Inflationary Magnetogenesis and the Gravitational Wave Signal

Sayan Mandal

Department of Physics, Carnegie Mellon University

13^{th} September, 2019

Collaborators

- Axel Brandenburg
- Ruth Durrer
- **Tina Kahniashvili**
- Shinji Mukohyama
- Alberto Roper Pol
- Alexander Tevzadze
- Tanmay Vachaspati

Based On

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Statistical properties of scale-invariant helical magnetic fields and applications to cosmology

Axel Brandenburg,[a,b,c,d,e] Ruth Durrer,[f] Tina Kahniashvili,[c,g,h]
Sayan Mandal[c,1] and Weichen Winston Yin[i,c]

[arXiv:**1804.01177**]

Magnetohydrodynamic evolution of inflationary homogeneous magnetic field during the radiation dominated epoch

Axel Brandenburg,[1,2,3,4,5] Ruth Durrer,[6] Yiwen Huang,[2,7]
Tina Kahniashvili,[2,8,9] Sayan Mandal *,[2,†] and Shinji Mukohyama[10,11]

[In preparation]

Introduction

- Magnetic fields ($\sim \mu$G) are detected at different scales in the universe.[1]
- Small seed (primordial) fields can be amplified by various mechanisms.
- What is the origin of these PMFs?
- Generation mechanism affects the statistical properties.

[1] Lawrence M. Widrow. "Origin of galactic and extragalactic magnetic fields". In: *Rev. Mod. Phys.* 74 (3 2002), pp. 775–823.

Generation Mechanisms

Inflationary Magnetogenesis

- PMFs arise from vacuum fluctuations[a] - very large correlation lengths.
- Involves the breaking of conformal symmetry.
- Scale invariant (or nearly) power spectrum.
- Large correlation lengths.
- Typically involves couplings like $R^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ or $f(\phi)F_{\mu\nu}F^{\mu\nu}$.

[a] Michael S. Turner and Lawrence M. Widrow. "Inflation-produced, large-scale magnetic fields". In: *Phys. Rev. D* 37 (10 1988), pp. 2743–2754; B. Ratra. "Cosmological 'seed' magnetic field from inflation". In: *Astrophysical Journal Letters* 391 (May 1992), pp. L1–L4.

Phase Transition Magnetogenesis

Several talks in this program!

There is Helicity!

- Generation mechanisms can involve significant parity (P) violation.
- This can lead to *helical* PMFs – the evolution is affected[2].
- *Fractional helicity* can grow with evolution[3].
- Can help us understand phenomena like *Baryogenesis*.

[2] Robi Banerjee and Karsten Jedamzik. "The Evolution of cosmic magnetic fields: From the very early universe, to recombination, to the present". In: *Phys. Rev.* D70 (2004), p. 123003.

[3] Alexander G. Tevzadze et al. "Magnetic Fields from QCD Phase Transitions". In: *Astrophys. J.* 759 (2012), p. 54.

Magnetic Helicity

This is given by

$$H = \frac{1}{V}\int_V \mathbf{A}\cdot\mathbf{B}\,d^3\mathbf{r} = \frac{1}{\tilde{V}}\int_{\tilde{V}} \tilde{\mathbf{A}}\cdot\tilde{\mathbf{B}}\,d^3\mathbf{x} \tag{1}$$

Tilde – comoving quantities. V denotes a closed volume with fully contained fields lines.

H is invariant under $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\Lambda$.

The evolution of H is,

$$\frac{dH}{d\tau} = -2\tilde{\eta}\int_{\tilde{V}} \tilde{\mathbf{B}}\cdot(\boldsymbol{\nabla}\times\tilde{\mathbf{B}})\,d^3\mathbf{x} \tag{2}$$

Modeling Magnetic Fields

We assume the PMFs to be **stochastic**, and statistically **isotropic**, **homogeneous**, and **gaussian**.

