Gravitational wave generation in a viable scenario of inflationary magnetogenesis

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"Gravitational Waves from the Early Universe"
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Outline of the Talk

Part 1 : A viable model for the generation of large scale magnetic field in the early universe

Based on:

- R. Sharma, S. Jagannathan, T. R. Seshadri, and K. Subramanian, Phys. Rev. D 96, 083511 (2017), arXiv:1708.08119
- R. Sharma, K. Subramanian, and T. R. Seshadri, Phys. Rev. D97, 083503 (2018), arXiv:1802.04847

Part 2 : Stochastic background of gravitational wave from the anisotropic stress due to these fields

Based on:

- R. Sharma, K. Subramanian, and T. R. Seshadri (In preparation)

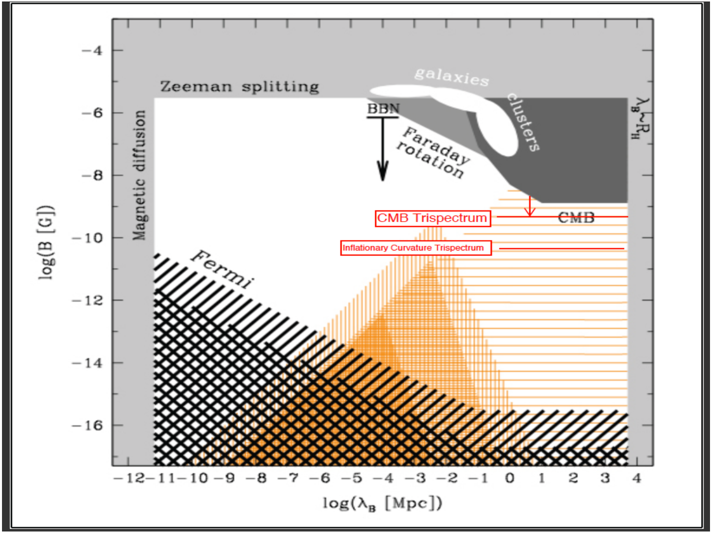
- **Observational evidences of magnetic fields**

Observational evidences of Magnetic Fields

- $\vec{B}$ over galactic scales (ordered on kpc) $\sim$ order of 10μG : Both coherent and stochastic [Beck 2001; Beck and Wielebinski 2013]
- Observed in clusters with a few μG strength, coherence length of the order of 10-20 kpc [Clarke et al. 2001, Govoni and Feretti 2004]
- Evidence for equally strong $\vec{B}$ in high redshift ($z \sim 1.3$) galaxies [Bernet et al. 08]
- FERMI/*LAT* observations of GeV photons from Blazars
  - **Lower** limit: $\vec{B} \geq 10^{-16}$ G on intergalactic $\vec{B}$ at scale above 1 Mpc [Neronov & Vovk, *Science* 10]

Summary of Observational Constraints

[Neronov & Vovk, *Science* 10]



- Observational evidences
- **Generation mechanism of the magnetic fields**

Origin and Growth: Broad Picture

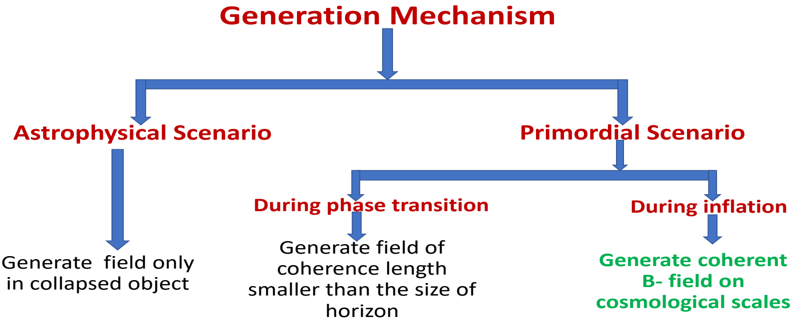
- Amplification $\longrightarrow$ growth (flux freezing, Dynamo mechanism) Governing equation for these mechanisms is magnetic induction equation,

$$\frac{\partial \vec{B}}{\partial \tau} = \vec{\nabla} \times (\vec{V} \times \vec{B} - \eta \vec{\nabla} \times \vec{B})$$

Here τ and η are the time parameter and plasma registivity, respectively.

