Sound shell model of acoustic gravitational wave production

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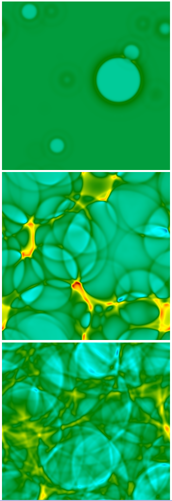
MH and M Hijazi, in preparation, MH arXiv:1608.04735

D. Cutting, MH, D. Weir arXiv:1906.00480

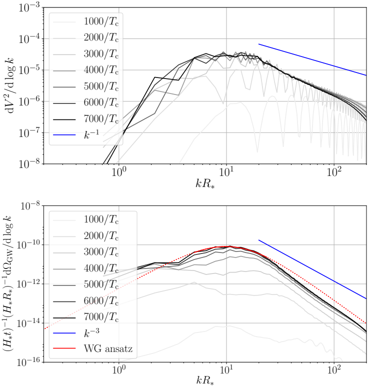
MH, S. Huber, K. Rummukainen, D. Weir arXiv:1703.06696, arXiv:1504.03291, arXiv:1304.2433

Outline

Introduction



$\overset{?}{\longrightarrow}$



Bubble nucleation temperature, transition rate parameter

- Below T_c, bubble nucleation rate/volume:

  $$p(T) = p_0(T)e^{-S(T)}$$

- Bubbles grow at speed v_w (S. Huber yesterday)
- Fractional volume in metastable phase:

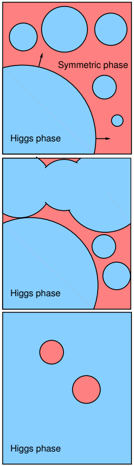
  $$h(t) = \exp\left(-\int^t dt' \frac{4\pi}{3} v_w^3 (t-t')^3 p(t') dt'\right)$$

- Reference time t_f such that $h(t_f) = 1/e$, evaluate by steepest descent:

  $$h(t) = \exp\left(-e^{\beta(t-t_f)}\right)$$

- Nucleation temperature: $T_n = T(t_f)$
- Transition rate parameter definition:

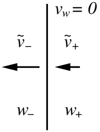
$$\beta = -\left.\frac{d}{dt}\ln p\right|_{T_n}$$



Self-similar fluid “shell” around bubble

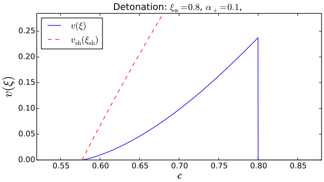
- Transition strength parameter: $\alpha(T_{\rm n}) = \Delta\theta/\bar{e}_Q$
  - Mean thermal energy $\bar{e}_Q = 3\bar{w}/4$, trace anomaly $\theta = (\bar{e} - 3\bar{p})/4$
- Junction conditions at wall

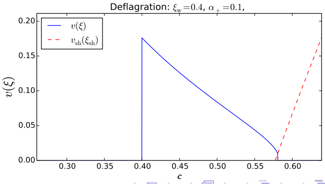
$$\tilde{v}_+\tilde{v}_- = \frac{1-(1-3\alpha_+)r}{3-3(1+\alpha_+)r}, \qquad \frac{\tilde{v}_+}{\tilde{v}_-} = \frac{3+(1-3\alpha_+)r}{1+3(1+\alpha_+)r}$$



NB $\alpha_+ = 3(\theta_+ - \theta_-)/4w_+$, $r = w_+/w_-$

- Similarity solution for bubble growth: detonation, deflagration





Kinetic energy production

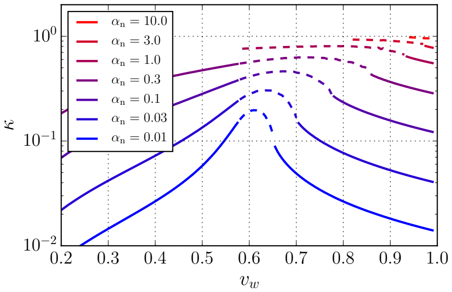
- Kinetic energy fraction

$$K \frac{\langle w\gamma^2 v^2 \rangle}{\bar{e}} = \Gamma \overline{U}_{\mathrm{f}}^2$$

Adiabatic index $\Gamma = \bar{w}/\bar{e}$
Weighted mean square fluid 4- velocity $\overline{U}_{\mathrm{f}}$

- "Efficiency parameter" κ

$$\frac{\kappa \alpha_\theta}{1 + \alpha_\theta} = K$$



Gravitational wave equation (and an auxiliary equation)

- Assume processes happen much faster than Hubble rate ($\beta/H \gg 1$)
- Assume metric perturbations are small
- Linearised GR, neglect expansion[(1)]

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j$$

- Linearised Einstein eqn for transverse-traceless part:

$$\ddot{h}_{ij} - \nabla^2 h_{ij} = 16\pi G \Pi_{ij}$$

- Convenient to avoid TT for evolution[(2)]

$$\ddot{u}_{ij} - \nabla^2 u_{ij} = 16\pi G \tau_{ij}$$

where $h_{ij} = u_{ij}^{TT}$ and $\Pi_{ij} = \tau_{ij}^{TT}$.
- Can neglect scalar if supercooling not extreme

$$\tau_{ij} = (e+p)\gamma^2 v_i v_j,$$

Gravitational wave spectral density

- GW energy density (average over many wavelengths, periods)

$$\rho_{\rm gw} = \frac{1}{32\pi G}\overline{\dot{h}_{ij}(x)\dot{h}_{ij}(x)}$$

- Fourier transform: $\dot{h}_{ij}(\mathbf{k},t) = \int d^3x \dot{h}_{ij}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}}$
- Define spectral density $P_{\dot{h}}$ through

$$\langle \dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k}',t)\rangle = P_{\dot{h}}(\mathbf{k})\,\bar{\delta}^3(\mathbf{k}-\mathbf{k}')$$

