1708.08701]

$$\sum_{\pi \in S_{n-2}} N_\pi \, \mathrm{PT}(1,\pi,n) \, \frac{d^n z}{\mathrm{Vol\ SL}(2)} = \sum_{T \in \mathbf{T}_1} N_T \, \mathcal{C}[T] \, d^{n-3} z$$

- gauge fixing: $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$
- $\mathbf{T}_1$: labeled trees rooted on 1, $\frac{1}{z_{ij}} \to \; i \bullet\!\!\rightarrow\!\!\bullet j$
- $\mathrm{PT}(1,\rho,n)$: labeled line $\mathcal{L}(1,\rho)$ starting at 1

$\mathcal{C}\Big[$  $\Big]$ $= \dfrac{1}{z_{12} z_{13} z_{14}} = \mathcal{C}[\, 1 \to 2 \to 3 \to 4 \,] + \mathrm{perm}(234)$

$$= \mathcal{L}(1,2,3,4) + \mathrm{perm}(234)$$

$\mathcal{C}\Big[$  $\Big]$ $= \dfrac{1}{z_{12} z_{13} z_{34}} = \mathcal{C}[\, 1 \to 2 \to 3 \to 4 \,] + \mathcal{C}[\, 1 \to 3 \to 2 \to 4 \,]$

$$+\, \mathcal{C}[\, 1 \to 3 \to 4 \to 2 \,]$$

$(z_5 \to \infty)$

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[Gao, He, Zhang, 1708.08701]

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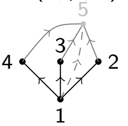
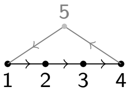
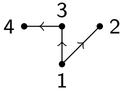
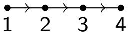
$\mathcal{C}\Big[$ 

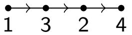
$\Big] = \frac{1}{z_{12}z_{13}z_{14}} \frac{z_{15}}{z_{25}z_{35}z_{45}} = \mathcal{C}\Big[$ 

$\Big] + \mathrm{perm}(234)$

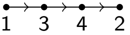
$= \mathcal{L}(1,2,3,4) + \mathrm{perm}(234)$

$\mathcal{C}\Big[$ 

$\Big] = \frac{1}{z_{12}z_{13}z_{34}} = \mathcal{C}\Big[$ 

$\Big] + \mathcal{C}\Big[$ 

$\Big]$

$+ \mathcal{C}\Big[$ 

$\Big]$

$(z_5 \to \infty)$

Labeled trees, PT factors and BCJ numerators

Labeled-tree expansion is equivalent to PT-factor expansion:

[Gao, He, Zhang, 1708.08701]

$$\mathcal{C}[T] = \sum_{\rho \in \mathcal{O}(T)} \mathrm{PT}(1, \rho, n)\,, \qquad\qquad N_\pi = \sum_{T \in \mathrm{IT}(\pi)} N_T$$

- $\mathcal{O}(T) = \{\rho \in S_{n-2} |\ i \bullet\!\!\to\!\!\bullet \cdots \bullet\!\!\to\!\!\bullet\, j \text{ in } T \Rightarrow i < j \text{ in } \rho\}$
- $\mathrm{IT}(\pi) = \{\, T \in \mathbf{T}_1 |$ All edges $i \bullet\!\!\to\!\!\bullet\, j$ satisfy $i < j$ in $\pi\}$ [increasing trees]

logarithmic functions $\Longleftrightarrow$ labeled trees $\Longleftrightarrow$ no PT subcycles

Streamlined algorithm to obtain N_T and N_π:

- Yang-Mills and NLSM [Du, **FT**, 1703.05717]
- Single-trace Yang-Mills-scalar [Fu, Du, Huang, Feng, 1702.08158] [Chiodaroli, Günaydin, Johansson, Roiban, 1703.00421] [**FT**, Feng, 1703.01269]
- Multitrace Yang-Mills-scalar [Du, Feng, **FT**, 1708.04514]

Open bosonic/heterotic string integrand

- Single trace:

$$\mathcal{M}^{\text{het}}_{m,r}(\rho) = \int |d\mu^{\text{string}}_n|^2 \text{PT}(\rho) R_m(z) \mathcal{K}_n(\bar{z})$$

- Multitrace:

$$\mathcal{M}^{\text{het}}_{m,r}(\{\rho_i\}) = \int |d\mu^{\text{string}}_n|^2 \left[\prod_{i=1}^{t} \text{PT}(\rho_i)\right] \mathcal{R}_m(z) \mathcal{K}(\bar{z})$$

- Reduced graviton correlator:

$$R_m(z) = \sum_{(I)(J)\cdots(K)\in S_m} \mathcal{R}_{(I)} \mathcal{R}_{(J)} \cdots \mathcal{R}_{(K)}$$

$$\mathcal{R}_{(a)} = \sum_{b\neq a} \frac{\epsilon_a \cdot k_b}{z_{ab}} \equiv C_a \,, \quad \mathcal{R}_{(ab)} = \frac{\epsilon_a \cdot \epsilon_b}{\alpha' z_{ab}^2} \,, \quad \text{and} \quad \mathcal{R}_{(I)} = 0 \text{ for } |I| \geqslant 3$$

[Schlotterer, 1608.00130]

Examples

- One graviton: $R_1(z) = C_1$
- Two gravitons:

$$R_2(z) = C_1 C_2 + \frac{\epsilon_1 \cdot \epsilon_2}{2\alpha' z_{12}^2}$$

- Three gravitons:

$$R_3(z) = C_1 C_2 C_3 + \frac{\epsilon_1 \cdot \epsilon_2}{\alpha' z_{12}^2} C_3 + \frac{\epsilon_1 \cdot \epsilon_3}{\alpha' z_{13}^2} C_2 + \frac{\epsilon_2 \cdot \epsilon_3}{\alpha' z_{23}^2} C_1$$

- Four gravitons:

$$\begin{aligned} R_4(z) = C_1 C_2 C_3 C_4 &+ \frac{\epsilon_1 \cdot \epsilon_2}{\alpha' z_{12}^2} C_3 C_4 + \frac{\epsilon_1 \cdot \epsilon_3}{\alpha' z_{13}^2} C_2 C_4 + \frac{\epsilon_2 \cdot \epsilon_3}{\alpha' z_{23}^2} C_1 C_4 \\ &+ \frac{\epsilon_1 \cdot \epsilon_4}{\alpha' z_{14}^2} C_2 C_3 + \frac{\epsilon_3 \cdot \epsilon_4}{\alpha' z_{34}^2} C_1 C_2 + \frac{\epsilon_2 \cdot \epsilon_4}{\alpha' z_{24}^2} C_1 C_3 \\ &+ \frac{\epsilon_1 \cdot \epsilon_2}{\alpha' z_{12}^2} \frac{\epsilon_3 \cdot \epsilon_4}{\alpha' z_{34}^2} + \frac{\epsilon_1 \cdot \epsilon_3}{\alpha' z_{13}^2} \frac{\epsilon_2 \cdot \epsilon_4}{\alpha' z_{24}^2} + \frac{\epsilon_1 \cdot \epsilon_4}{\alpha' z_{14}^2} \frac{\epsilon_2 \cdot \epsilon_3}{\alpha' z_{23}^2} \end{aligned}$$