- However dynamo requires an initial seed field $\sim 10^{-20}$ G.
- Origin of seed field $\longrightarrow$ Astrophysical or Primordial

Generation Mechanism of magnetic field



Astrophysical origin of seeds may not be able to explain the presence of magnetic field in voids

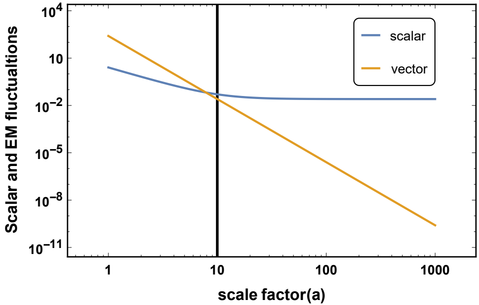
Worth considering primordial origin possibly during inflationary process. (*Durrer and Neronov 2013; K. Subramamnian, 2010, 2016*)

- Observational evidences
- Generation Mechanism of the magnetic fields
- **Inflationary Magnetogenesis**

Inflation

- An era of exponential expansion of space in the early Universe.
- Introduced to solve Horizon and Flatness problems.
- Also provides a natural explanation to initial density fluctuations.
- These initial density fluctuations arise due to the quantum mechanical nature of the field which causes inflation or some other field present during inflation.
- As different modes cross the horizon, the nature of fluctuations over these modes becomes classical.

Scalar field vs EM field fluctuations during inflation



Scalar field fluctuations $\langle 0|\hat{\delta\phi}(\vec{x},\eta)\hat{\delta\phi}(\vec{y},\eta)|0\rangle \approx \Delta_\phi(k)|_{k\sim 1/L}$

EM field fluctuations $\langle 0|B_i(\vec{x},\eta)B^i(\vec{y},\eta)|0\rangle \approx \Delta_B(k)|_{k\sim 1/L}$

For inflationary scale $H_f = 10^{14}$ GeV, the value of Δ_B for 1 Mpc mode at horizon crossing $\approx 10^{-10}$ G.

However this value at the end of inflation becomes $\approx 10^{-10} \times 10^{-46}$ G

Scalar field vs EM field fluctuations during inflation

- **Scalar fluctuations:**

$$S_\phi = \frac{-1}{2}\int d^4x\sqrt{-g}(\partial^\nu\phi\partial_\nu\phi - V(\phi))$$

$$(a\delta\phi(k,\eta))'' + \left(k^2 - \frac{a''}{a}\right) a\delta\phi(k,\eta) = 0$$

- **EM fluctuations:**

$$S_{EM} = -\int \sqrt{-g}\,d^4x \frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$$

$$(aA(k,\eta))'' + k^2 aA(k,\eta) = 0$$

  This implies $B \propto \frac{1}{a^2}$.

- This happens due to the conformal invariance of the EM action and the conformal flatness of the background spacetime.

Breaking conformal invariance

Action: Modified electromagnetic action + interaction with charged particles/current

$$S_{EM} = -\int \sqrt{-g}\, d^4x \left(f^2(\phi) \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu \right)$$

In Coulomb Gauge, $$A_i'' + 2\frac{f'}{f} A_i' - a^2 \partial_j \partial^j A_i = 0.$$

Define $\bar{A} \equiv aA(k,\eta)$ $$\bar{A}'' + 2\frac{f'}{f}\bar{A}' + k^2 \bar{A} = 0.$$

Define $\mathcal{A} \equiv f\bar{A}(k,\eta)$ $$\mathcal{A}''(k,\eta) + \left(k^2 - \frac{f''}{f}\right)\mathcal{A}(k,\eta) = 0.$$

Energy density of the EM field

- Energy momentum tensor

$$T_{\mu\nu} = f^2\left[g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - g_{\mu\nu}\frac{F_{\alpha\beta}F^{\alpha\beta}}{4}\right]$$