- Assume gravitational waves are generated by a process which is
  - isotropic: $P_{\dot{h}}(\mathbf{k}) \to P_{\dot{h}}(k)$, homogeneous (hence $\bar{\delta}^3(\mathbf{k}-\mathbf{k}')$)
  - Random, Gaussian (Wick's theorem)
  - Rapid compared with Hubble rate (neglect expansion)
- GW energy density

$$\rho_{\rm gw} = \frac{1}{32\pi G}\int \bar{d}^3k P_{\dot{h}}(k) = \frac{1}{32\pi G}\frac{1}{2\pi^2}\int dk k^2 P_{\dot{h}}(k)$$

Gravitational wave power spectrum

- Recall GW energy density from spectral density

$$\rho_{\rm gw} = \frac{1}{32\pi G}\frac{1}{2\pi^2}\int dk k^2 P_{\dot h}(k)$$

- Convenient to introduce power spectrum $\mathcal{P}_{\dot h} = \frac{k^3}{2\pi^2} P_{\dot h}(k)$

$$\rho_{\rm gw} = \frac{1}{32\pi G}\int \frac{dk}{k}\mathcal{P}_{\dot h}(k)$$

- Cosmology: characterisation better in terms of $\Omega_{\rm gw} = \rho_{\rm gw}/\bar{e}$.
- Define gravitational wave power spectrum

$$\mathcal{P}_{\rm gw}(k) \equiv \frac{d\Omega_{\rm gw}}{d\ln(k)} = \frac{1}{\bar{e}}\frac{1}{32\pi G}\mathcal{P}_{\dot h}(k) = \frac{1}{12H^2}\mathcal{P}_{\dot h}(k)$$

GW from stochastic sources

- Equation for auxiliary tensor $u_{ij}(\mathbf{x}, t)$

$$(\partial_t^2 - \nabla^2)u_{ij}(\mathbf{x}, t) = (16\pi G)\tau_{ij}(\mathbf{x}, t)$$

- Solution with oscillator Green's function[3]

$$u_{ij}(\mathbf{k}, t) = (16\pi G)\int_0^t dt' \frac{\sin[k(t-t')]}{k}\tau_{ij}(\mathbf{k}, t')$$

- Gravitational wave from TT projector $\dot{h}_{ij}(\mathbf{k}, t) = \lambda_{ij,kl}(\mathbf{k})\dot{u}_{kl}(\mathbf{k}, t)$
- Projector: $\lambda_{ij,kl}(\mathbf{k}) = P_{ik}(\mathbf{k})P_{jl}(\mathbf{k}) - \frac{1}{2}P_{ij}(\mathbf{k})P_{kl}(\mathbf{k})$ with $P_{ij}(\mathbf{k}) = \delta_{ij} - \hat{k}_i\hat{k}_j$.
- GW power spectrum obtained from

$$\langle \dot{h}^{ij}_{\mathbf{k}}(t)\dot{h}^{ij}_{\mathbf{k}'}(t)\rangle =$$

$$(16\pi G)^2\int_0^t dt_1 dt_2 \cos[k(t-t_1)]\cos[k(t-t_2)]\lambda_{ij,kl}(\mathbf{k})\langle\tau^{ij}_{\mathrm{f}}(\mathbf{k}, t_1)\tau^{kl}_{\mathrm{f}}(\mathbf{k}', t_2)\rangle.$$

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[3] Boundary condition: $u_{ij}(\mathbf{k}, t) \to 0$ as $t \to 0$

GW from stochastic sources

- Equation for auxiliary tensor $u_{ij}(\mathbf{k}, t)$

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$$(16\pi G)^2 \int_0^t dt_1 dt_2 \cos[k(t - t_1)] \cos[k(t - t_2)] \lambda_{ij,kl}(\mathbf{k}) \langle \tau_{\mathrm{f}}^{ij}(\mathbf{k}, t_1) \tau_{\mathrm{f}}^{kl}(\mathbf{k}', t_2) \rangle .$$

(3) Boundary condition: $u_{ij}(\mathbf{k}, t) \to 0$ as $t \to 0$

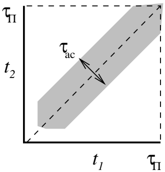
Unequal time correlator (UETC) for shear stress

- Define shear stress UETC U_Π by

$$\lambda_{ij,kl}(\mathbf{k})\langle\tau^{ij}(\mathbf{k},t_1)\tau^{kl}(\mathbf{k}',t_2)\rangle = U_\Pi(k,t_1,t_2)\,\delta^3(\mathbf{k}+\mathbf{k}')$$

- Form of UETC

- source is "on" for a time τ_Π
- auto-correlated for a time τ_{ac}
- peak at wavenumber $k \sim 1/L_\Pi$
- assume $\tau_\Pi \gg \tau_{ac}$ (approx. stationary)



GW spectral density from shear stress UETC

- Averaging over a many periods of the wave[(4)]

$$P_{\dot{h}}(k,t) = (16\pi G)^2 \tfrac{1}{2} \int_0^t dt_1 dt_2 \cos[k(t_1 - t_2)] U_\Pi(k, t_1, t_2).$$

