Chiral Effects as the Source of MHD Turbulence and Large-Scale Magnetic Field

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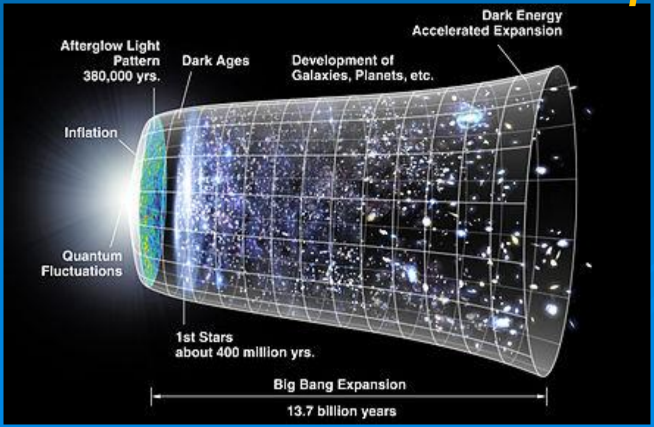

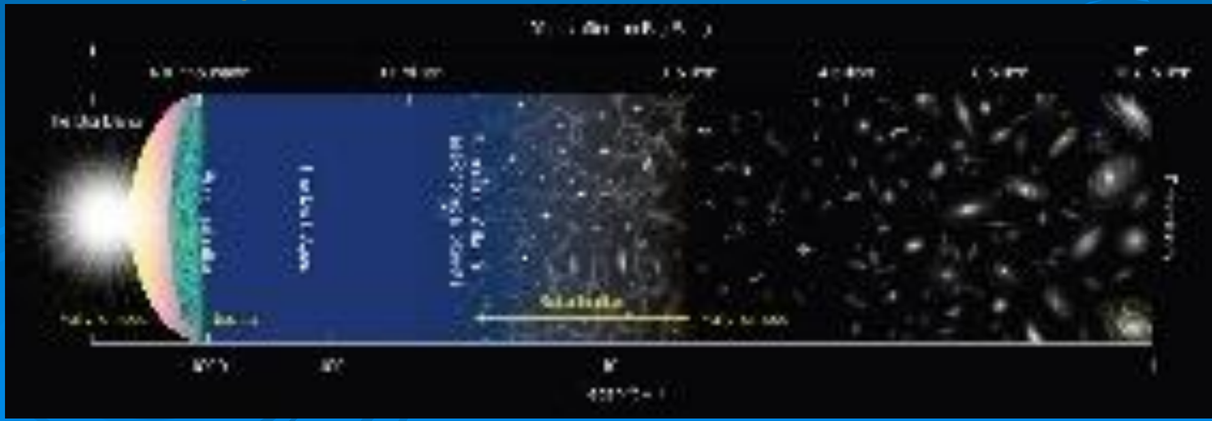

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- **Rogachevskii, I., Ruchayskiy, O., Boyarsky, A., Fröhlich, J., Kleeorin, N., Brandenburg, A., & Schober, J., 2017, Astrophys. J., v. 846, 153; THEORY**
- **Brandenburg, A., Schober, J., Rogachevskii, I., Kahniashvili, T., Boyarsky, A., Fröhlich, J., Ruchayskiy, O., & Kleeorin, N., 2017, Astrophys. J. Letters, v. 845, L21; DNS + scalings**
- **Schober, J., Rogachevskii, I., Brandenburg, A., Boyarsky, A., Fröhlich, J., Ruchayskiy, O., & Kleeorin, N., 2018, Astrophys. J., v. 858, 124; DNS + astrophysical applications**
- **Schober, J., Brandenburg, A., Rogachevskii, I., & Kleeorin, N. 2019, Geophys. Astrophys. Fluid Dynamics, v. 113, 107-130; DNS + energetic analysis of chiral turbulence**
- **Schober, J., Brandenburg, A., Rogachevskii, I., 2019, Geophys. Astrophys. Fluid Dynamics, in press; arXiv: 1808.06624; DNS + chiral magnetic waves+chiral turbulence**

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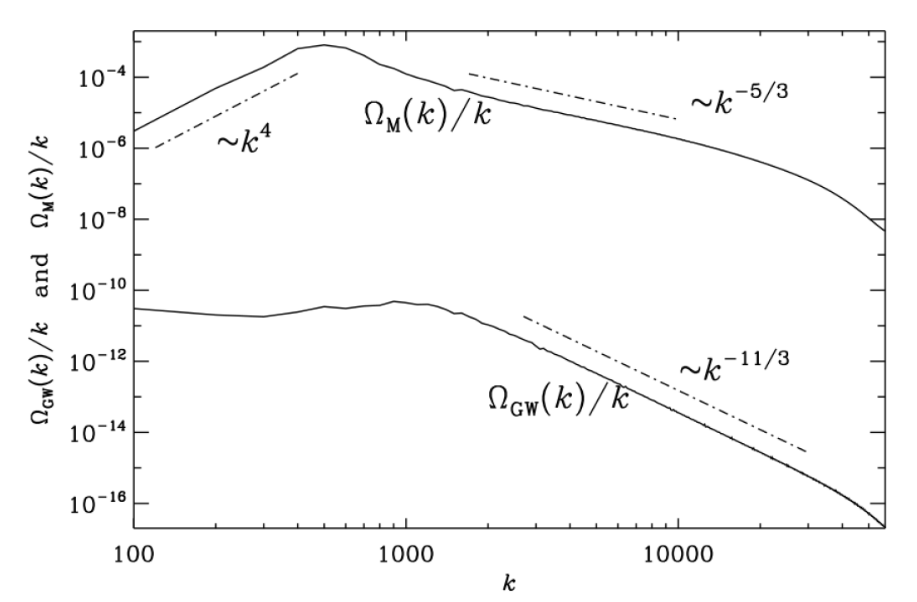
GWs equation

$$\left(\partial_t^2 - c^2\nabla^2\right) h_{ij} = \frac{16\pi G}{ac^2} T_{ij}^{\rm TT}$$