- Energy density

$$\rho = \langle 0|T_{\mu\nu}u^\mu u^\nu|0\rangle$$

$$\begin{aligned}
T_{\mu\nu}u^\mu u^\nu =& \frac{f^2}{2}B^iB_i + \frac{f^2}{2}E^iE_i \\
\langle 0|\frac{f^2}{2}B^iB_i|0\rangle =& \int d\ln k\,\frac{1}{2\pi^2}\frac{k^5}{a^4}|\mathcal{A}(k,\eta)|^2 \equiv \int d\ln k\,\frac{d\rho_B}{d\ln k} \\
\langle 0|\frac{f^2}{2}E^iE_i|0\rangle =& \int d\ln k\,\frac{f^2}{2\pi^2}\frac{k^3}{a^4}\left|\left[\frac{\mathcal{A}(k,\eta)}{f}\right]'\right|^2 \equiv \int d\ln k\,\frac{d\rho_E}{d\ln k}
\end{aligned}$$

Generated magnetic field

- For, $f = f_i \ a^{\alpha}$ and $a = -\frac{1}{H_f \eta}$, there are two possibility for a scale invariant magnetic field spectral energy density; $\alpha = 2$ and $\alpha = -3$.

- For scale invariant spectrum $\frac{d\rho_B}{d \ln k} \approx \frac{9}{4\pi^2} H_f^4$

- After generation, magnetic energy density varies with time as $\rho_B \propto 1/a^4$.

- Corresponding magnetic field strength

$$B_0 = 2\sqrt{\left.\frac{d\rho_B}{d \ln k}\right|_f} \left(\frac{a_f}{a_0}\right)^2 \quad \sim 5 \times 10^{-10} G \left(\frac{H_f}{10^{-5} M_{pl}}\right)$$

Back reaction and strong coupling problems

- Scale invariant spectral magnetic energy density: $\alpha = 2$ and $\alpha = -3$
- For $\alpha = -3$, Electric energy density spectrum $\propto (\frac{k}{aH})^{-2}$
  - Electric energy density diverges towards the end of inflation.
  - Electrical energy density dominates over inflation energy density.

  This is known as **back reaction problem**.
- In the usual approach with conformal breaking, the final value of f is made unity to match with the standard EM theory.
- Since f grows as $a^2 \implies$ initial value of f is very small.
- Effective coupling parameter $e_N = e/f^2$ becomes very large.

  This is known as **strong coupling problem**.

- Observational evidences
- Generation Mechanism of the Magnetic fields
- Inflationary Magnetogenesis
- **Viable model of magnetic field generation**

Addressing the strong coupling problem

- In our model, we bring the system back to the standard form not at end of inflation but some time after it before reheating.

  $f_i = 1 \implies, \quad f = a^2 > 1$(during inflation) & $f \gg 1$ at end of inflation.

  Hence no strong coupling problem.

- As coupling parameter is very small at the end of inflation. Hence, f need to be brought back to unity post inflation.

$$\text{During Inflation,} \qquad f = a^2$$

$$\text{Post Inflation,} \qquad f = f_f \left(a/a_f\right)^{-\beta}$$

- Models are constrained by the requirement of how fast the factor f falls to 1 from a large value.

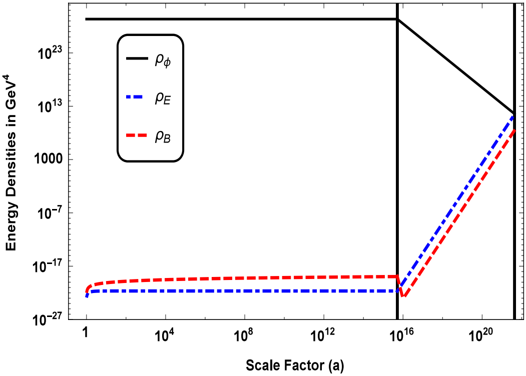
Post Inflationary era

- We assume a matter dominated universe after inflation till reheating.
- For $f \propto a^{-\beta}$, we solved vector potential by demanding the continuity of vector potential and its time derivative at the end of inflation.
- Energy density in magnetic and electric field can be calculated as before.
- **At reheating, for super horizon modes $\frac{d\rho_B}{d\ln k} \propto k^4$ for** $\alpha = 2$

Constraints from Post Inflationary Pre-reheating phase

Total energy in electric and magnetic field should be less that in inflation field at reheating.