- Assumed form of UETC (write $t_+ = (t_1 + t_2)/2$, $t_- = t_1 - t_2$)
  $U_\Pi(k, t_-/\tau_{ac}, t_+/\tau_\Pi)$, with $\tau_\Pi \gg \tau_{ac}$.
- Results in spectral density for $\dot{h}$

$$P_{\dot{h}}(k,t) = (16\pi G)^2 \tau_\Pi \tau_{ac} \mathcal{C}_\Pi(k,t)$$

where $\mathcal{C}_\Pi(k,t) = \frac{1}{2} \int^t \frac{dt_+}{\tau_\Pi} \int \frac{dt_-}{\tau_{ac}} \cos(kt_-) U_\Pi(k, t_-/\tau_{ac}, t_+/\tau_\Pi)$

- Available scales for auto-correlation time τ_{ac}:
  - $\tau_{ac} \sim k^{-1}$
  - $\tau_{ac} \sim L_\Pi$
  - $\tau_{ac} \sim \tau_\Pi$ (stationary approx breaks down)

[(4)] So that $\cos[k(t - t_1)] \cos[k(t - t_2)] \to \frac{1}{2} \cos[k(t_1 - t_2)]$

Estimates of $\Omega_{\rm gw}$

- Total gravitational wave power:
$$\Omega_{\rm gw} = \frac{1}{12H^2}\int \bar{d}^3 k P_{\dot{h}}(k) = \frac{(16\pi G)^2}{12H^2}\tau_\Pi \tau_{\rm ac}\int \bar{d}^3 k \mathcal{C}_\Pi(k)$$
- Shape function $\mathcal{C}_\Pi \sim \int\int \langle T_{ij}T_{ij}\rangle \sim (\bar{e}K)^2$
- Hence:
$$\Omega_{\rm gw} = 3K^2(H\tau_\Pi)(H\tau_{\rm ac})\tilde{\Omega}_{\rm gw},$$
where $\tilde{\Omega}_{\rm gw}$ is a GW production efficiency
- Acoustic production: $\Omega^{\rm ac}_{\rm gw} \sim 10^{-2}$

Estimates of $\Omega_{\rm gw}$: weak acoustic source ($K \ll (H_{\rm n} R_*)^2$)

- General:
  - fluid kinetic energy fraction K, ($\sim \overline{U}_{\rm f}^2$, mean square fluid velocity),
  - autocorrelation time $\tau_{\rm ac}$ ($\sim L_{\rm f}$, velocity field length scale),
  - Mean bubble separation R_*: expect $L_{\rm f} \sim R_*$
  - source lifetime τ_Π,
  - Hubble parameter H

$$\Omega_{\rm gw} \sim K^2 (H\tau_\Pi)(H\tau_{\rm ac}),$$

- Sound: $\tau_\Pi = \min(\tau_{\rm sh}, H_{\rm n}^{-1})$
- Shock appearance/dissipation time: $\tau_{\rm sh} \sim L_{\rm f}/\overline{U}_{\rm f}$
- Suppose $\tau_{\rm sh} \ll H_{\rm n}^{-1}$ (meaning low kinetic energy $K \ll (H_{\rm n} R_*)^2$),

$$\Omega_{\rm gw}^{\rm ac} \sim K^2 (H_{\rm n} R_*),$$

Estimates of Ω_{gw}: strong acoustic source ($K \gtrsim (H_n R_*)^2$)

- General:
  - fluid kinetic energy fraction K,
  - autocorrelation time τ_{ac},
  - source lifetime τ_Π,
  - Hubble parameter H:

$$\Omega_{gw} \sim K^2 (H\tau_\Pi)(H\tau_{ac}),$$

- Sound waves $\to$ shock waves and dissipation: $\tau_\Pi = \tau_{sh}$, $\Omega_{gw}^{ac} \sim K^2 (H_n L_f)(H_n \tau_{sh})$,
- $\tau_{sh} \sim R_* / \overline{U}_f$ ($\overline{U}_f \simeq K^{1/2}$)
- Hence:

$$\Omega_{gw}^{ac} \sim \Omega_{gw}^{sh} \sim K^{3/2} (H_n R_*)^2$$

Estimates of Ω_{gw}: pure turbulent source

- General:
  - fluid kinetic energy fraction K,
  - autocorrelation time τ_{ac},
  - source lifetime τ_Π,
  - Hubble parameter H:

$$\Omega_{gw} \sim K_\perp^2 (H\tau_\Pi)(H\tau_{ac}),$$

- Eddy turn-over time, $\tau_{tu} \sim R_* / \sqrt{K_\perp}$,
- Turbulence lifetime: $\tau_\Pi \sim \min(\tau_{tu}, H_n^{-1})$
- Turbulence autocorrelation time: $\tau_{ac} \sim \tau_{tu}$,
- Long-lasting turbulence ($H_n R_* / \sqrt{K_\perp} \gg 1$): $\Omega_{gw}^{ac} \sim K_\perp^2 (H_n \tau_{tu}) \sim K_\perp^{3/2} (H_n R_*)$
- Short-lasting turbulence ($H_n R_* / \sqrt{K_\perp} \ll 1$): $\Omega_{gw}^{ac} \sim K_\perp^2 (H_n \tau_{tu}) \sim K_\perp (H_n R_*)^2$

