Recap of stuff at end of Lecture 1

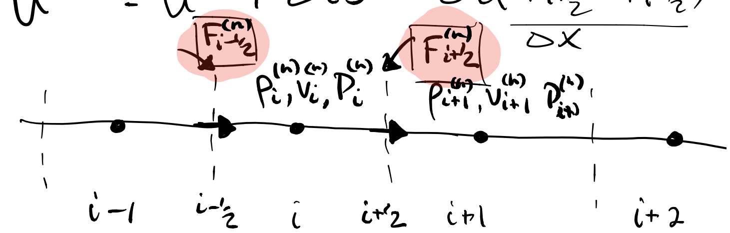
~ finite volume - like

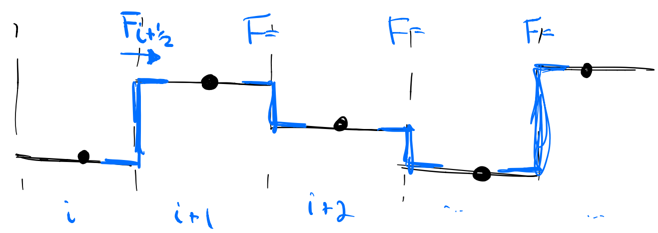
$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{F}_j}{\partial x_j} = \vec{S} \Rightarrow \frac{u^{(n+1)} - u^{(n)}}{\Delta t} = S^{(n)} - \frac{F^{(n)}_{i+\frac{1}{2}} - F^{(n)}_{i-\frac{1}{2}}}{\Delta x}$$

$$\vec{u} = \begin{Bmatrix} \rho \\ \rho v_i \\ E \end{Bmatrix} \quad \vec{F} = \begin{Bmatrix} \rho v_j \\ \rho v_i v_j + P\delta_{ij} \\ (E+P)v_j \end{Bmatrix}$$

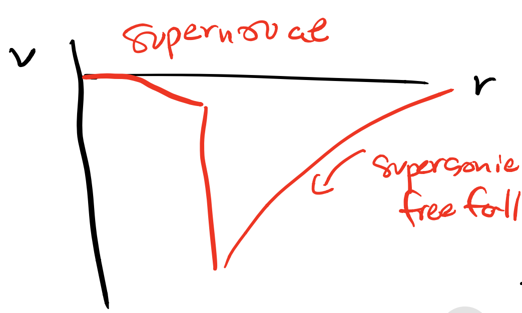
$$\vec{S} = \begin{Bmatrix} 0 \\ \rho a_i \\ \rho a_j v_j \end{Bmatrix}$$

$$u^{(n+1)} = u^{(n)} + \Delta t S^{(n)} - \Delta t \left(\frac{F^{(n)}_{i+\frac{1}{2}} - F^{(n)}_{i-\frac{1}{2}}}{\Delta x} \right)$$





Some info on Shocks.

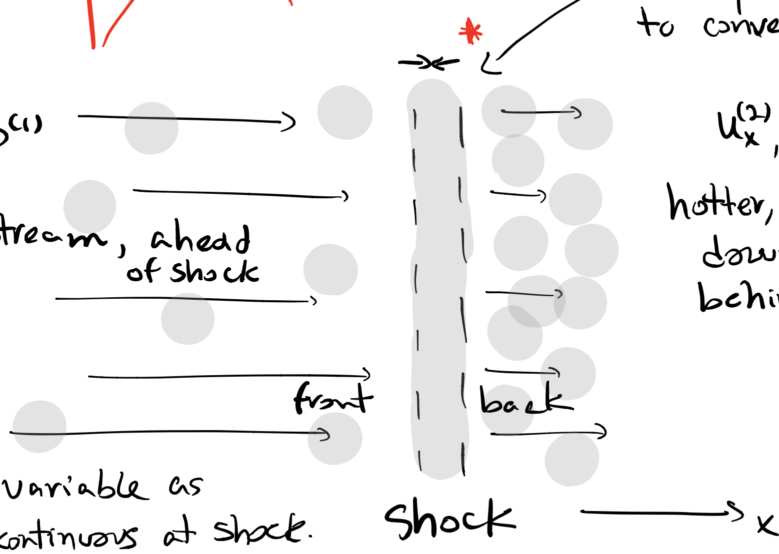


Astrophysically shocks are important too!

viscosity is crucial to converting KE to heat

$u_x^{(1)}, \rho^{(1)} P^{(1)}$

cold, upstream, ahead of shock



$u_x^{(2)}, \rho^{(2)}, P^{(2)}$

hotter, shocked region downstream behind

treat variable as discontinuous at shock.

