Lecture 1 Hydrodynamics (Eulerian)

→ study of the movement of fluids

- ideal/perfect fluid (no viscosity, no heat conduction, no shears)
- gases, i.e. compressible & <u>not</u> incompressible.

For the astrophysical problem we are interested in, this means solving the Euler equations, and it must be done numerically.

show some movies

Derivation of the Euler equations

Euler's equations define how the density, momentum and energy of the system evolves over time due to external forces or pressure gradients

i.e. → neutron stars orbiting each other

→ box with high pressure on left, low pressure on the right

→ core-collapse where gravity overtakes electron degeneracy pressure.

fundamentally the motion of all the individual particles is governed by the Boltzmann equation, so lets start there. (following F. Shu. P. of Ap. Vol 2)

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + a_i \frac{\partial f}{\partial v_i} = \left.\frac{df}{dt}\right|_{coll}$$

f = particle distribution function.

$$dN = f\,dx\,dy\,dz\,dp_x\,dp_y\,dp_z\,dt$$

$$f = \frac{dN}{d\vec{x}\,d\vec{p}\,dt}$$

$$\langle Q \rangle = \frac{\int Q f (\quad)}{\int f (\quad)}$$

Could solve your hydrodynamical problem by solving the boltzmann equation, fortunately for hydrodynamics we don't have to. Instead, we can determine how bulk properties of the fluid change with time by intergrating out the velocity space components of the distribution function.

$$\frac{\partial n}{\partial t} = \frac{dN}{d\vec{x}\,dt} = \int f\,d^3\vec{v} \Longrightarrow$$ total number of particle (irrespective of their velocity) at a given spatial location; i.e. the density.

$$\int d^3\vec{v}\,m\,(\text{B.E.}) \longrightarrow \text{zeroth moment}$$

$$\int m \frac{\partial f}{\partial t}\,d^3\vec{v} + \int m v_i \frac{\partial f}{\partial x_i}\,d^3\vec{v} + \int m\, a_i \frac{\partial f}{\partial v_i}\,d^3\vec{v} = \int m \left.\frac{df}{dt}\right|_{coll} d^3\vec{v}$$

$$\int m f\,d^3\vec{v} = \rho$$

$$\int m v_i f\,d^3\vec{v} = \rho \langle v_i \rangle \equiv \rho u_i$$

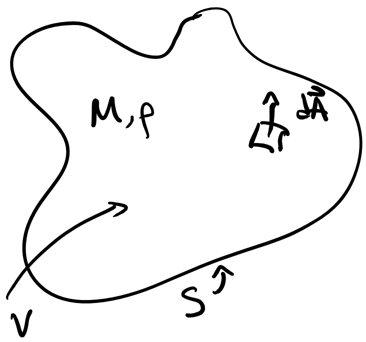
$$m \int \frac{\partial}{\partial v_i}(a_i f)\,d^3\vec{v} = 0$$ (b/c total # isn't changing, just moving around)

Since particle only move around velocity space due to collisions (and not real space) $m \int \left(\frac{df}{dt}\right)_{coll} d^3\vec{v} = 0$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0} \qquad (1)$$

"continuity equation"

other derivation



$$M = \int \rho \, dV$$

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\left(\int \rho \, dV\right) \qquad \frac{\partial M}{\partial t} = -\int_S \rho \, \vec{u} \cdot d\vec{A}$$

$$\frac{\partial}{\partial t}\left(\int \rho \, dV\right) = -\int_S \rho \vec{u} \cdot d\vec{A}$$

$$\int \frac{\partial}{\partial t} \rho \, dV = -\int \vec{\nabla} \cdot (\rho \vec{u}) \, dV$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{u})$$

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0}$$

Momentum equation.

take the "first moment"

$$\int m v_i \, (B.E.) \, d^3\vec{v}$$

$$\int m v_j \frac{\partial f}{\partial t} d^3\vec{v} + \int m v_j v_i \frac{\partial}{\partial x_i} f \, d^3\vec{v} + \int m v_j a_i \frac{\partial f}{\partial v_i} d^3\vec{v} = \int m v_j \frac{\delta f}{\delta t} d^3\vec{v}$$