Sources of shear stress: fluid vs. scalar field

- Estimate size of shear stress correlator: $U_\Pi \sim \langle \tau\tau \rangle$
  - Fluid source tensor $\tau_{\rm f}^{ij} = w\gamma^2 v^i v^j$
  - Field source tensor $\tau_\phi^{ij} = \partial^i\phi\partial^j\phi$
- Kinetic energies $K_{\rm f} = \int d^3x \tau_{ii}^{\rm f}$, $K_\phi = \int d^3x \tau_{ii}^\phi$
  - Fluid: $K_{\rm f} = \int d^3x\, w\gamma^2 v^2 = \frac{4\pi}{3} R^3 \bar{w} \frac{3}{4}\alpha\kappa$
  - Field: $K_\phi = \int d^3x\, (\nabla\phi)^2 = 4\pi R^2 \sigma$
- Ratio $K_{\rm f}/K_\phi \sim R\bar{w}/\sigma \sim R/\ell \gg 1$
  - Bubble size R grows to Hubble length, ℓ is a microscopic scale (wall width).
- Scalar coupled to fluid, similarity solution: fluid shear stress dominant
  - Fluid shear stresses come from compression/rarefaction: sound waves
- Runaway: field shear stress also grows as R^3 (not considered here)

Sound waves

Consider EM tensor for perturbations with z dependence only

$$T^{tt} = w\gamma^2 - p, \qquad T^{tz} = w\gamma^2 v^z, \qquad T^{zz} = w\gamma^2 (v^z)^2 + p$$

Perturbations: $\delta e = e - \bar{e}$, $\delta p = p - \bar{p}$, v^z all $\ll 1$

$$\partial_t T^{tt} + \partial_z T^{zt} = 0 \implies \partial_t(\delta e) + \bar{w}\partial_z v^z = 0 \quad (1)$$

$$\partial_t T^{tz} + \partial_z T^{zz} = 0 \implies \bar{w}\partial_t v^z + \partial_z(\delta p) = 0 \quad (2)$$

Note that δp and δe both depends temperature T: $\delta p = \left(\frac{\partial p}{\partial T} / \frac{\partial e}{\partial T} \right) \delta e = c_s^2 \delta e$

Hence equations (1) and (2) can be combined

$$\left(\partial_t^2 - c_s^2 \partial_z^2 \right) v^z = 0, \qquad \left(\partial_t^2 - c_s^2 \partial_z^2 \right) \lambda = 0$$

Sound wave is a collective mode of fluid velocity v^i and pressure $\lambda = \delta p / \bar{w}$.
It is longitudinal: v^i is in direction of travel of wave.

Shear stress UETC from sound waves 1

- Recall shear stress UETC U_Π:
  $$\lambda_{ij,kl}(\mathbf{k}_1)\langle\tau^{ij}(\mathbf{k}_1,t_1)\tau^{kl}(\mathbf{k}_2,t_2)\rangle = U_\Pi(k_1,t_1,t_2)\,\delta^3(\mathbf{k}_1+\mathbf{k}_2)$$
- Source tensor dominated by fluid: $\tau^{ij} = \tau^{ij}_\mathrm{f} = w\gamma^2 v^i v^j$
- Non-relativistic fluid velocities: $\tau^{ij}_\mathrm{f} \simeq \bar{w} v^i v^j$
- Fourier transform of velocity field $\tilde{v}^i(\mathbf{q},t) = \int d^3x\, v^i(\mathbf{x},t)e^{-i\mathbf{q}\cdot\mathbf{x}}$
- Hence
  $$\tau^{ij}_\mathrm{f}(\mathbf{k},t) = \bar{w}\int \bar{d}^3q\, \tilde{v}^i(\mathbf{q},t)\tilde{v}^j(\tilde{\mathbf{q}},t), \qquad \tilde{\mathbf{q}} = \mathbf{q}-\mathbf{k}$$
- Assume velocity field is Gaussian: $\langle\tau\tau\rangle \sim \langle vvvv\rangle = \sum\langle vv\rangle\langle vv\rangle$
- Velocity unequal time correlator:
  $$\langle\tilde{v}^i_{\mathbf{q}_1}(t_1)\tilde{v}^{*j}_{\mathbf{q}_2}(t_2)\rangle = \left[P_{ij}(q)F(q,t_1,t_2) + \hat{q}^i\hat{q}^j G(q,t_1,t_2)\right]\,\delta^3(\mathbf{q}_1-\mathbf{q}_2).$$
  - Tranverse projector $P_{ij}(q) = \delta_{ij} - \hat{q}^i\hat{q}^j$
  - Sound waves contribute only to longitudinal part G

Shear stress UETC from sound waves 2

- Recall shear stress UETC U_Π:
  $$\lambda_{ij,kl}(\mathbf{k}_1)\langle\tau^{ij}(\mathbf{k}_1,t_1)\tau^{kl}(\mathbf{k}_2,t_2)\rangle = U_\Pi(k_1,t_1,t_2)\,\bar{\delta}^3(\mathbf{k}_1+\mathbf{k}_2)$$
- With $\tau_\mathrm{f}^{ij}(\mathbf{k},t) = \bar{w}\int \bar{d}^3q\,\tilde{v}^i(\mathbf{q},t)\tilde{v}^j(\tilde{\mathbf{q}},t), \qquad \tilde{\mathbf{q}} = \mathbf{q}-\mathbf{k}$
- And the velocity unequal time correlator from sound waves:
  $$\langle\tilde{v}^i_{\mathbf{q}_1}(t_1)\tilde{v}^{*j}_{\mathbf{q}_2}(t_2)\rangle = \hat{q}^i\hat{q}^j G(q,t_1,t_2)\,\bar{\delta}^3(\mathbf{q}_1-\mathbf{q}_2).$$
- A long calculation gives:[5]

$$U_\Pi(k,t_1,t_2) = \bar{w}^2\int \bar{d}^3q\,\frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 G(q,t_1,t_2)G(\tilde{q},t_1,t_2)$$

where $\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{q}}$, $\tilde{q}^2 = q^2 - 2qk\mu + k^2$.