- From fluid motions
  $T_{ij} = \left(p/c^2 + \rho\right)\gamma^2 u_i u_j + p\delta_{ij}$
  Relativistic equation of state:
  $p = \rho c^2/3$
- From magnetic fields:
  $T_{ij} = -B_i B_j + \delta_{ij} B^2/2$

Numerical results for decaying MHD turbulence



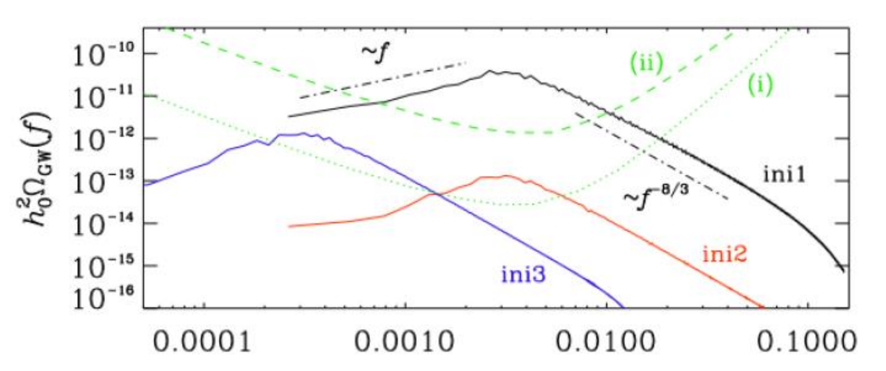
- h_{ij} are rescaled $h_{ij} = a h_{ij}^{\rm phys}$
- Comoving spatial coordinates ∇ and conformal time t
- Comoving stress-energy tensor components T_{ij}
- Radiation-dominated epoch such that $a'' = 0$

GWs energy density:

$$\Omega_{\rm GW}(t) = \mathcal{E}_{\rm GW}/\mathcal{E}_{\rm rad}^*, \quad \mathcal{E}_{\rm rad}^* = \frac{3H_*^2 c^2}{8\pi G}$$

$$\Omega_{\rm GW}(t) = \int_{-\infty}^{\infty} \Omega_{\rm GW}(k,t)\,{\rm d}\ln k$$

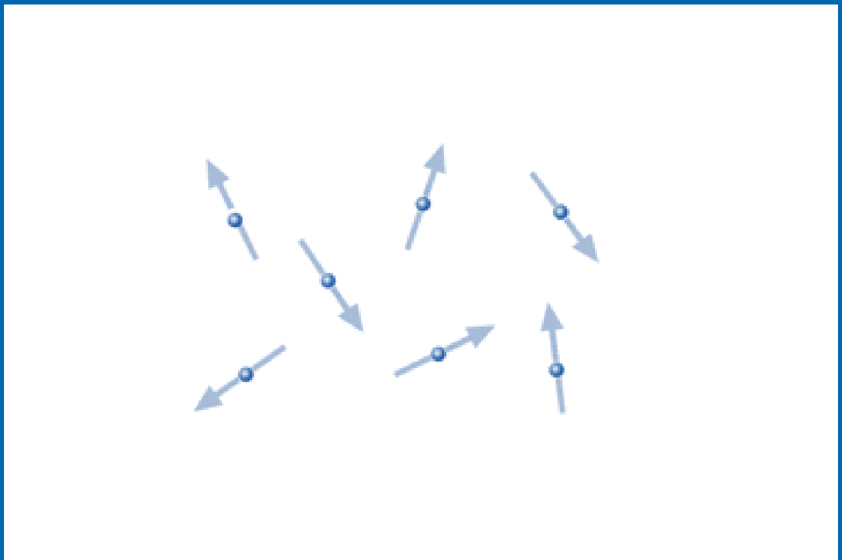
$$\Omega_{\rm GW}(k,t) = \frac{k}{6H_*^2}\int_{4\pi}\left(\left|\dot{\tilde{h}}_+^{\rm phys}\right|^2 + \left|\dot{\tilde{h}}_\times^{\rm phys}\right|^2\right) k^2\,{\rm d}\Omega_k$$



$E_{\rm GW} \propto E_{\rm M}/k^2$

Normalized GW equation $\left(\partial_t^2 - \nabla^2\right) h_{ij} = 6\,T_{ij}^{\rm TT}/t$

Chiral Magnetic Effect

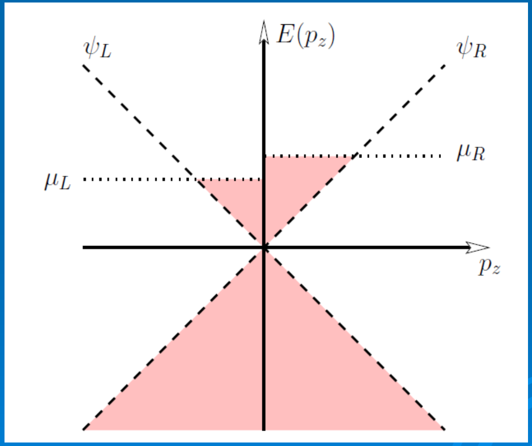


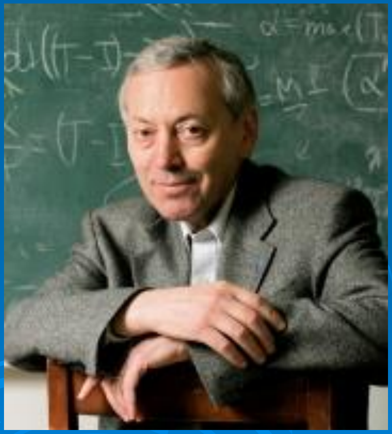
1. We consider fermions in magnetic field.
2. The particles with the momentum ***p*** along magnetic field have positive projection of spin onto momentum
(the right-chiral particles).
3. The particles with the momentum ***p*** opposite to magnetic field have negative projection of the spin onto momentum
(the left-chiral particles).