$= 0$ b/c momentum is conserved in collisions

$$\frac{\partial}{\partial t}\int m v_j f \, d^3\vec{v} = \frac{\partial}{\partial t}(\rho u_j)$$

$$v_j \frac{\partial f}{\partial v_i} = \frac{\partial}{\partial v_i}(v_j f) - f \frac{\partial v_j}{\partial v_i} = \frac{\partial}{\partial v_i}(v_j f) - f \delta^{ij}$$

$$\frac{\partial}{\partial x_i}\int m v_j v_i f \, d^3\vec{v} = \frac{\partial}{\partial x_i}(\rho \langle v_i v_j \rangle)$$

$$v_i = u_i + w_i$$

$$\rho \langle v_i v_j \rangle = \rho u_i u_j + \rho \langle w_i w_j \rangle$$

$$\int m v_j a_i \frac{\partial f}{\partial v_i} d^3\vec{v} = -\int m a_i \delta^{ij} f \, d^3\vec{v} = -a_j \rho$$

mean velocity, zero if not moving, even if zero, there can still be non-zero $\langle v_i v_j \rangle$

$$\rho\langle w_i w_j\rangle \equiv P\delta_{ij} - \pi_{ij}$$

where $P \equiv \frac{1}{3}\rho\langle w^2\rangle$

viscous stress tensor

$$\boxed{\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}\left(\rho u_i u_j + P\delta_{ij}\right) = \rho a_i} \qquad (2)$$

(dropped viscous terms)

"momentum equation"

Energy Equation

→ take the second moment

$$\frac{1}{2}\int m v_i v_j \delta^{ij} (B.E.)\, d^3\vec{v}$$

$$\frac{1}{2}\int m v_i v^i \frac{\partial f}{\partial t} d^3\vec{v} + \frac{1}{2}\int m v_i v^i v_k \frac{\partial}{\partial x_k} f\, d^3\vec{v} + \frac{1}{2}\int m v_i v^i a_k \frac{\partial f}{\partial v_k} d^3\vec{v} + \frac{1}{2}\int m v_i v^i \left.\frac{df}{dt}\right|_{coll} d^3\vec{v}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\rho u_i u^i + P\delta_{ii}\right]$$

"0 b/c collision conserve energy."

$$\cancel{\frac{\partial}{\partial v_k}\left[\int d^3\vec{v}\, \frac{1}{2} m v_i v^i a_k f\right]} \; 0$$

like before

$$-\int d^3\vec{v}\, \frac{1}{2} m a_k f (2 v_k)$$

$$= -m a_k \int f v_k\, d^3\vec{v}$$

$$= -\rho a_k u_k$$

$$\frac{1}{2}\frac{\partial}{\partial x_k}\left[\int m v_i v^i v_k f\, d^3\vec{v}\right]$$

"$\rho\langle v_i v^i v_k\rangle$"

$$\rho\langle (u_i + w_i)(u_i + w_i)(u_k + w_k)\rangle$$

$$\rho\langle [u^2 + w^2 + 2u_i w_i] u_k\rangle + \rho\langle [u^2 + w^2 + 2u_i w_i] w_k\rangle$$

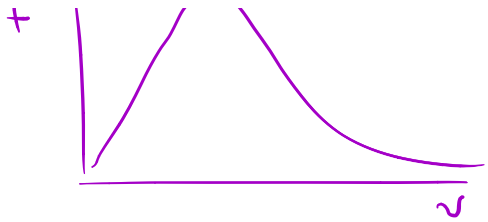
$$= \rho\langle u^2 u_k\rangle + \rho\langle w^2 u_k\rangle + 2\rho\langle \cancel{u_i w_i} u_k\rangle + \rho\langle \cancel{u^2 w_k}\rangle + \rho\langle w^2 w_k\rangle + 2\rho\langle u_i w_i w_k\rangle$$

$$= \rho u^2 u_k + 2 u_k \rho\mathcal{E} + \cancel{\rho\langle w^2 w_k\rangle} + 2\rho u_i \langle w_i w_k\rangle$$

$\rho\mathcal{E} = \rho\langle \frac{1}{2}|w^2|\rangle = \frac{3P}{2}$

"heat flux"

$$\rho\langle w_i w_k\rangle \equiv P\delta_{ik} - \pi_{ik}$$



$$\frac{\partial}{\partial x_k}\left[\left[\frac{\rho u^2}{2}+\rho\varepsilon\right]u_k + \vec{F}_H + u_i\left[P\delta_{ik}-\pi_{ik}\right]\right]$$

$$\frac{\rho}{2}\langle w^2 w_k\rangle$$

Simplify w/ no viscosity or heat flux.