We work with the correlation function

$$\mathcal{B}_{ij}(r) \equiv \langle B_i(\mathbf{x}) B_j(\mathbf{x}+\mathbf{r})\rangle = M_\mathrm{N}(r)\delta_{ij} + \left[M_\mathrm{L}(r) - M_\mathrm{N}(r)\right]\hat{r}_i\hat{r}_j + M_\mathrm{H}(r)\epsilon_{ijl}r_l \tag{3}$$

In Fourier space,

$$\mathcal{F}_{ij}^{(B)}(\mathbf{k}) = \int d^3\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \mathcal{B}_{ij}(r)$$

This gives the *symmetric* and *helical* parts,

$$\frac{\mathcal{F}_{ij}^{(B)}(\mathbf{k})}{(2\pi)^3} = P_{ij}(\hat{\mathbf{k}})\frac{E_M(k)}{4\pi k^2} + i\epsilon_{ijl}k_l\frac{H_M(k)}{8\pi k^2} \tag{4}$$

Here $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j$.

Modeling Magnetic Fields

Mean magnetic energy density: $\mathcal{E}_{\mathrm{M}} = \int dk\, E_{\mathrm{M}}(k)$.

Magnetic integral scale: $\xi_{\mathrm{M}}(t) = \dfrac{\int_0^\infty dk\, k^{-1} E_{\mathrm{M}}(k)}{\mathcal{E}_{\mathrm{M}}}$.

Magnetic Helicity: $\mathcal{H}_{\mathrm{M}} = \dfrac{1}{V}\int_V \mathbf{A}\cdot\mathbf{B}\, d^3\mathbf{r} = \int dk\, H_{\mathrm{M}}(k)$.

Current Helicity: $\mathcal{H}_{\mathrm{C}} = \dfrac{1}{V}\int_V (\nabla\times\mathbf{B})\cdot\mathbf{B}\, d^3\mathbf{r} = \int dk\, k^2\, H_{\mathrm{M}}(k)$.

The Realizability Condition

V.I. Arnold (1986), *The asymptotic Hopf invariant and its applications*

Theorem. *The eigenfield of* curl^{-1} *corresponding to the eigenvalue* v *of largest modulus has minimum energy in the class of divergence-free fields obtained from the eigenfield under the action of volume-preserving diffeomorphisms.*

Proof. Let v_- and v_+ be the smallest and largest eigenvalues of the operator curl^{-1}. Then for every field ξ that is homologous to zero we have

$$v_-\langle \xi, \xi\rangle \leq \langle \operatorname{curl}^{-1} \xi, \xi\rangle \leq v_+\langle \xi, \xi\rangle, \qquad v_- < 0 < v_+ .$$

Consequently, we have the following bound for the energy in terms of the Hopf invariant:

$$\langle \xi, \xi\rangle \geq \langle \operatorname{curl}^{-1} \xi, \xi\rangle / v,$$

where v denotes the value v_+ or v_- of larger modulus.

The Realizability Condition

In the early universe, $\mathcal{H}_M$ is conserved. This leads to (for divergenceless $\mathbf{B}$),

$$L_-|\mathbf{B}(\mathbf{x})|^2 \leq \left(\mathrm{curl}^{-1}\mathbf{B}\right) \cdot \mathbf{B} \leq L_+|\mathbf{B}(\mathbf{x})|^2$$

where $L_- < 0 < L_+$ are the eigenvalues of curl^{-1}. This implies,

$$\left|\frac{\left(\mathrm{curl}^{-1}\mathbf{B}\right) \cdot \mathbf{B}}{L_+}\right| \leq |\mathbf{B}(\mathbf{x})|^2$$

Taking the ensemble average, we get $\left|\frac{\mathcal{H}_M}{L_+}\right| \leq 2\mathcal{E}_M$ Reasonable to expect: $L_+ \sim \xi_M$.