Vilenkin, A. 1980, Phys. Rev., D22, 3080

$$Q_5 = \text{Number of left particles} - \text{Number of right particles}$$

$$\frac{dQ_5}{dt} = \frac{2\alpha_{\rm em}}{\pi\hbar} \int d^3\boldsymbol{x}\, \boldsymbol{E}\cdot\boldsymbol{B} = -\frac{\alpha_{\rm em}}{\pi\,\hbar c}\frac{d}{dt}\int d^3\boldsymbol{x}\, \boldsymbol{A}\cdot\boldsymbol{B},$$





$$f_L(p) = \frac{1}{\exp\left(\frac{c|p_z| - \mu_L}{k_B T}\right) + 1}, \quad p_z < 0\,,$$

$$f_R(p) = \frac{1}{\exp\left(\frac{c|p_z| - \mu_R}{k_B T}\right) + 1}, \quad p_z > 0\,,$$

$$\alpha_{\rm em} \equiv \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

$$J_\mu = \frac{eB}{(2\pi\hbar)^2}\Big[\int_0^\infty \frac{dp_z}{2\pi\hbar}\,\psi^\dagger_{p_z}\gamma_z\psi_{p_z} f_R(p_z) + \int_{-\infty}^0 \frac{dp_z}{2\pi\hbar}\,\psi^\dagger_{p_z}\gamma_z\psi_{p_z} f_L(p_z)\Big]\,,$$

$$\mu_5^{\rm phys} \equiv \mu_{\rm L}^{\rm phys} - \mu_{\rm R}^{\rm phys}\,,$$

$$J_{\rm CME} = \frac{\alpha_{\rm em}}{\pi\hbar}\mu_5^{\rm phys} B,$$

Chiral Magnetic Effect

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Applications:

1. Astrophysics:
   the early Universe,
   neutron stars;
2. Heavy ion collisions;
3. *Condensed Matter Physics*
   new materials (Weyl semimetals);

Equations in Chiral MHD: Small Magnetic Diffusion

$$\nabla \times B = \frac{4\pi}{c} J_{\rm tot} + \frac{1}{c}\frac{\partial E}{\partial t},$$
$$\nabla \cdot E = 4\pi \varrho_{\rm tot},$$
$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t},$$
$$\nabla \cdot B = 0,$$

$$\varrho_{\rm tot} = \varrho_{\rm el} - B \cdot \nabla\Theta.$$

$$J_{\rm tot} \equiv \varrho_{\rm el} U + J_{\rm Ohm} + J_{\Theta}.$$

Quasi-neutrality:

$$\varrho_{\rm tot} = 0.$$

Fröhlich & Pedrini (2000, 2002).

Review:

Kharzeev, D. E. 2014, Prog. Part. Nucl. Phys., 75, 133

$$J_{\rm CME} \to J_\Theta = B\frac{\partial\Theta}{\partial t} + c\,\nabla\Theta \times E.$$

The total electric current:

$$J_{\rm tot} = (B \cdot \nabla\Theta)U + \sigma\left(E + \frac{1}{c}U \times B\right) + (\dot{\Theta}B + c\,\nabla\Theta \times E).$$

$$J_{\rm Ohm} = \sigma\left(E + \frac{1}{c}U \times B\right)$$

$$U(B \cdot \nabla\Theta) = B(U \cdot \nabla\Theta) + \nabla\Theta \times (U \times B)$$

The resulting total electric current:

$$J_{\rm tot} = \sigma\left(E + \frac{1}{c}U \times B\right) + \frac{D\Theta}{Dt}B + c\,\nabla\Theta \times \left(E + \frac{1}{c}U \times B\right),$$

Electric and Magnetic Fields for Small Magnetic Diffusion in Chiral MHD

$$\frac{\partial B}{\partial t} = \nabla \times \left[U \times B - \eta\,(\nabla \times B - \mu_5 B)\right],$$

$$J_{\rm CME} = \frac{\alpha_{\rm em}}{\pi\hbar}\mu_5^{\rm phys} B,$$

$$E = -U \times B + \eta\nabla \times B - \eta\,\mu_5\,B + {\rm O}(\eta^2)$$

$$\mu_5 \equiv (4\alpha_{\rm em}/\hbar c)\mu_5^{\rm phys}$$

$$\alpha_{\rm em} \equiv \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

$$\mu_5^{\rm phys} \equiv \mu_{\rm L}^{\rm phys} - \mu_{\rm R}^{\rm phys},$$

Chiral Magnetic Effect

$$J_{\mathrm{CME}} = \frac{\alpha_{\mathrm{em}}}{\pi\hbar}\mu_5^{\mathrm{phys}}\boldsymbol{B},$$

$$\mu_5^{\mathrm{phys}} \equiv \mu_{\mathrm{L}}^{\mathrm{phys}} - \mu_{\mathrm{R}}^{\mathrm{phys}},$$