[5] Caprini, Durrer, Servant 2007, 2009

Velocity UETC from sound waves

- Following disappearance of bubble at t_i, sound waves propagate freely
- General sound wave solution:

$$v^i(\mathbf{x},t) = \int \bar{d}^3 q \left(v^i_{\mathbf{q}} e^{-i\omega t + i\mathbf{q}\cdot\mathbf{x}} + v^{*i}_{\mathbf{q}} e^{i\omega t - i\mathbf{q}\cdot\mathbf{x}} \right), \qquad \omega = c_s q$$

- Spectral densities of velocity plane wave amplitudes

$$\langle v^i_{\mathbf{q}_1} v^{j*}_{\mathbf{q}_2} \rangle = \hat{q}^i \hat{q}^j P_v(q)\, \bar{\delta}^3(\mathbf{q}_1 - \mathbf{q}_2)$$
$$\langle v^i_{\mathbf{q}_1} v^{j}_{-\mathbf{q}_2} \rangle = \hat{q}^i \hat{q}^j Q_v(q) e^{2i\bar{\theta}_{q_1}}\, \bar{\delta}^3(\mathbf{q}_1 - \mathbf{q}_2)$$

- Recall velocity UETC $\langle \tilde{v}^i_{\mathbf{q}}(t_1) \tilde{v}^{*j}_{\mathbf{q}'}(t_2) \rangle = \hat{q}^i \hat{q}^j G(q,t_1,t_2)\, \bar{\delta}^3(\mathbf{q} - \mathbf{q}')$.
- Plane wave amplitudes related to Fourier transform
- Hence (recall $t_+ = (t_1+t_2)/2$, $t_- = t_1 - t_2$)

$$G(q,t_1,t_2) = 2P_v(q)\cos(\omega t_-) + 2Q_v(q)\cos(2\omega t_+ - 2\bar{\theta}_q)$$

Shear stress UETC from sound waves 3

- Recall expressions for shear stress UETC and velocity UETC

$$U_\Pi(k, t_1, t_2) = \bar{w}^2 \int \bar{d}^3 q \, \frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 G(q, t_1, t_2) G(\tilde{q}, t_1, t_2)$$

$$G(q, t_1, t_2) = 2P_v(q)\cos(\omega t_-) + 2Q_v(q)\cos(2\omega t_+ - 2\bar{\theta}_q)$$

- Argument:
  - Random process creates sound waves with random phases
  - Superposition gives non-oscillatory ETC $G(q, t, t)$, i.e. $Q_v(q) \simeq 0$.
  - $\implies U_\Pi$ dominated by $\cos(\omega t_-)\cos(\tilde{\omega} t_-)$ terms

$$U_\Pi(k, t_1, t_2) = 4\bar{w}^2 \int \bar{d}^3 q \, \frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 P_v(q) P_v(\tilde{q}) \cos(\omega t_-)\cos(\tilde{\omega} t_-)$$

- Conclusions:
  - $U_\Pi(k, t_1, t_2)$ depends mostly on $t_- = t_1 - t_2$ (“stationary”)
  - Autocorrelation time of mode with wavenumber k is $\tau_{ac} \sim k^{-1}$

GW from UETC

- Recall in spectral density for $\dot{h}$

$$P_{\dot{h}}(k,t) = (16\pi G)^2 \tau_\Pi \tau_{\rm ac} \mathcal{C}_\Pi(k,t)$$

where $\mathcal{C}_\Pi(k,t) = \frac{1}{2}\int^t \frac{dt_+}{\tau_\Pi} \int \frac{dt_-}{\tau_{\rm ac}} \cos(kt_-) U_\Pi(k, t_-/\tau_{\rm ac}, t_+/\tau_\Pi)$

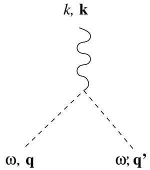
- Assume (simplicity) U_Π constant for time τ_Π, then zero
- We argued that $\tau_{\rm ac} = k^{-1}$, so
  $\mathcal{C}_\Pi(k,t) = \frac{1}{2} k \int dt_- \cos(kt_-) U_\Pi(k, kt_-)$
- Hence

$$\dot{\mathcal{C}}_\Pi(k,t) = 4\bar{w}^2 \int \bar{d}^3 q \, \frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 P_v(q) P_v(\tilde{q}) k \Delta(k,\omega,\tilde{\omega})$$

where $\Delta(k,\omega,\tilde{\omega}) = \frac{1}{2}\int dt_- \cos(kt_-)\cos(\omega t_-)\cos(\tilde{\omega} t_-)$