We can relate the symmetric and helical components,

$$|\mathcal{H}_M| \leq 2\xi_M \mathcal{E}_M \qquad \Rightarrow \qquad |H_M(k)| \leq 2k^{-1}E_M(k) \tag{5}$$

However...

For scale-invariant spectrum, at large length scales, $E_M \sim k^{-1}$; so, ξ_M is unbounded.

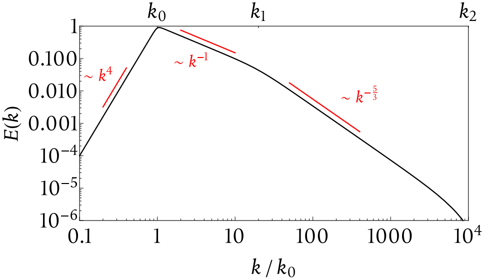
Then Helicity is divergent!?!?

However...

For scale-invariant spectrum, at large length scales, $E_M \sim k^{-1}$; so, ξ_M is unbounded.

Then Helicity is divergent!?!?

One must have $E_M \sim k^4$ at superhorizon scales.



Scale Invariant Spectrum.

Numerical Simulations

The PENCIL CODE is used to study the evolution of $\mathcal{E}_M(t)$.

Resolution of 2304^3 meshpoints (early) and 1152^3 meshpoints (late).

At early times, the small-k spectrum does not change.

At late times, we have the usual inverse cascade.

The Pencil Code Equations

We solve the hydromagnetic equations for an isothermal relativistic gas with pressure $p = \rho/3$

$$\frac{\partial \ln \rho}{\partial t} = -\frac{4}{3}\left(\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \ln \rho\right) + \frac{1}{\rho}\left[\mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) + \eta \mathbf{J}^2\right], \tag{6}$$

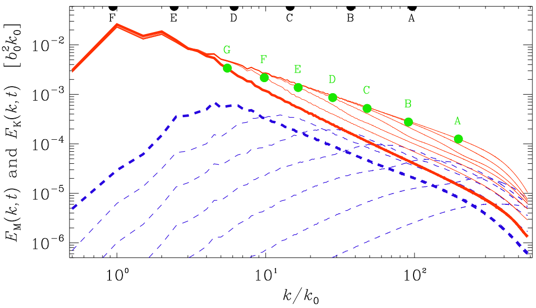
$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= -\mathbf{u} \cdot \nabla \mathbf{u} + \frac{\mathbf{u}}{3}\left(\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \ln \rho\right) - \frac{\mathbf{u}}{\rho}\left[\mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) + \eta \mathbf{J}^2\right] \\ &\quad -\frac{1}{4}\nabla \ln \rho + \frac{3}{4\rho}\mathbf{J} \times \mathbf{B} + \frac{2}{\rho}\nabla \cdot \left(\rho \nu \mathsf{S}\right), \end{aligned} \tag{7}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \mathbf{J}), \tag{8}$$

where $\mathsf{S}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3}\delta_{ij}\nabla \cdot \mathbf{u}$ is the rate-of-strain tensor, ν is the viscosity, and η is the magnetic diffusivity.

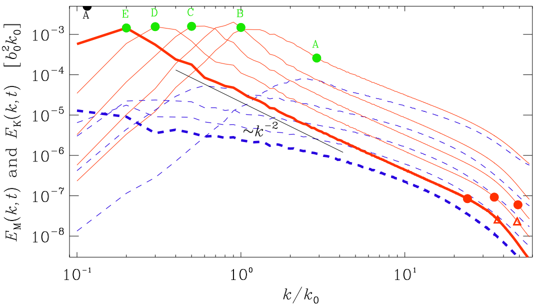
Early Times

$$k_*(t) \approx \xi_M(t)(\eta_{\mathrm{turb}} t)^{-1/2}, \qquad k_{\mathrm{hor}}(t) = (ct)^{-1} \tag{9}$$



Magnetic (red) and kinetic (blue) energy spectra at early times. The green symbols denote the position of $k_*(t)$. Black symbols denote the location of the horizon wavenumber $k_{\mathrm{hor}}(t)$.