$Q_5 =$ Number of left particles – Number of right particles

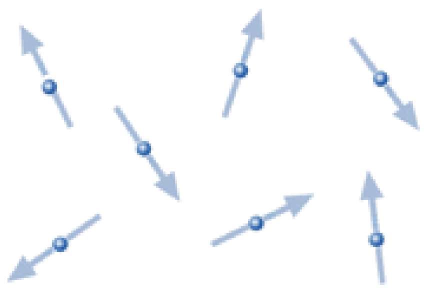
$$\frac{dQ_5}{dt} = \frac{2\alpha_{\mathrm{em}}}{\pi\hbar}\int d^3x\,\boldsymbol{E}\cdot\boldsymbol{B} = -\frac{\alpha_{\mathrm{em}}}{\pi\,\hbar c}\frac{d}{dt}\int d^3x\,\boldsymbol{A}\cdot\boldsymbol{B},$$

$$\frac{d}{dt}\left(Q_5 + \frac{\alpha_{\mathrm{em}}}{\pi\hbar c}\chi_m\right) = 0$$

$$\alpha_{\mathrm{em}} \equiv \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

$$\chi_{\mathrm{m}} \equiv \int d^3x\,\boldsymbol{A}\cdot\boldsymbol{B}$$

$$\boldsymbol{E} = -\boldsymbol{U}\times\boldsymbol{B} + \eta\boldsymbol{\nabla}\times\boldsymbol{B} - \eta\,\mu_5\,\boldsymbol{B} + \mathrm{O}(\eta^2)$$



$$\frac{D\mu_5}{Dt} = D_5\,\Delta\mu_5 + \lambda\,\eta\,\left[\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) - \mu_5\boldsymbol{B}^2\right]$$

$$\eta \equiv \frac{c^2}{4\pi\sigma}$$

$$\lambda = 3\hbar c\left(\frac{8\alpha_{\mathrm{em}}}{k_B T}\right)^2$$

Equations in Chiral MHD

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \left[\boldsymbol{U} \times \boldsymbol{B} - \eta \left(\boldsymbol{\nabla} \times \boldsymbol{B} - \mu_5 \boldsymbol{B}\right)\right],$$

$$\rho \frac{D\boldsymbol{U}}{Dt} = (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} - \boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot (2\nu\rho \mathsf{S}) + \rho \boldsymbol{f},$$

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{U},$$

$$\frac{D\mu_5}{Dt} = D_5 \, \Delta\mu_5 + \lambda \eta \left[\boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) - \mu_5 \boldsymbol{B}^2\right] - C_5 (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \mu - \Gamma_{\mathrm{f}} \mu_5,$$

$$\frac{D\mu}{Dt} = D_\mu \, \Delta\mu - C_\mu (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \mu_5,$$

$$\mu_5 \equiv (4\alpha_{\mathrm{em}}/\hbar c)\mu_5^{\mathrm{phys}} \qquad \mu_5^{\mathrm{phys}} \equiv \mu_{\mathrm{L}}^{\mathrm{phys}} - \mu_{\mathrm{R}}^{\mathrm{phys}}, \qquad \lambda = 3\hbar c \left(\frac{8\alpha_{\mathrm{em}}}{k_B T}\right)^2$$

$$D/Dt \equiv \partial/\partial t + \boldsymbol{U} \cdot \boldsymbol{\nabla}, \qquad \mu \equiv (4\alpha_{\mathrm{em}}/\hbar c)\mu^{\mathrm{phys}} \qquad \mu^{\mathrm{phys}} \equiv \mu_{\mathrm{R}}^{\mathrm{phys}} + \mu_{\mathrm{L}}^{\mathrm{phys}}$$

Conservation Law: Magnetic Helicity + Chiral Chemical Potential

$$\frac{\partial}{\partial t}\left(\frac{\lambda}{2} \boldsymbol{A} \cdot \boldsymbol{B} + \mu_5\right) + \boldsymbol{\nabla} \cdot \left[\frac{\lambda}{2} \left(\boldsymbol{E} \times \boldsymbol{A} - \boldsymbol{B}\,\Phi\right) - D_5 \boldsymbol{\nabla} \mu_5 + C_5 \boldsymbol{B} \mu\right] = 0,$$

Conservation Law:

Magnetic Helicity + Chiral Chemical Potential

$$\frac{\partial}{\partial t}\left(\frac{\lambda}{2}\boldsymbol{A}\cdot\boldsymbol{B}+\mu_5\right)+\boldsymbol{\nabla}\cdot\left[\frac{\lambda}{2}\left(\boldsymbol{E}\times\boldsymbol{A}-\boldsymbol{B}\,\Phi\right)-D_5\boldsymbol{\nabla}\mu_5+C_5\boldsymbol{B}\mu\right]=0,$$

Magnetic Helicity:

$$\frac{\partial\boldsymbol{A}\cdot\boldsymbol{B}}{\partial t}+\boldsymbol{\nabla}\cdot\left(\boldsymbol{E}\times\boldsymbol{A}+\boldsymbol{B}\,\Phi\right)=-2\boldsymbol{E}\cdot\boldsymbol{B}$$

$$\boldsymbol{E}\cdot\boldsymbol{B}\propto\eta$$

$$\eta\to 0$$

Chiral Chemical Potential:

$$\frac{\partial(2\mu_5/\lambda)}{\partial t}+\boldsymbol{\nabla}\cdot\left[-\left(2D_5/\lambda\right)\boldsymbol{\nabla}\mu_5\right]=2\boldsymbol{E}\cdot\boldsymbol{B},$$

$$\lambda=3\hbar c\left(\frac{8\alpha_{\mathrm{em}}}{k_BT}\right)^2$$

$$\boldsymbol{E}=-\boldsymbol{U}\times\boldsymbol{B}+\eta\boldsymbol{\nabla}\times\boldsymbol{B}-\eta\,\mu_5\,\boldsymbol{B}+\mathrm{O}(\eta^2)$$