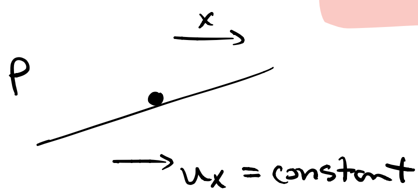
$$\frac{\partial}{\partial x_k}\left[\left(\frac{\rho u^2}{2}+\rho\varepsilon+P\right)u_k\right]$$

$$\frac{\partial}{\partial t}\Big(\overbrace{\frac{\rho u^2}{2}+\rho\varepsilon}^{E}\Big)+\frac{\partial}{\partial x_k}\left[\left(\frac{\rho u^2}{2}+\rho\varepsilon+P\right)u_k\right]=\rho a_k u_k$$

"Energy equation": can write the energy equation in several way in order to include different energy sources (like gravity) in the $\frac{\partial}{\partial t}$ term. Could have other source terms, like nuclear burning, neutrino losses, ...

These equations were writen in the Eulerian frame. This means we are tracking how the properties change at a fixed location, rather than how the properties change moving along with a particle in the flow. That would be the lagrangian view point.

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i)=0$$



in Eulerian

$$\frac{\partial\rho}{\partial t}\neq 0$$

in Lagrangian

$$\frac{\partial\rho}{\partial t}=0$$

$$\frac{df}{dt} = \partial_t f + \vec{v}\cdot\vec{\nabla} f$$

$$\partial_t f = \frac{df}{dt} - \vec{v}\cdot\vec{\nabla} f$$

$$\partial_t \rho = \frac{d\rho}{dt} - \vec{v}\cdot\vec{\nabla}\rho \implies \frac{d\rho}{dt} - \vec{u}\cdot\vec{\nabla}\rho + \partial_{\vec{x}}(\rho\vec{u}) = 0$$

$$\frac{d\rho}{dt} - \vec{u}\cdot\vec{\nabla}\rho + \rho\,\vec{\nabla}\cdot\vec{v} + \vec{u}\cdot\vec{\nabla}\rho = 0$$

$$\boxed{\frac{d\rho}{dt} + \rho\,\vec{\nabla}\cdot\vec{v} = 0}$$

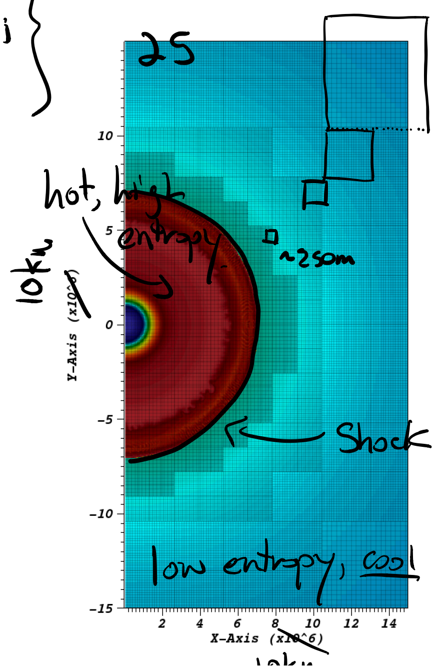
Beginnings of a numerical solution

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{F}_j}{\partial x_j} = \vec{S} \qquad \vec{S} = \begin{Bmatrix} 0 \\ \rho a_i \\ \rho a_i v_i \end{Bmatrix}$$

$$\vec{u} = \begin{Bmatrix} \rho \\ \rho v_i \\ E \end{Bmatrix} \qquad \vec{F} = \begin{Bmatrix} \rho v_j \\ \rho v_i v_j + P\delta_{ij} \\ (E+P)v_j \end{Bmatrix}$$

How to solve numerically?
both in space and time

Need a method of discretizing
→ Finite difference
→ Finite Volume
→ spectral methods
→ Method of lines



each method has pros/cons 10km
each problem may be best suited with one method

Method of lines is quite appealing because the methods can be easily extended to high order time integrations. (i.e. RK4)

1. solve spatial derivatives using so-called High-resolution shock capturing methods. / finite volume method.