mass is conserved across shock, mass does not accumulate in the shock.

$$\boxed{\rho^{(1)} u_x^{(1)} = \rho^{(2)} u_x^{(2)}} \rightarrow [\rho u_x] = 0$$

mass flux in = mass flux out

* strictly speaking this is the normal component of the velocity only. Tangential components of velocity do not change

momentum flux conservation

$$c_s^2 = \frac{\Gamma P}{\rho}$$

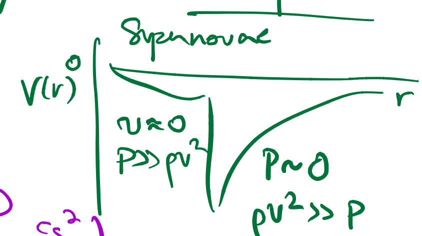
$$[\rho u_x u_j + P\delta_{xj}] = 0$$

$x=j$ $\quad [\rho u_x^2 + P] = 0 \rightarrow \boxed{\rho^{(1)} u_x^{(1)} u_x^{(1)} + P^{(1)} = \rho^{(2)} u_x^{(2)} u_x^{(2)} + P^{(2)}}$ ← ram pressure

$x \neq j$ $\quad \left.\begin{matrix}[\rho u_x u_y] = 0 \\ [\rho u_x u_z] = 0\end{matrix}\right\} \rightarrow \rho^{(1)} u_x^{(1)} u_y^{(1)} = \rho^{(2)} u_x^{(2)} u_y^{(2)}$

$$\boxed{u_y^{(1)} = u_y^{(2)}}$$

Supernovae



energy flux conservation

$$\left[u_x\left(\frac{\rho u^2}{2} + \rho\varepsilon + P\right)\right] = 0 \Rightarrow \left[\frac{1}{2}v^2 + h\right] = 0 \quad \left(h = \frac{c_s^2}{\gamma - 1}\right)$$

→ if one has an EOS, these can be simplified

ex. if $P = K\rho^\gamma$, for <u>strong</u> shock $(v_x^{(1)} \gg c_s^{(1)})$

$$c_s^2 = \frac{\gamma P}{\rho} \Rightarrow P = \frac{c_s^2 \rho}{\gamma}$$

ex. (1)

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_2 u_1} = \frac{M_1^2 c_1^2}{u_2 u_1} = \frac{M_1^2 h_0 (\gamma - 1)}{u_2 u_1 \left(1 + \frac{(\gamma-1)}{2} M_1^2\right)}$$

$$\hookrightarrow \frac{M_1^2(\gamma+1)}{2 + (\gamma - 1) M_1^2}$$

(3)

$$h_0 = \frac{1}{2}v_1^2 + \frac{c_{s_1}^2}{\gamma - 1}$$

$$c_{s_1}^2 = \frac{h_0(\gamma - 1)}{\left(1 + \frac{(\gamma-1)}{2} M_1^2\right)}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{P_1}{\rho_1 v_1} + v_1 = \frac{P_2}{\rho_2 v_2} + v_2$$

$$(v_1 - v_2) = \frac{1}{\gamma}\left[\frac{c_2^2}{v_2} - \frac{c_1^2}{v_1}\right]$$

solve for

$$\frac{h_0}{v_1 v_2} = \frac{1}{2}\frac{(\gamma+1)}{(\gamma-1)}$$

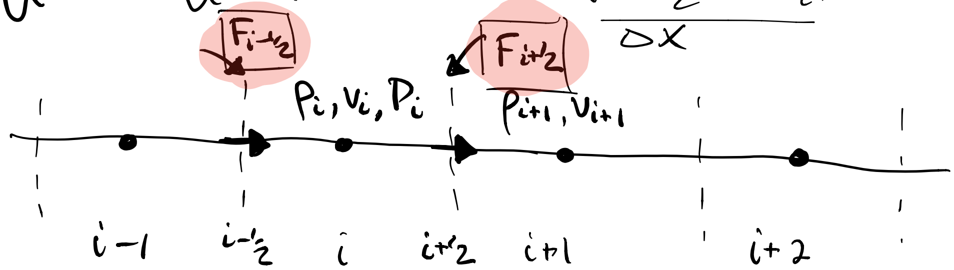
$M_1 \gg 1$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)}{(\gamma-1)} \sim 7$$

$\Gamma = 4/3$

Recall

$$U^{(n+1)} = U^{(n)} + \Delta t S^{(n)} - \Delta t \frac{\left(F^{(n)}_{i+\frac{1}{2}} - F^{(n)}_{i-\frac{1}{2}} \right)}{\Delta x}$$



Steps to get $U^{(n+1)}$: Godunov method (for Hydro but also used in some rad trans)