Laminar V_μ^2-Dynamo in Chiral MHD

Joyce and Shaposhnikov, PRL 79, 1193 (1997)

$$\mu_{\rm eq} \equiv \mu_0 = {\rm const\ and}\ \boldsymbol{U}_{\rm eq} = 0.$$

$$\boldsymbol{B}(t,x,z) = B_y(t,x,z)\boldsymbol{e}_y + \boldsymbol{\nabla} \times [A(t,x,z)\boldsymbol{e}_y]$$

$$\gamma = |v_\mu\, k| - \eta k^2,$$

Chiral velocity

$$v_\mu \equiv \eta\mu_{5,0}$$

$$\frac{\partial A(t,x,z)}{\partial t} = v_\mu\, B_y + \eta\Delta A,$$
$$\frac{\partial B_y(t,x,z)}{\partial t} = -v_\mu\, \Delta A + \eta\Delta B_y,$$

$$\gamma^{\rm max} = v_\mu^2/4\eta,$$

$$k^{\rm max} = \frac{1}{2}|\mu_0|.$$

Nonlinear dynamos Equations

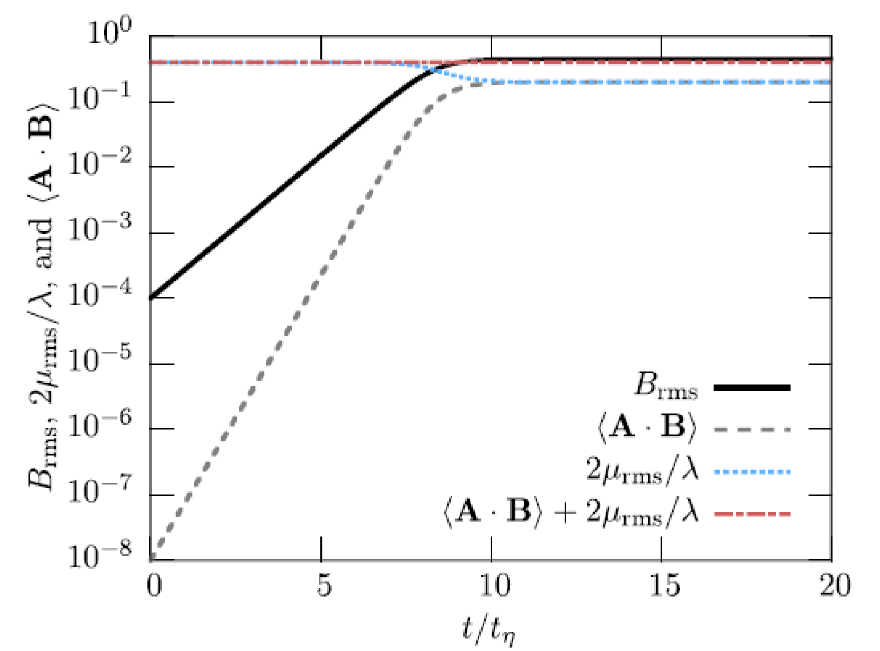
$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times [\boldsymbol{U} \times \boldsymbol{B} - \eta\,(\boldsymbol{\nabla} \times \boldsymbol{B} - \mu_5 \boldsymbol{B})],$$
$$\rho\frac{D\boldsymbol{U}}{Dt} = (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} - \boldsymbol{\nabla} p + \boldsymbol{\nabla}\boldsymbol{\cdot}(2\nu\rho\mathsf{S}) + \rho\boldsymbol{f},$$
$$\frac{D\rho}{Dt} = -\rho\,\boldsymbol{\nabla}\cdot\boldsymbol{U},$$
$$\frac{D\mu_5}{Dt} = D_5\,\Delta\mu_5 + \lambda\,\eta\,\left[\boldsymbol{B}\boldsymbol{\cdot}(\boldsymbol{\nabla} \times \boldsymbol{B}) - \mu_5 \boldsymbol{B}^2\right] - C_5(\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla})\mu - \Gamma_{\rm f}\mu_5,$$
$$\frac{D\mu}{Dt} = D_\mu\,\Delta\mu - C_\mu(\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla})\mu_5,$$

$$D/Dt \equiv \partial/\partial t + \boldsymbol{U}\boldsymbol{\cdot}\boldsymbol{\nabla},$$

$$\lambda = 3\hbar c\left(\frac{8\alpha_{\rm em}}{k_B T}\right)^2$$

$$\alpha_{\rm em} \equiv \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

Nonlinear dynamos



Waves in chiral MHD

$U_{\rm eq} = 0,\ \rho_{\rm eq} \equiv \rho_0 = {\rm const},\ \text{and}\ \mu_{\rm eq} \equiv \mu_0 = {\rm const}.$

$\boldsymbol{B}_{\rm eq} \equiv \boldsymbol{B}_0 = {\rm const},$

Chiral Magnetic Effect

$$\frac{\partial \boldsymbol{B}}{\partial t} = (\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}) \boldsymbol{U} + v_\mu\, \boldsymbol{\nabla} \times \boldsymbol{B} - \boldsymbol{B}_0 (\boldsymbol{\nabla} \cdot \boldsymbol{U}) + \eta \Delta \boldsymbol{B},$$
$$\rho_0 \frac{\partial \boldsymbol{U}}{\partial t} = (\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}) \boldsymbol{B} - \boldsymbol{\nabla} (p + \boldsymbol{B}_0 \cdot \boldsymbol{B}),$$
$$\frac{\partial \rho}{\partial t} = -\rho_0 (\boldsymbol{\nabla} \cdot \boldsymbol{U}),$$

$$v_\mu = \eta\, \mu_0$$

Incompressible flow

$$\omega = \left[\omega_A^2 - (1/4)(V_\mu k)^2\right]^{1/2}$$

$$\omega_{\rm A} = \boldsymbol{k} \cdot \boldsymbol{v}_A$$

$$v_A = B_0/\sqrt{\rho}$$

Compressible flow

$$(\omega^2 - \omega_{\rm A}^2) \left[\omega^4 - \omega^2 (v_A^2 + c_{\rm s}^2) k^2 + \omega_{\rm A}^2 c_{\rm s}^2 k^2\right] - \omega^2 (v_\mu\, k)^2 \left(\omega^2 - c_{\rm s}^2 k^2\right) = 0,$$

The chiral magnetic effect:

(i) decreases the frequency of Alfven waves for an incompressible fluid; (ii) increases the frequencies of Alfven waves and fast magnetosonic waves for a compressible flow, and (iii) decreases the frequency of slow magnetosonic waves.