$$\partial_t u + \partial_{x_i} \vec{F}_i = \vec{S}$$

$$\int_{\Delta t} dt \int_{\Delta V} dV \left(\qquad \right)$$

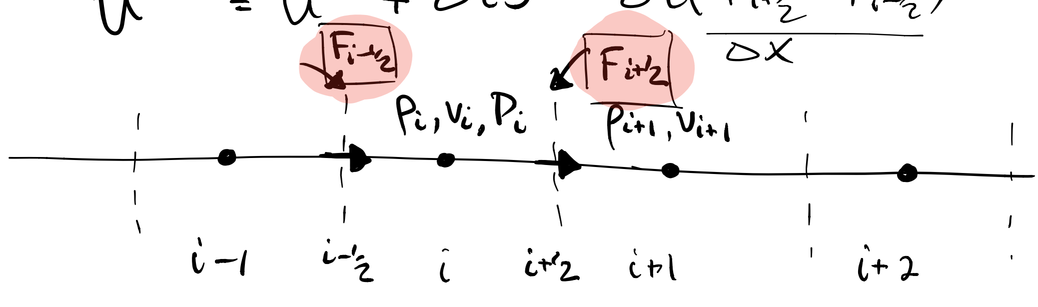
$$= \int_{\Delta t} dt \int_{\Delta V} dV\, \partial_t u + \int_{\Delta t} dt \int_{\Delta V} dV\, \partial_{x_i} \vec{F}_i = \int_{\Delta t}\int_{\Delta V} dt\, dV\, S$$

take $\Delta V = \Delta x$

$$\Delta x \left(u^{(n+1)} - u^{(n)} \right) + \Delta t \left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right) = \Delta t \Delta x\, S$$

$$\frac{u^{(n+1)} - u^{(n)}}{\Delta t} = S - \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x}$$

$$u^{(n+1)} = u^{(n)} + \Delta t S - \Delta t \frac{\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right)}{\Delta x}$$



combine with high order MoL.

$$U^{(n+1)} = U^{(n)} + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = \left[S - \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x}\right]\bigg|_{U^{(n)}} \sim \dot{U}^{(n)}$$

$$k_2 = \left[S - \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x}\right]\left(U^{(n)} + \frac{\delta t k_1}{2}\right)$$

$$k_3 = \dots$$

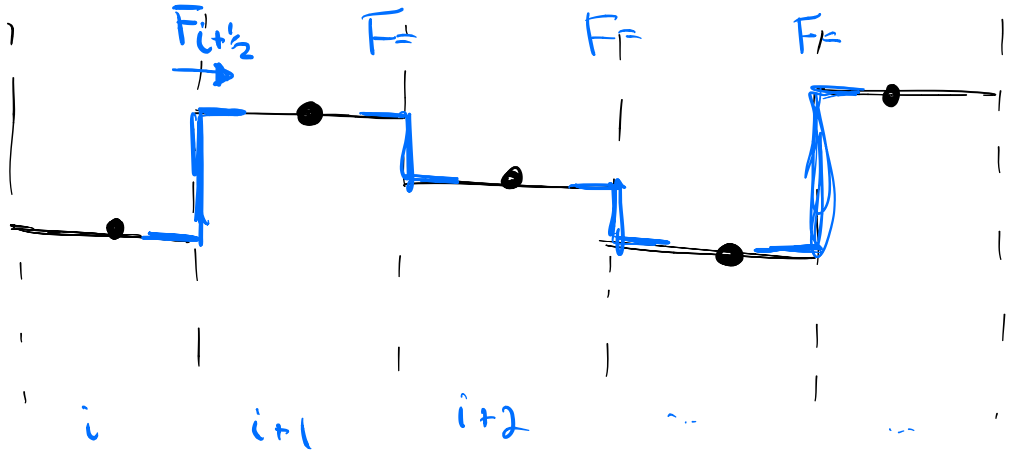
$$k_4 = \dots$$

Big part of HRSC methods is determining the the value of $F_{i+\frac{1}{2}}$ and $F_{i-\frac{1}{2}}$

this involves typically two steps.

1. reconstruction: determine variable (ρ, v_i, P) at the interface by interpolating center values to edges.

2. using reconstructed values, solve a riemann problem to determine fluxes.



$F_{i+\frac{1}{2}}$
i
i+1
i+2