Kinematics of GW production from sound waves

- Spectral density of metric perturbation $P_{\dot{h}}(k,t) = (16\pi G)^2 \tau_\Pi k^{-1} \mathcal{C}_\Pi(k)$
  $\mathcal{C}_\Pi(k) = 4\bar{w}^2 \int \bar{d}^3 q \, \frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 P_v(q) P_v(\tilde{q}) k \Delta(k,\omega,\tilde{\omega})$
  $\Delta(k,\omega,\tilde{\omega}) = \frac{1}{2}\int dt_- \cos(kt_-)\cos(\omega t_-)\cos(\tilde{\omega} t_-)$
- For large time differences:
  $$\Delta(k,\omega,\tilde{\omega}) \to \frac{\pi}{8} \sum_{\pm\pm\pm} \delta(\pm k \pm \omega \pm \tilde{\omega})$$
  - Recall $\omega = c_s q$,
    $\tilde{\omega} = c_s \tilde{q} = c_s(q^2 + k^2 - 2kq\mu)$
  - Kinematics: only $k - \omega - \tilde{\omega}$ can vanish
  - Conservation of energy for production of GWs



- Hence

$$\mathcal{C}_\Pi(k) = \pi \bar{w}^2 \int \bar{d}^3 q \, \frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 P_v(q) P_v(\tilde{q}) k \delta(k - \omega - \tilde{\omega})$$

Kinematics of GW production from sound waves

- Spectral density of metric perturbation $P_{\dot{h}}(k,t) = (16\pi G)^2 \tau_\Pi k^{-1} \mathcal{C}_\Pi(k)$

$$\mathcal{C}_\Pi(k) = \pi \bar{w}^2 \int \bar{d}^3 q \, \frac{q^2}{\tilde{q}^2} (1-\mu^2)^2 P_v(q) P_v(\tilde{q}) k \delta(k - \omega - \tilde{\omega})$$

- Use δ function to perform integral over $\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}$
  - Solution: $\mu = \mu_* = \frac{1}{c_s}\left(1 - \frac{1}{2}(1 - c_s^2)\frac{k}{q}\right)$
  - Giving $\tilde{q} = \tilde{q}_* = \frac{1}{c_s} k - q$
- Only $q_- < q < q_+$ can produce GW with frequency k: with $q_\pm = k\frac{1 \pm c_s}{2c_s}$

$$\mathcal{C}_\Pi(k) = \frac{\bar{w}^2}{4\pi c_s} \int_{q_-}^{q_+} dq \, q^2 \frac{q}{\tilde{q}} (1 - \mu_*^2)^2 P_v(q) P_v(\tilde{q}_*)$$

Kinematics of GW production from sound waves

- Assume that sound waves have length scale L_f
- Scale out mean square velocity $\overline{U}_\mathrm{f}^2$
- Hence $P_v(q) = \overline{U}_\mathrm{f}^2 L_\mathrm{f}^3 \tilde{P}_v(qL_\mathrm{f})$, where $\tilde{P}_v$ is dimensionless.
- Define $z = qL_\mathrm{f}$ and $z_\pm = (kL_\mathrm{f})\frac{1 \pm c_\mathrm{s}}{2c_\mathrm{s}}$

$$\mathcal{C}_\Pi(k) = \frac{\bar{w}^2 \overline{U}_\mathrm{f}^4}{4\pi c_\mathrm{s}} L_\mathrm{f}^3 \int_{z_-}^{z_+} \frac{dz}{z} \frac{(z-z_+)^2(z-z_-)^2}{z_+ + z_- - z} \tilde{P}_v(z)\tilde{P}_v(z_+ + z_- - z)$$

- Define dimensionless function

$$\tilde{\mathcal{C}}_\Pi(kL_\mathrm{f}) = \int_{z_-}^{z_+} \frac{dz}{z} \frac{(z-z_+)^2(z-z_-)^2}{z_+ + z_- - z} \tilde{P}_v(z)\tilde{P}_v(z_+ + z_- - z)$$

  - is dimensionless and a function of $y = kL_\mathrm{f}$
  - peaks at $y \sim 1$ (definition of L_f)
  - Magnitude O(1)

Power spectrum of GWs from sound waves

- The final pieces are

$$\mathcal{P}_{\rm gw}(k) \equiv \frac{d\Omega_{\rm gw}}{d\ln(k)} = \frac{1}{12H^2}\frac{k^3}{2\pi^2}P_{\dot h}(k)$$

$$P_{\dot h}(k,t) = (16\pi G)^2 \tau_\Pi k^{-1}\,\mathcal{C}_\Pi(k)$$

$$\mathcal{C}_\Pi(k) = \frac{\bar w^2 \overline{U}_{\rm f}^4}{4\pi c_{\rm s}} L_{\rm f}^3 \tilde{\mathcal{C}}_\Pi(kL_{\rm f})$$

$$\tilde{\mathcal{C}}_\Pi(kL_{\rm f}) = \int_{z_-}^{z_+} \frac{dz}{z}\frac{(z-z_+)^2(z-z_-)^2}{z_+ + z_- - z}\tilde{P}_v(z)\tilde{P}_v(z_+ + z_- - z)$$

- The gravitational wave power spectrum is

$$\mathcal{P}_{\rm gw}(k) = 3\Gamma^2\overline{U}_{\rm f}^4 (H\tau_\Pi)(HL_{\rm f})\frac{(kL_{\rm f})^3}{2\pi^2}\frac{\tilde{\mathcal{C}}_\Pi(kL_{\rm f})}{4\pi c_{\rm s} kL_{\rm f}}.$$

where $\Gamma = \bar w/\bar e$ (adiabatic index, $\Gamma \simeq 4/3$)