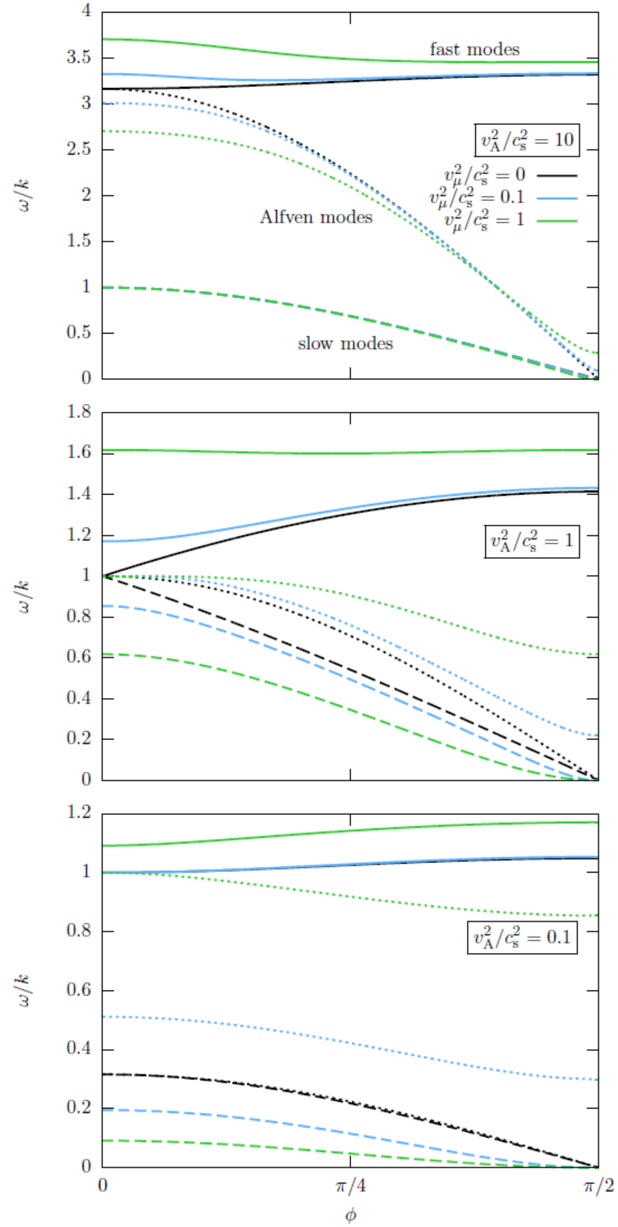


FIG. 1.— The influence of the chiral magnetic effect on the MHD waves in a compressible flow. The ratios ω/k versus the angle ϕ between the wave vector $\boldsymbol{k}$ and the equilibrium magnetic field $\boldsymbol{B}_0$ for the Alfvén wave (dotted lines), the slow magnetosonic wave (dashed lines) and the fast magnetosonic wave (solid lines) for different values of $v_{\rm A}^2/c_{\rm s}^2$ (shown in legends) and different values of $v_\mu^2/c_{\rm s}^2 = 0$ (black lines); 0.1 (blue lines) and 1 (green lines).

Chiral Magnetic Wave

$$\frac{\partial B}{\partial t} = (B_0 \cdot \nabla)\, U - B_0\, (\nabla \cdot U) + \eta \Delta B$$
$$+ v_\mu\, (\nabla \times B - \mu_0\, \Delta U)\,,$$
$$\rho_0 \frac{\partial U}{\partial t} = (B_0 \cdot \nabla)\, B - \nabla\, (p + B_0 \cdot B)\,,$$
$$\frac{\partial \rho}{\partial t} = -\rho_0\, (\nabla \cdot U)\,,$$
$$\frac{\partial \tilde{\mu}_5}{\partial t} = \lambda \left[\eta B_0 \cdot (\nabla \times B) - 2 v_\mu B_0 \cdot B - \eta \tilde{\mu}_5 B_0^2 \right]$$
$$-C_5 (B_0 \cdot \nabla) \mu,$$
$$\frac{\partial \mu}{\partial t} = -C_\mu (B_0 \cdot \nabla) \tilde{\mu}_5 .$$

Alfven wave

$$\omega_A = k \cdot v_A$$

$$v_A = B_0 / \sqrt{\rho}$$

Chiral magnetic wave:

$$\Omega_{\rm CMW} = \sqrt{C_5\, C_\mu}\, |k \cdot B_0|\,,$$

Kharzeev and Yee (2011)

The dispersion relation for the MHD waves is not modified by the chiral magnetic wave, and vise versa.

However, the MHD waves can play a source for the chiral magnetic wave.