$$\rho_E + \rho_B < \rho_\phi \,|_{reheat} = g_r \frac{\pi^2}{30} T_r^4$$



Post reheating evolution of magnetic field

- $$B_0^{NL}[L_{c0}^{NL}] = B_0[L_{c0}]\left(\frac{a_m}{a_r}\right)^{-p}, \quad L_{c0}^{NL} = L_{c0}\left(\frac{a_m}{a_r}\right)^{q},$$

  where $a_m \implies$ scale factor at radiation-matter equality,

  $$p \equiv (n+3)/(n+5) \quad \text{and} \quad q \equiv 2/(n+5)$$

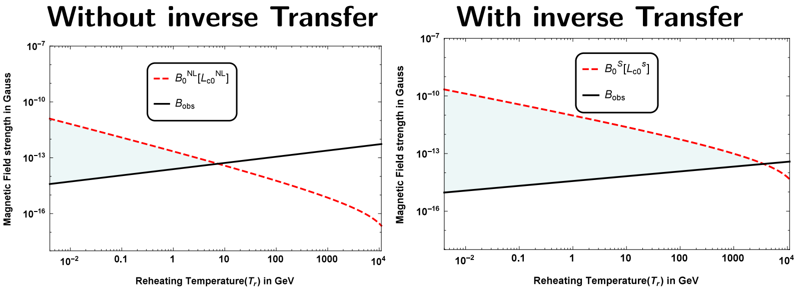
  here n is defined in such a way that $\frac{d\rho_B}{d\ln k} \propto k^{n+3}$
  (Banerjee and Jedamzik, 2004; Brandenburg et al. 2015)

- After incorporating the results of magnetic field evolution suggested by simulation,

  $$B_0^S[L_{c0}^S] = B_0[L_{c0}]\,(a_m/a_r)^{-0.5}, \quad L_{c0}^S = L_{c0}\,(a_m/a_r)^{0.5}$$

  (Brandenburg et al. 2015; Brandenburg and Kahniashvili 2016)

Results taking nonlinear effects into account



- For $T_R = 100$ GeV, $B_0 \sim 10^{-15}$G and coherence length $\sim 10^{-5}$ Mpc.
- $B_0 \sim 10^{-13}$G and coherence length $\sim 10^{-3}$ Mpc (with inverse transfer).

- Observational evidences
- Generation Mechanism of the Magnetic fields
- Inflationary Magnetogenesis
- Viable model of magnetic field generation
- **Helical magnetic field generation**

EM action for the generation of helical magnetic field

- Action

$$S_{EM} = -\int \sqrt{-g} d^4x \left(\frac{f^2(\phi)}{16\pi} \left(F_{\mu\nu}F^{\mu\nu} + F_{\mu\nu}\tilde{F}^{\mu\nu} \right) + j^\mu A_\mu \right)$$

- Modified Maxwell's Equation

$$A_i'' + 2\frac{f'}{f} \left(A_i' + \epsilon_{ijk}\partial_j A_k \right) - a^2 \partial_j \partial^j A_i = 0$$

- In terms of circular polarisation basis

$$\bar{A}_h'' + 2\frac{f'}{f} \left(\bar{A}_h' + hk\bar{A}_h \right) + k^2 \bar{A}_h = 0$$

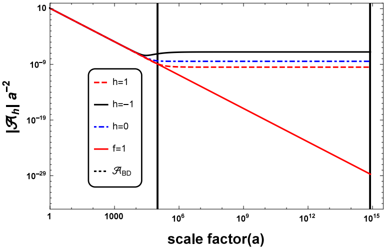
here $h = \pm 1$

Magnetic field energy spectrum

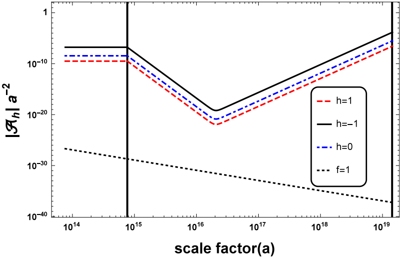
$$\boxed{\frac{d\rho_B(k,\eta)}{d\ln k} = \frac{1}{(2\pi)^2}\frac{k^5}{a^4}\left(|\mathcal{A}_+(k,\eta)|^2 + |\mathcal{A}_-(k,\eta)|^2\right)}$$