Power spectrum of GWs from sound waves: amplitude

- From last slide

$$\mathcal{P}_{\rm gw}(k) = 3\Gamma^2 \overline{U}_{\rm f}^4 (H\tau_\Pi)(HL_{\rm f})\tilde{\mathcal{P}}_{\rm gw}(kL_{\rm f})$$

$$\tilde{\mathcal{P}}_{\rm gw}(kL_{\rm f}) = \frac{(kL_{\rm f})^3}{2\pi^2}\frac{\tilde{\mathcal{C}}_\Pi(kL_{\rm f})}{4\pi c_{\rm s} kL_{\rm f}}$$

$$\tilde{\mathcal{C}}_\Pi(kL_{\rm f}) = \int_{z_-}^{z_+}\frac{dz}{z}\frac{(z-z_+)^2(z-z_-)^2}{z_+ + z_- - z}\tilde{P}_v(z)\tilde{P}_v(z_+ + z_- - z)$$

where $z_\pm = (kL_{\rm f})^{\frac{1\pm c_{\rm s}}{2c_{\rm s}}}$

- Power proportional to square of kinetic energy: $\Gamma\overline{U}_{\rm f}^2 \simeq \kappa\alpha$
  - Phase transition strength parameter α
  - Scalar potential to kinetic energy conversion efficiency κ
- Power proportional to length scale $L_{\rm f}$
  - Length scale set by bubble separation: $L_{\rm f} \sim R_* = (8\pi)^{\frac{1}{3}} v_{\rm w}/\beta$
- Power proportional to time sound waves last τ_Π, which is the shorter of
  - Hubble time $\tau_H = H^{-1}$
  - Shock appearance time $\tau_{\rm sh} \sim L_{\rm f}/\overline{U}_{\rm f}$

Power spectrum of GWs from sound waves: shape

- From last-but-one slide

$$\mathcal{P}_{\rm gw}(k) = 3\Gamma^2 \overline{U}_{\rm f}^4 (H\tau_\Pi)(HL_{\rm f})\tilde{\mathcal{P}}_{\rm gw}(kL_{\rm f})$$

$$\tilde{\mathcal{P}}_{\rm gw}(kL_{\rm f}) = \frac{(kL_{\rm f})^3}{2\pi^2}\frac{\tilde{\mathcal{C}}_\Pi(kL_{\rm f})}{4\pi c_{\rm s} kL_{\rm f}}$$

$$\tilde{\mathcal{C}}_\Pi(kL_{\rm f}) = \int_{z_-}^{z_+}\frac{dz}{z}\frac{(z-z_+)^2(z-z_-)^2}{z_+ + z_- - z}\tilde{P}_v(z)\tilde{P}_v(z_+ + z_- - z)$$

where $z_\pm = (kL_{\rm f})\frac{1\pm c_{\rm s}}{2c_{\rm s}}$

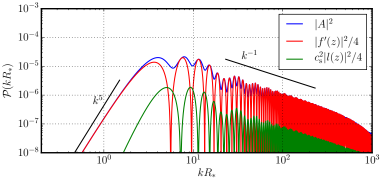
- If velocity power spectrum $\mathcal{P}_v \sim q^n$, then $\tilde{P}_v(z) \sim z^{n-3}$
- $\implies \tilde{\mathcal{C}}_\Pi(y) \sim y^3 (y^{n-3})^2 \sim y^{2n-3}$
- $\implies \tilde{\mathcal{P}}_{\rm gw}(y) \sim y^3 y^{2n-4} \sim y^{2n-1} \propto k^{2n-1}$
- Prediction: $\mathcal{P}_{\rm gw}(k) \sim k^{2n-1}$ from $\mathcal{P}_v(q) \sim q^n$

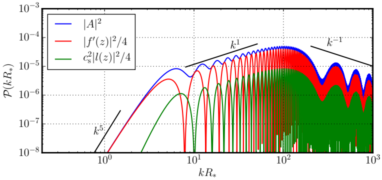
1-bubble velocity power spectra

- Sound Shell Model: velocity power spectrum is a weighted superposition of 1-bubble velocity plane-wave power spectra $A(kR)$

$$\left\langle v^i_{\mathbf{q}_1} v^{*j}_{\mathbf{q}_2} \right\rangle = \int dT_{\rm i} n(T_{\rm i}) T_{\rm i}^6 \hat{z}^i \hat{z}^j |A(z)|^2 \tilde{\delta}^3(\mathbf{q}_1 - \mathbf{q}_2).$$

- Bubble lifetime $T_{\rm i}$ - crudely modelled as disappearing all at once
- Bubble lifetime distribution function $n(T_{\rm i})$

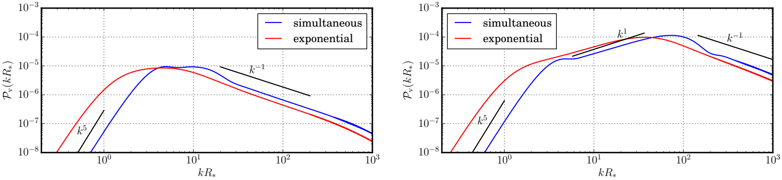




Single-bubble plane wave power spectra. Left: $v_{\rm w} = 0.92$, right: $v_{\rm w} = 0.56$. Transition strength $\alpha_\theta = 0.0046$