Late Times



The late time evolution. We have the usual inverse cascade, with an increase of $\xi_M(t) \sim t^q$.

Consider an axisymmetric Bianchi Type-I universe:

$$ds^2 = -N^2\, dt^2 + a^2\left(e^{4\sigma}\, dx^2 + e^{-2\sigma}\, dy^2 + e^{-2\sigma}\, dz^2\right)$$

With the Lagrangian:

$$\mathcal{L} = \frac{R}{2\kappa} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \xi\tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma}R_{\mu\nu\rho\sigma} + \mathcal{L}_\varphi$$

Here, as usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, with:

$$A_t = 0, \qquad A_x = \int^t \frac{N(t')\, e^{4\sigma(t')}}{a(t')}E(t')\, dt', \qquad A_y = \frac{1}{2}Bz, \qquad A_z = -\frac{1}{2}By$$

The k-essence term $\mathcal{L}_\varphi = p(\varphi, X)$, where $X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi$.

Helicity Conservation

For an *imposed* field, helicity is not conserved[4]. The magnetic field can be split,

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$$

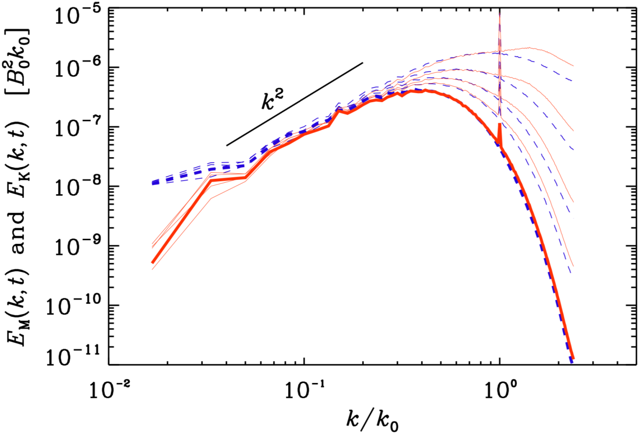
The evolution of helicity is,

$$\frac{d}{dt} \langle \mathbf{a} \cdot \mathbf{b} \rangle = -2\alpha \mathbf{B}_0^2 - 2\eta \langle \mathbf{j} \cdot \mathbf{b} \rangle$$

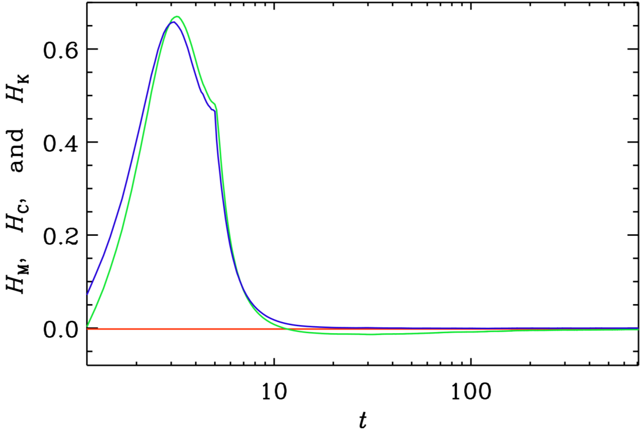
A "generalized" helicity is conserved – but that is not gauge invariant!!

[4] Axel Brandenburg and William H. Matthaeus. "Magnetic helicity evolution in a periodic domain with imposed field". In: *Phys. Rev.* E69 (2004), p. 056407.