Evolution of a mode $k = 10^5 H_f$

During inflation

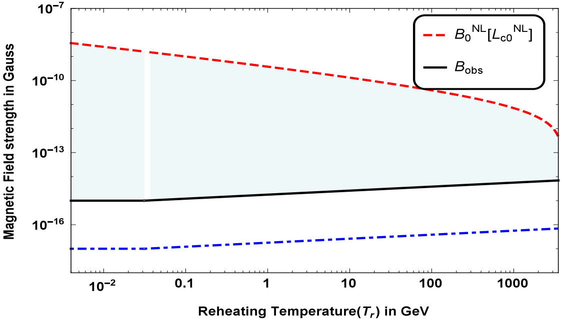


Post Inflation



Present strength of magnetic fields

$$L_{c0}^{NL} = L_{c0}\left(\frac{a_m}{a_r}\right)^{2/3} \qquad B_0^{NL}[L_{c0}^{NL}] \quad = B_0[L_{c0}]\left(\frac{a_m}{a_r}\right)^{-1/3}.$$



For $T_R = 100$ GeV, $B_0 \sim 10^{-11}$ G and coherence length ~ 0.07 Mpc.

Part 2 : Production of stochastic background of Gravitational Waves from EM fields anisotropic stress

- Before reheating, $\frac{d\rho_B}{d\ln k} \propto k^4$ and $\frac{d\rho_E}{d\ln k} \propto k^2$ for wavenumbers below to the value corresponding to horizon size.
- Electric spectral energy density dominates over the magnetic spectral energy desnity.
- After reheating $\frac{d\rho_B}{d\ln k} \propto k^4$ and electric field gets shorted.

Gravitational waves

- **Gravitational waves** $\implies$ Represented by the traceless transverse part of the space-time metric perturbation.

- The metric for homogeneous, isotropic and spatially flat universe.

$$ds^2 = a^2(\eta)(-d\eta^2 + (\delta_{ij} + 2h_{ij})dx^i dx^j).$$

where h_{ij} satisfies: $\partial^i h_{ij} = 0$ and $h^i_i = 0$.

- The energy density of the stochastic GW in terms of tensor perturbations,

$$\begin{aligned}
\rho_{GW} &= \frac{1}{16\pi G a^2}\langle h'_{ij} h'^{ij}\rangle \\
&= \frac{1}{16\pi G a^2}\int \frac{d^3 k}{(2\pi)^3}\int \frac{d^3 q}{(2\pi)^3}\langle h'_{ij}(\vec{k},\eta) h'^{*}_{ij}(\vec{q},\eta)\rangle e^{i(\vec{k}-\vec{q})\cdot\vec{x}} \\
&\equiv \int d\ln k \frac{d\rho_{GW}}{d\ln k}
\end{aligned}$$

Evolution of Gravitational Waves

- The evolution equation for h_{ij} in presence of a source,

$$h''_{ij} + \frac{2a'}{a} h'_{ij} + k^2 h_{ij} = 8\pi G a^2 \overline{T}_{ij}.$$

here $a^2 \overline{T}_{ij}$ is the transverse traceless part of energy momentum tensor of the source.

- For statistically homogeneous and isotropic EM fields,

$$\langle \overline{T}_{ij}(\vec{k}, \eta) \overline{T}^{ij}(\vec{k'}, \eta) \rangle \propto \delta(\vec{k} - \vec{k'})$$

using this,

$$\langle h'_{ij}(\vec{k}, \eta) h'^{ij}(\vec{k'}, \eta) \rangle \propto \delta(\vec{k} - \vec{k'})$$

we obtained,

$$\frac{d\rho_{GW}}{d \ln k} = \frac{k^3}{4(2\pi)^3 G a^2} \sum_{\aleph} \left(\left| \frac{dh^{\aleph}(k, \eta)}{d\eta} \right|^2 \right)$$

where $(\aleph = T, \times)$ or $(\aleph = +, -)$ for linear and circular polarisation basis respectively.