SSM velocity power spectra

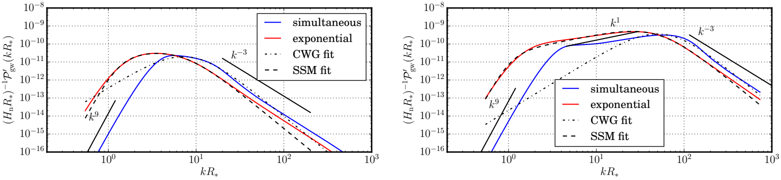
- Bubble lifetime distribution (exponential nucleation):
  $n_{\rm exp}(T_{\rm i}) = \frac{\beta}{R_*^3} \exp(-\beta T_{\rm i})$
- Bubble lifetime distribution (simultaneous nucleation):
  $n_{\rm sim}(T_{\rm i}) = \frac{\beta}{R_*^3} \frac{1}{2}(\beta T_{\rm i})^2 \exp\left(-\frac{1}{6}(\beta T_{\rm i})^3\right)$
- Results after convolution:



Velocity power spectra. Left: $v_{\rm w} = 0.92$, right: $v_{\rm w} = 0.56$. Transition strength $\alpha_\theta = 0.0046$

SSM gravitational wave power spectra

- Gravitational wave power spectra proportional to time and proportional to $H_{\mathrm{n}}R_*$
- Plot rate of increase divided by $H_{\mathrm{n}}R_*$



Scaled gravitational wave power spectra. Left: $v_{\mathrm{w}} = 0.92$, right: $v_{\mathrm{w}} = 0.56$. Transition strength $\alpha_\theta = 0.0046$

Modelling GWs from sound waves: "CWG" function

- Fitting function GW power spectrum from linear sound waves:
  - Bubble nucleation temperature T_n
  - Hubble rate H_n
  - Mean bubble separation R_*, peak of power spectrum at $z_\mathrm{p} = k_\mathrm{p} R_* \simeq 10$
-
$$\frac{d\Omega_{\mathrm{gw},0}}{d\ln(f)} = F_{\mathrm{gw},0} 3\Gamma^2 \overline{U}_\mathrm{f}^4 (H_\mathrm{n} R_*) \tilde{\Omega}_\mathrm{gw} C\left(\frac{f}{f_{\mathrm{p},0}}\right).$$

  where

$$C(s) = s^3 \left(\frac{7}{4+3s^2}\right)^{7/2}.$$

  dilution of GWs since matter-domination

$$F_{\mathrm{gw},0} = (3.57 \pm 0.05) \times 10^{-5} \left(\frac{100}{h_*}\right)^{\frac{1}{3}}.$$

  Peak frequency

$$f_{\mathrm{p},0} \simeq 26 \left(\frac{1}{H_\mathrm{n} R_*}\right) \left(\frac{z_\mathrm{p}}{10}\right) \left(\frac{T_\mathrm{n}}{10^2\,\mathrm{GeV}}\right) \left(\frac{h_*}{100}\right)^{\frac{1}{6}} \mu\mathrm{Hz},$$

Modelling GWs from sound waves: new Sound Shell Model prediction

- Fitting function GW power spectrum from linear sound waves:
  - Bubble nucleation temperature T_n
  - Hubble rate H_n
  - Mean bubble separation R_*, peak of power spectrum at $z_\mathrm{p} = k_\mathrm{p} R_* \simeq 10$
- double broken power law

$$\frac{d\Omega_{\mathrm{gw},0}}{d\ln(f)} = F_{\mathrm{gw},0} 3\Gamma^2 \overline{U}_\mathrm{f}^4 (H_\mathrm{n} R_*) \tilde{\Omega}_\mathrm{gw} M\left(\frac{f}{f_{\mathrm{p},0}}, r_\mathrm{b}\right).$$

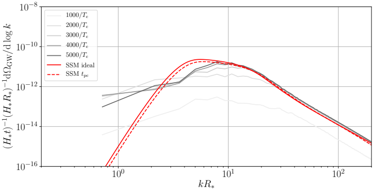
where

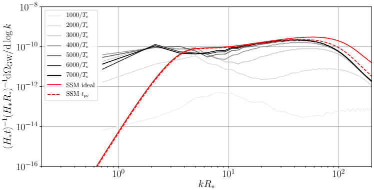
$$M(s, r_\mathrm{b}) = s^9 \left(\frac{r_\mathrm{b}^4 + 1}{r_\mathrm{b}^4 + s^4}\right)^2 \left(\frac{5}{5 - m + ms^2}\right)^{5/2},$$

and $m = (9r_\mathrm{b}^4 + 1)/(r_\mathrm{b}^4 + 1)$

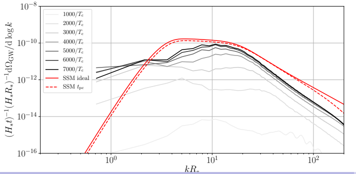
Comparison to numerical simulations

- Compare with numerical simulations.[(6)]
  Left: $v_\mathrm{w} = 0.92$, right: $v_\mathrm{w} = 0.56$. Transition strength $\alpha_\theta = 0.0046$





- but $v_\mathrm{w} = 0.44$, transition strength $\alpha_\theta = 0.0046$ (deflagration)



- GWs too high for deflagrations
- Kinetic energy suppression (see D. Cutting's talk next week)

Summary and outlook

- Sound Shell Model is a promising first step for semi-analytic calculations of GW power spectra in first order phase transitions
- Areas for further work:
  - Long-wavelength part of power spectrum
  - Understanding kinetic energy suppression
  - Generation of vorticity
  - GWs from turbulent velocity field
- Also necessary
  - Parameter extraction from GW power spectrum α, β, v_w, T_c
  - Improve accuracy of calculations of α, β, v_w, T_c from fundamental theory