Laminar V_μ^2-Dynamo in Chiral MHD

Joyce and Shaposhnikov, PRL 79, 1193 (1997)

$$\mu_{\rm eq} \equiv \mu_0 = \text{const and } \boldsymbol{U}_{\rm eq} = 0.$$

$$\boldsymbol{B}(t,x,z) = B_y(t,x,z)\boldsymbol{e}_y + \boldsymbol{\nabla} \times [A(t,x,z)\boldsymbol{e}_y]$$

$$\gamma = |v_\mu \, k| - \eta k^2,$$

Chiral velocity

$$v_\mu \equiv \eta \mu_{5,0}$$

$$\frac{\partial A(t,x,z)}{\partial t} = v_\mu \, B_y + \eta \Delta A,$$
$$\frac{\partial B_y(t,x,z)}{\partial t} = -v_\mu \, \Delta A + \eta \Delta B_y,$$

$$\gamma^{\rm max} = v_\mu^2/4\eta,$$

$$k^{\rm max} = \frac{1}{2}|\mu_0|.$$

Nonlinear dynamos Equations

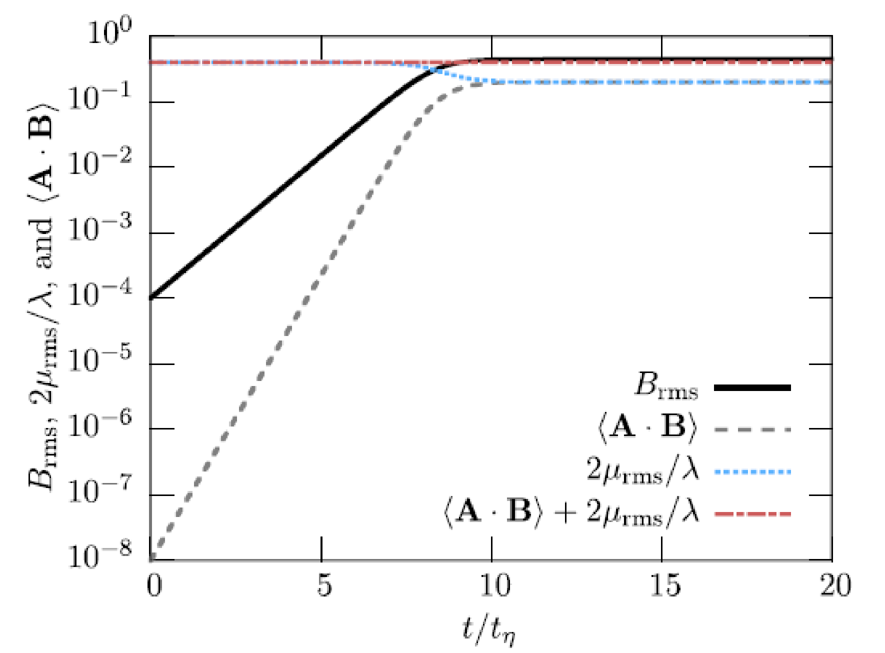
$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times [\boldsymbol{U} \times \boldsymbol{B} - \eta \, (\boldsymbol{\nabla} \times \boldsymbol{B} - \mu_5 \boldsymbol{B})] \, ,$$
$$\rho \frac{D\boldsymbol{U}}{Dt} = (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} - \boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot (2\nu \rho \mathbf{S}) + \rho \boldsymbol{f},$$
$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{U},$$
$$\frac{D\mu_5}{Dt} = D_5 \, \Delta\mu_5 + \lambda \eta \, [\boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) - \mu_5 \boldsymbol{B}^2] - C_5 (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \mu - \Gamma_{\rm f} \mu_5,$$
$$\frac{D\mu}{Dt} = D_\mu \, \Delta\mu - C_\mu (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \mu_5,$$

$$D/Dt \equiv \partial/\partial t + \boldsymbol{U} \cdot \boldsymbol{\nabla},$$

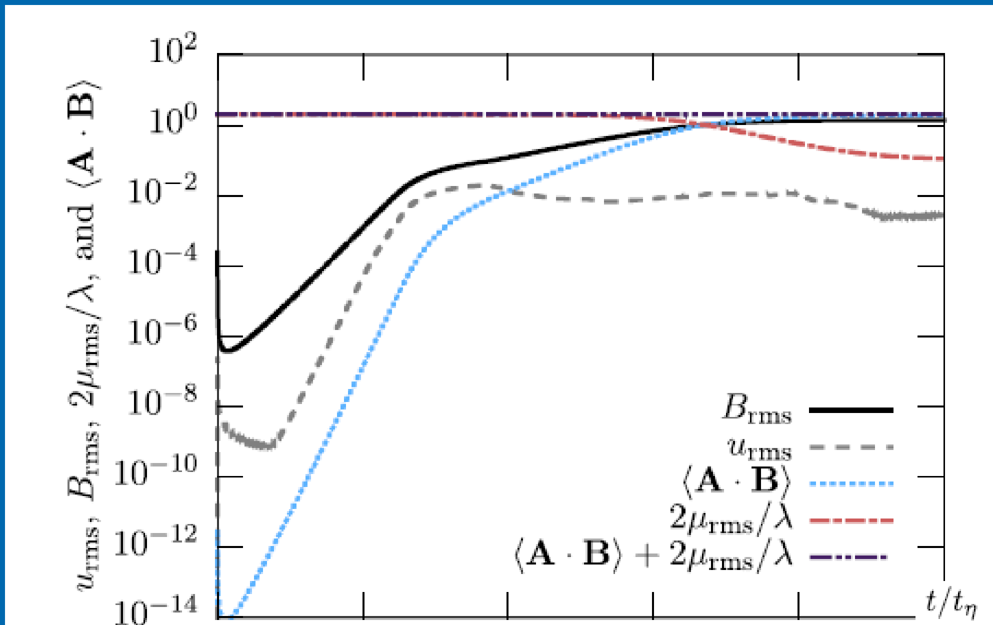
$$\lambda = 3\hbar c \left(\frac{8\alpha_{\rm em}}{k_B T} \right)^2$$