Evolution of Gravitational Waves

After normalising the gravitational energy density with background energy density at present

$$\left.\frac{d\Omega_{GW}}{d\ln k}\right|_0 = \left.\frac{d\Omega_{GW}}{d\ln k}\right|_\eta a^4(\eta) = \frac{k^3 a^2}{4(2\pi)^3 G\rho_{c_0}} \sum_{\aleph} \left(\left| \frac{dh^{\aleph}(k,\eta)}{d\eta} \right|^2 \right),$$

Energy momentum tensor of the EM field

- The energy momentum tensor of the EM field is given by,

$$T_{\mu\nu} = \frac{1}{4\pi}\left(g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{g_{\mu\nu}}{4}F^{\alpha\beta}F_{\alpha\beta}\right).$$

- Anisotropic stress tensor is given by the transverse traceless projection of the spatial part of the energy momentum tensor.

$$a^2\overline{T}_{ij}(\vec{k},\eta) = \frac{1}{4\pi}\int\frac{d^3q}{(2\pi)^3}P_{ij}^{mn}\Big(B_m(\vec{q},\eta)B_n^*(\vec{q}-\vec{k},\eta) + E_m(\vec{q},\eta)E_n^*(\vec{q}-\vec{k},\eta)\Big)$$

where
$E_i = \frac{1}{a}F_{i0} = -\frac{1}{a}A_i'$ and $B_i = \frac{1}{2a}\epsilon_{ijk}^*\delta^{jl}\delta^{km}F_{lm} = \frac{1}{a}\epsilon_{ijk}\delta^{jl}\delta^{km}\partial_l A_m$

GW power spectrum

- Before reheating, both electric and magnetic field contribute to the anisotropic stress and result in GW production with a dominant contribution from the electric field.
- After reheating, only anisotropic stress due to the magnetic field contributes since the electric field gets shorted out by the large conductivity of the plasma.
- To obtain $\left.\frac{d\Omega_{GW}}{d\ln k}\right|_0$, we need to calculate $\left|\frac{dh^{\aleph}(k,\eta)}{d\eta}\right|^2$ which further depends upon $\langle\overline{T}_{ij}(\vec{k},\eta)\overline{T}^{*ij}(\vec{k'},\eta')\rangle$.

GW power spectrum

- For non-helical EM fields,

$$\langle \overline{T}_{ij}(\vec{k},\eta)\overline{T}^{*ij}(\vec{k'},\eta')\rangle = \frac{1}{a^4(\eta)a^4(\eta')}\left(f_B(k,\eta,\eta') + f_E(k,\eta,\eta')\right)(2\pi)^3\delta(\vec{k}-\vec{k'}).$$

- For helical EM fields,

$$\langle \overline{T}_{ij}(\vec{k},\eta)\overline{T}^{*ij}(\vec{k'},\eta')\rangle = \frac{1}{a^4(\eta)a^4(\eta')}\left(g_B(k,\eta,\eta') + g_E(k,\eta,\eta')\right)(2\pi)^3\delta(\vec{k}-\vec{k'})$$

Where

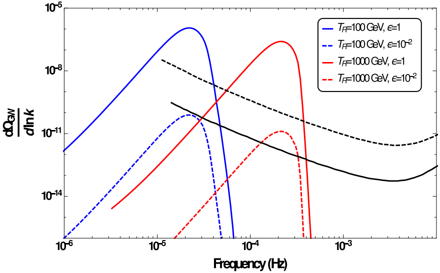
$$f_{B,E}(k,\eta,\eta') = \frac{1}{4(2\pi)^5}\int d^3q\left[P_{SB,SE}(q,\eta)P_{SB,SE}(|\vec{k}-\vec{q}|,\eta)(1+\gamma^2+\beta^2+\gamma^2\beta^2)\right] C_{B,E}(q,\eta,\eta')C_{B,E}(|\vec{k}-\vec{q}|,\eta,\eta')$$