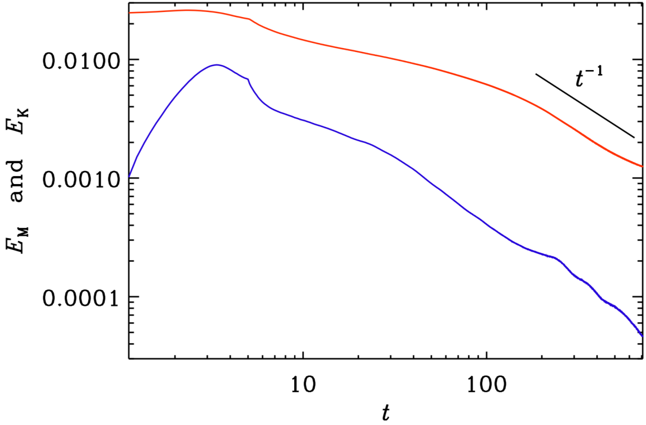
The Energy Spectra



The Evolution of Helicity



The Evolution of Energy



Applications to Cosmology

The Spectra

Decompose the temperature fluctuations:

$$\frac{\delta T}{\bar{T}}(\hat{\mathbf{n}}) = \sum_{l,m} a_{lm}^{T} Y_{lm}(\hat{\mathbf{n}})$$

Similarly for the Stokes parameters:

$$(Q \pm iU)(\hat{\mathbf{n}}) = \sum_{l,m} a_{lm}^{(\pm 2)} Y_{lm}^{(\pm 2)}(\hat{\mathbf{n}})$$

Then,

$$a_{lm}^{E} = -\frac{1}{2}\left(a_{lm}^{(2)} + a_{lm}^{(-2)}\right), \qquad a_{lm}^{B} = -\frac{1}{2i}\left(a_{lm}^{(2)} - a_{lm}^{(-2)}\right)$$

The power spectra are:

$$C_l^{XY} = \frac{1}{2l+1} \sum_m \left\langle \left(a_{lm}^{X}\right)^* \left(a_{lm}^{Y}\right) \right\rangle$$

where $X, Y = E, B, T$.

Signatures on CMB

- The Planck Collaboration[5] derived upper limits on PMFs.
- Parity-even CMB spectra depend on both E_{M} and H_{M}.
- Parity-odd spectra depend only on H_{M}.
- H_{M} also turns on the *bispectrum*.
- Parity-odd fluctuations are sourced if[6] $\mathcal{H}_{\mathrm{M}} \sim 10^{-14}\,\mathrm{G}^2\,\mathrm{MPc}$.
- This cannot come from PT – $\mathcal{H}_{\mathrm{M}} \lesssim 10^{-20}\,\mathrm{G}^2\,\mathrm{MPc}$.

[5] P. A. R. Ade et al. "Planck 2015 results. XIX. Constraints on primordial magnetic fields". In: *Astron. Astrophys.* 594 (2016), A19.

[6] Tina Kahniashvili et al. "Primordial Magnetic Helicity Constraints from WMAP Nine-Year Data". In: *Phys. Rev.* D90.8 (2014), p. 083004.

Signatures on CMB

- The Planck Collaboration[5] derived upper limits on PMFs.
- Parity-even CMB spectra depend on both E_{M} and H_{M}.
- Parity-odd spectra depend only on H_{M}.
- H_{M} also turns on the *bispectrum*.
- Parity-odd fluctuations are sourced if[6] $\mathcal{H}_{\mathrm{M}} \sim 10^{-14}\,\mathrm{G}^2\,\mathrm{MPc}$.
- This cannot come from PT – $\mathcal{H}_{\mathrm{M}} \lesssim 10^{-20}\,\mathrm{G}^2\,\mathrm{MPc}$.

If helicity leaves traces on CMB – PMFs from inflation!!

[5] P. A. R. Ade et al. "Planck 2015 results. XIX. Constraints on primordial magnetic fields". In: *Astron. Astrophys.* 594 (2016), A19.

[6] Tina Kahniashvili et al. "Primordial Magnetic Helicity Constraints from WMAP Nine-Year Data". In: *Phys. Rev.* D90.8 (2014), p. 083004.