0. determine timestep Δt (limited via CFL $\Delta t \sim \frac{\Delta x}{c_s}$)
1. Calculate $U^{(n)}$ (prim2con)
2. Calculate sources $S^{(n)}$ (sources)
3. Calculate Fluxes $F^{(n)}_{i+\frac{1}{2}}$ & $F^{(n)}_{i-\frac{1}{2}}$ (reconstruct & Riemann)
4. Determine $U^{(n+1)}$ (evolve)
5. Determine new primatives (con2prim)

Primative vs. conservative

$$\vec{u} = \begin{Bmatrix} \rho \\ \rho v_i \\ \frac{\rho v^2}{2} + \rho \varepsilon \end{Bmatrix}$$

$\vec{u}$ are called the conservative variables, but the fundamental variable are what we would like to have, ρ, v_i, ϵ, P.

prim2con take $\rho, v_i, \epsilon, P \to \vec{u}$

con2prim takes $\vec{u} \to \rho, v_i, \epsilon, P$.

for these equations here these steps are trivial

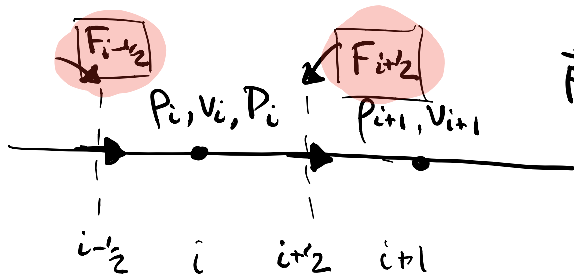
if $\vec{u} = (u_1, u_2^i, u_3)$
$\vec{Q} = (\rho, v_i, \varepsilon, P)$

$\rho = u_1 \qquad \varepsilon = \frac{u_3}{u_1} - \left(\frac{u_2^i}{u_1}\right)^2 \frac{1}{2}$

$v_i = u_2^i / u_1 \qquad P = \frac{2}{3} \varepsilon.$

as we shall see, for relativistic hydrodynamics this is not the case.

Fluxes



$$\vec{F} = \begin{Bmatrix} \rho v_j \\ \rho v_i v_j + P\delta_{ij} \\ (E+P)v_j \end{Bmatrix}$$

Godunov method

1. Use some method to "reconstruct" cell centered variables to the $\pm$ interface of each zone

$\rho_{i+1/2} \ngtr \max(\rho_i, \rho_{i+1})$
$\nless \min(\rho_i, \rho_{i+1})$
"enforcing monotonicity"

Piecewise reconstruction 0^{th}

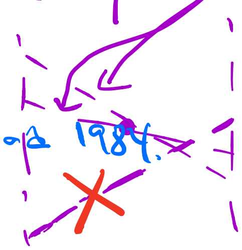
Piecewise linear + limiting (no new maximum) 1^{st}

Piecewise parabolic + limiting (PPM) 2^{nd}

→ lots of extra stuff

flattening, steepening, fancy limiter

require cell average to stay the same



Colella & Woodward 1984

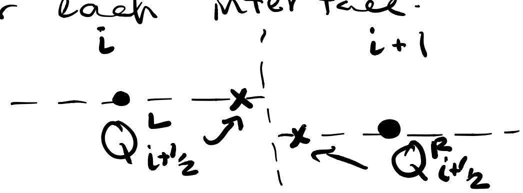
ENO/WENO methods

essentially non-oscillitory methods

(W = weigthed) 3^{rd}, 5^{th}, 7^{th}, 9^{th}

orders are in smooth flows

this method leaves you with two sets of reconstructed variables for each interface.



2. Solve the Riemann problem.

for HLL family of solvers
Solver determine
$F = R(F^L, u^L, F^R, u^R, \lambda)$

$Q^L_{i+1/2}$

L $\quad Q_{i+1/2}$? $\quad$ R

$Q^R_{i+1/2}$

How much flux is travelling across the boundary.