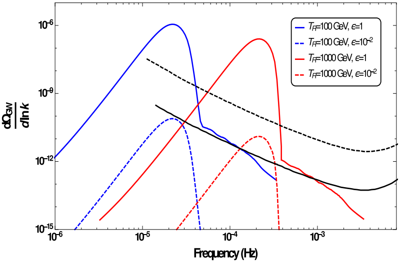
$$g_{B,E}(k,\eta,\eta') = \frac{1}{4(2\pi)^5}\int d^3q\Bigg[P_{SB,SE}(q,\eta)P_{SB,SE}(|\vec{k}-\vec{q}|,\eta)(1+\gamma^2+\beta^2+\gamma^2\beta^2) + 4\gamma\beta P_{AB,AE}(q,\eta)P_{AB,AE}(|\vec{k}-\vec{q}|,\eta)\Bigg] C_{B,E}(q,\eta,\eta')C_{B,E}(|\vec{k}-\vec{q}|,\eta,\eta')$$

In the above expression $\gamma = \hat{k}\cdot\hat{q}$ and $\beta = \hat{k}\cdot\widehat{k-q}$.

GW energy spectrum for nonhelical EM field

GW energy spectrum for $T_R = 100$ GeV and $T_R = 1000$ GeV and also for the different fraction ϵ along with the LISA sensitivity curve.



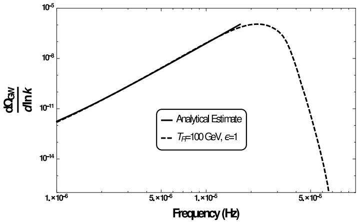


- The peak value of $d\Omega_{GW}/d\ln(k) \approx 1.2 \times 10^{-6}$ for $T_R = 100$ GeV and 2.5×10^{-7} for $T_R = 1000$ GeV assuming $\epsilon = 1$. For $\epsilon = 10^{-2}$, the peak value changes to 7.8×10^{-11} for $T_R = 100$ GeV and to 1.3×10^{-11} for $T_R = 1000$ GeV, respectively.
- Strong gap between the GW power for wavenumbers below k_{peak} from the GW power which arises due to the Kolmogorov branch, above k_{peak}. This feature of the GW spectrum is unique to our model of magnetogenesis compared to the GW spectrum in case of phase transition.

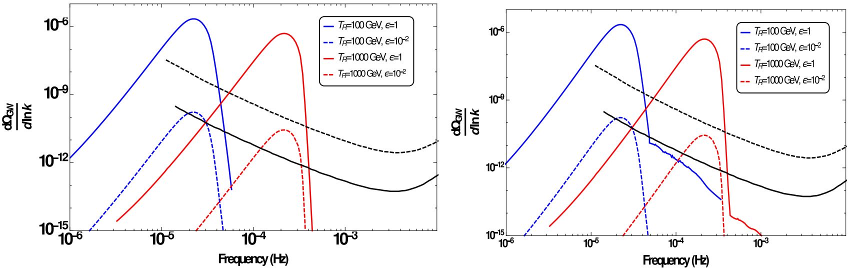
Analytical estimate

$$\left.\frac{d\Omega_{GW}}{d\ln k}\right|_0 = \frac{7\Omega_R}{5}\left(\frac{k}{k_0}\right)^3\left(\frac{D_2}{\tilde{\rho}+\tilde{p}}\right)^2\left(\frac{1}{(1-2\beta)^2(4\beta+1)^2}+\frac{4x_R^2}{(4\beta+1)^2}\right)$$

- For $T_R = 1000$ GeV, the peak value of $\left.\frac{d\Omega_{GW}}{d\ln k}\right|_0 \approx 1.3\times 10^{-7}\epsilon^2$.



GW energy spectrum for helical EM field



- The peak value of the generated GW spectrum in this case is of the same order as in non-helical case.

Summary

- We obtained the GW spectrum for both magnetogenesis models where the generated EM fields are non-helical or helical.
- The generated GW spectrum $d\Omega_{GW}/d\ln(k) \propto k^3$, till $k \leq k_{peak}$ determined by the Hubble radius at reheating.
- Non-linear evolution of the magnetic field after reheating develops a tail of the stochastic GW spectrum, for the modes with $k > k_{peak}$.
- The generated GW background lies within the sensitivity of LISA for $T_R \geq 100$ GeV.
- A possible detection of GW spectrum of the nature calculated here by LISA will provide important probe of the scenarios of magnetogenesis discussed in Part 1.
- For reheating scale around $T_R = 150$ MeV, PTA may provide important constraints to our models.

Thank you