Duration of Inflation (JCAP 1712 (2017) no.12, 002)

Largest length scale in the system:

$$k_{\mathrm{min}}^{-1} = \xi_{\mathrm{M,max}} \simeq \left(\frac{|\mathcal{H}_{\mathrm{M}}|}{2\mathcal{E}_{\mathrm{M}}} \right)_{\mathrm{end}}$$

Natural $\xi_{\mathrm{M,max}}$ – Horizon scale at the beginning.

This scale exits first – then H_0 exits. This gives number of e-folds *before* current horizon exits:

$$N_H = \log \left(\frac{H_0}{k_{\mathrm{min}}} \right)$$

We know number of e-folds *after* current horizon exits:

$$N_e \sim 55 - 60$$

So,

$$\text{Total } e\text{-folds during inflation} = N_H + N_e$$

Supplementary Slides

Some Definitions

Magnetic dissipation wavenumber is,

$$k_{\mathrm{MD}}^4 = \frac{2}{\eta^2} \int_0^\infty dk\, k^2\, \tilde{E}_M(k)$$

Weak turbulence dissipation wavenumber is,

$$\mathrm{Lu}(k_{\mathrm{WT}}) = \frac{v_{\mathrm{A}}(k_{\mathrm{WT}})}{\eta k_{\mathrm{WT}}} = 1$$

with $v_{\mathrm{A}}^2 = 2kE_M(k)$.

Loitsiansky Integral: $\mathcal{L} = \int r^2 \langle \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}+\mathbf{r}) \rangle\, d\mathbf{r} \propto \ell^5 u_\ell^2$

Saffman Integral: $\mathcal{S} = \int \langle \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}+\mathbf{r}) \rangle\, d\mathbf{r} \propto \ell^3 u_\ell^2$

$$\mathrm{Re} = \frac{u_{rms}\xi_M}{\nu}$$

Time Dependence of Homogeneous B

We consider a zero electric field,

$$E = 0, \qquad B = bH_0$$

The above definition **does not imply** a constant value of the magnetic field.

A freely-falling observer has a velocity $u^\mu = (1, 0, 0, 0)$; such an observer sees the following magnetic field,

$$B_\mu = \left(0, \frac{Be^{4\sigma}}{a}, 0, 0\right)$$

This implies

$$B^\mu B_\mu = \frac{B^2 e^{4\sigma}}{a^4}$$

Decay Laws

We take the maximum *comoving* correlation length at the epoch of EW Phase transition,

$$\xi_\star \equiv \xi_{\max} = H_\star^{-1}\left(\frac{a_0}{a_\star}\right) \sim 6 \times 10^{-11}\,\mathrm{Mpc}$$

and the maximum *mean* energy density as,

$$\mathcal{E}_\star = 0.1 \times \frac{\pi^2}{30} g_\star T_\star^4 \sim 4 \times 10^{58}\,\mathrm{eV\,cm^{-3}}$$

We use $\eta(a) = \frac{2}{\sqrt{\Omega_{m,0}}H_0}\left[\sqrt{a_{\mathrm{eq}}+a} - \sqrt{a_{\mathrm{eq}}}\right]$.

Non-helical case: $\dfrac{\xi}{\xi_\star} = \left(\dfrac{\eta}{\eta_\star}\right)^{\frac{1}{2}}, \qquad \dfrac{\mathcal{E}}{\mathcal{E}_\star} = \left(\dfrac{\eta}{\eta_\star}\right)^{-1}.$

Helical case: $\dfrac{\xi}{\xi_\star} = \left(\dfrac{\eta}{\eta_\star}\right)^{\frac{2}{3}}, \qquad \dfrac{\mathcal{E}}{\mathcal{E}_\star} = \left(\dfrac{\eta}{\eta_\star}\right)^{\frac{-2}{3}}.$

Partial: Turnover when $\left(\dfrac{\eta_{\frac{1}{2}}}{\eta_\star}\right) = \exp\left(\dfrac{1}{2\sigma}\right).$