need for step #3

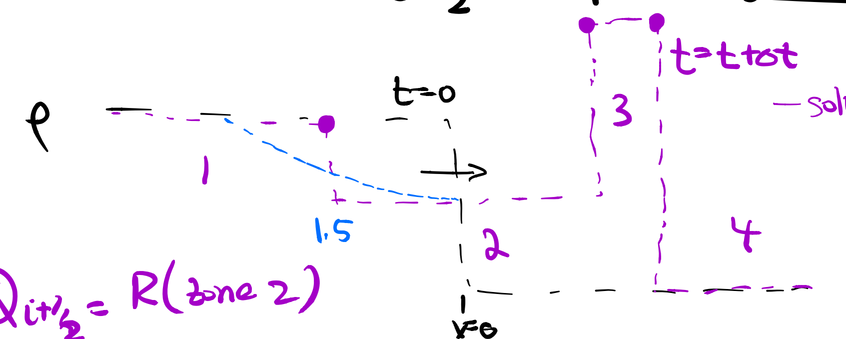
$i+1/2$

$$Q_{i+1/2} = R\left(Q^L_{i+1/2}, Q^R_{i+1/2}\right)$$

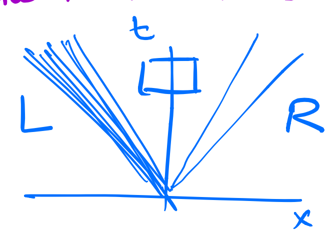
compute

$F_{i+1/2}$ given $Q_{i+1/2}$

Simple average is unstable

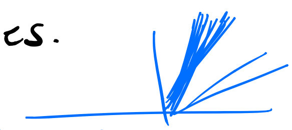


$Q_{i+1/2} = R(\text{zone } 2)$



Open-source examples of Hydrocodes.

Pyro2, GRID, FLASH, E.T.
ZEUS, ENZO, VH1, Athena, Castro



$$F_{HLL} = \frac{\lambda_R F_L - \lambda_L F_R}{\lambda_R - \lambda_L} + \frac{\lambda_R \lambda_L}{\lambda_R - \lambda_L}(U_R - U_L)$$

Relativistic Hydrodynamics.

→ techniques are all the same, equations are different (but similar). Here, I'll assume the gravity is handled seperately, therefore I'll take the metric as a given

$$g_{\mu\nu} \quad : \quad g = \det|g^{\mu\nu}|$$

also assume we have a stress-energy tensor and a current density.

$$T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu} \qquad J^\mu = \rho u^\mu$$

u = 4-velocity of the fluid $u^\mu = (\gamma, \gamma v^i, \gamma v^j, \gamma u^k)$ flat, not moving $u = (1, 0, 0, 0)$

$h = \left(1 + \frac{\epsilon}{c^2} + P/\rho c^2\right)$

primatives: $w = (\rho, v^i, \epsilon)$

ϵ = specific internal energy

P = pressure

Concept of conservation of energy, mass, and momentum fall out of

$$\nabla_\mu T^{\mu\nu} = 0 \quad \& \quad \nabla_\mu J^\mu = 0$$

$$\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\, J^\mu\right) = 0$$

Valencia formalism

define a set of conservative variables

$$D = \rho W \qquad S_j = \rho h W^2 v_j \qquad E = \rho h W^2 - P$$

$$\tau = E - D$$

$W = \alpha u^0 = (1 - v^2)^{-1/2}$; $v^2 = \gamma_{ij} v^i v^j$

$v^i = \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha}$; $h = \left(1 + \frac{\epsilon}{c^2} + P/\rho c^2\right)$

then expand $\nabla \cdot (\;) = 0$ above to get:

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial}{\partial t}\left[\sqrt{\gamma}\, U\right] + \frac{\partial}{\partial x^i}\left[\sqrt{-g}\, F^i\right]\right] = S$$

$$\vec{U} = \begin{Bmatrix} D \\ S_i \\ \tau \end{Bmatrix} \qquad F^i = \begin{Bmatrix} D(v^i - \beta^i/\alpha) \\ S_j(v^i - \beta^i/\alpha) + p\,\delta^i_j \\ \tau(v^i - \beta^i/\alpha) + p v^i \end{Bmatrix}$$

$$\tilde{S} = \begin{cases} T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma^{\delta}_{\nu\mu} g_{\delta j}\right) \\ \alpha\left(T^{\mu 0}\frac{\partial \ln(\alpha)}{\partial x^\mu} - T^{\mu\nu}\Gamma^{0}_{\nu\mu}\right) \end{cases}$$

* Only complicated part compared to before is
con2prim

the lorentz factors & h complicate everything.
con2prim requires a non-linear solve that should be iterate until convergeon. added complication is the EOS needs to be inverted at each iteration.

$$D = \rho W$$
$$S_j = \rho h W^2 v_j$$
$$E = \rho h W^2 - P$$
$$\tau = E - D$$

ex guess P

$$v_j^* = \frac{S_j}{\tau + D + P^*}$$

update W^*

$$\rho^* = D/W^*$$

$$e^* = \frac{S_j}{\rho^* W^{*2} v_j^*} - P^*/\rho^* - 1$$

Update EOS $P^* = f(\rho^*, \epsilon^*)$
iterate to converge on P.