$$\alpha_{\rm em} \equiv \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

Nonlinear dynamos



Chiral-Magnetically Produced Turbulence



$$\gamma = |v_\mu\, k| - \eta k^2,$$

$$\gamma^{\rm max} = v_\mu^2/4\eta,$$

$$k^{\rm max} = \frac{1}{2}|\mu_0|.$$

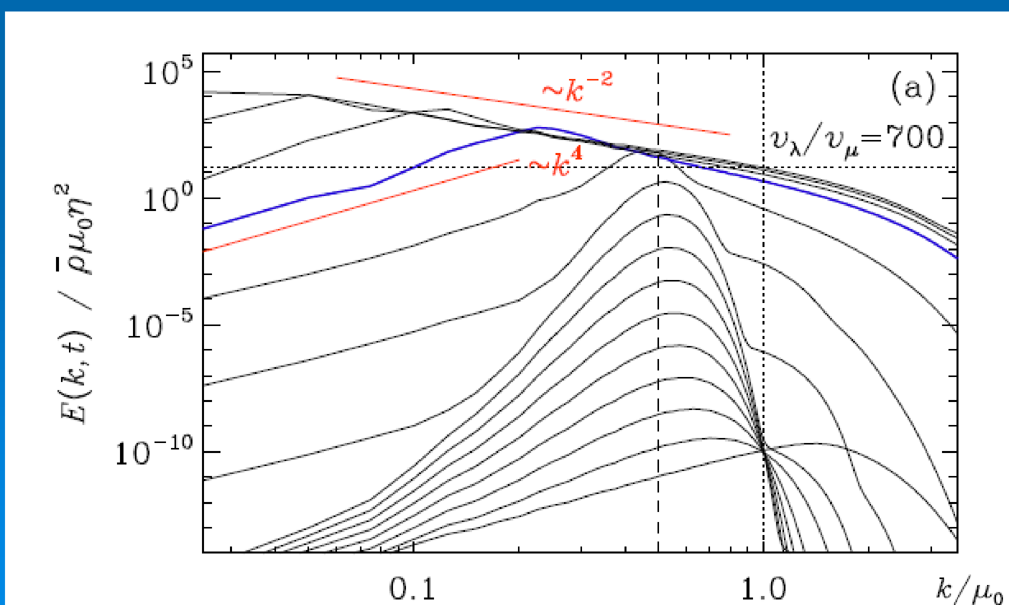
$$v_\mu = \mu\eta,$$

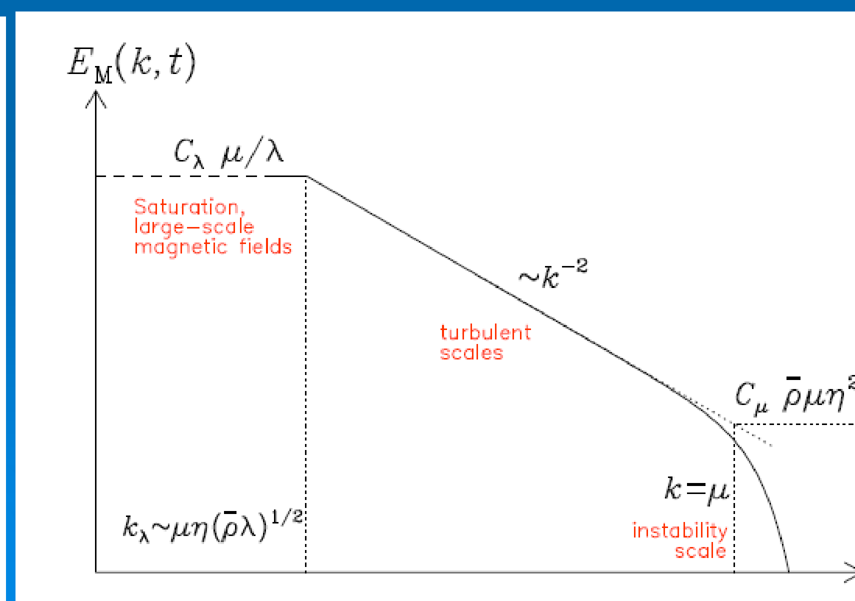
$$\tfrac{1}{2}\lambda\, \langle \boldsymbol{A}\cdot\boldsymbol{B}\rangle + \langle\mu\rangle = \text{const} \equiv \mu_0$$

$$\langle \boldsymbol{B}^2\rangle\, \xi_{\rm M} \lesssim \mu_0/\lambda,$$

$$E_{\rm M}(k,t) = C_\mu\, \overline{\rho}\mu^3\eta^2 k^{-2},$$

$$v_\lambda = \mu/(\overline{\rho}\lambda)^{1/2},$$





Equations in chiral MHD

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \left[\boldsymbol{U} \times \boldsymbol{B} - \eta\left(\boldsymbol{\nabla} \times \boldsymbol{B} - \mu_5 \boldsymbol{B}\right)\right],$$

$$\rho \frac{D\boldsymbol{U}}{Dt} = (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} - \boldsymbol{\nabla} p + \boldsymbol{\nabla}\cdot(2\nu\rho\mathsf{S}) + \rho \boldsymbol{f},$$

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{U},$$

$$\frac{D\mu_5}{Dt} = D_5\,\Delta\mu_5 + \lambda\,\eta\left[\boldsymbol{B}\cdot(\boldsymbol{\nabla} \times \boldsymbol{B}) - \mu_5 \boldsymbol{B}^2\right] - C_5(\boldsymbol{B}\cdot\boldsymbol{\nabla})\mu - \Gamma_{\mathrm{f}}\mu_5,$$

$$\frac{D\mu}{Dt} = D_\mu\,\Delta\mu - C_\mu(\boldsymbol{B}\cdot\boldsymbol{\nabla})\mu_5,$$

$$\mu_5 \equiv (4\alpha_{\mathrm{em}}/\hbar c)\mu_5^{\mathrm{phys}} \qquad \mu_5^{\mathrm{phys}} \equiv \mu_{\mathrm{L}}^{\mathrm{phys}} - \mu_{\mathrm{R}}^{\mathrm{phys}},$$

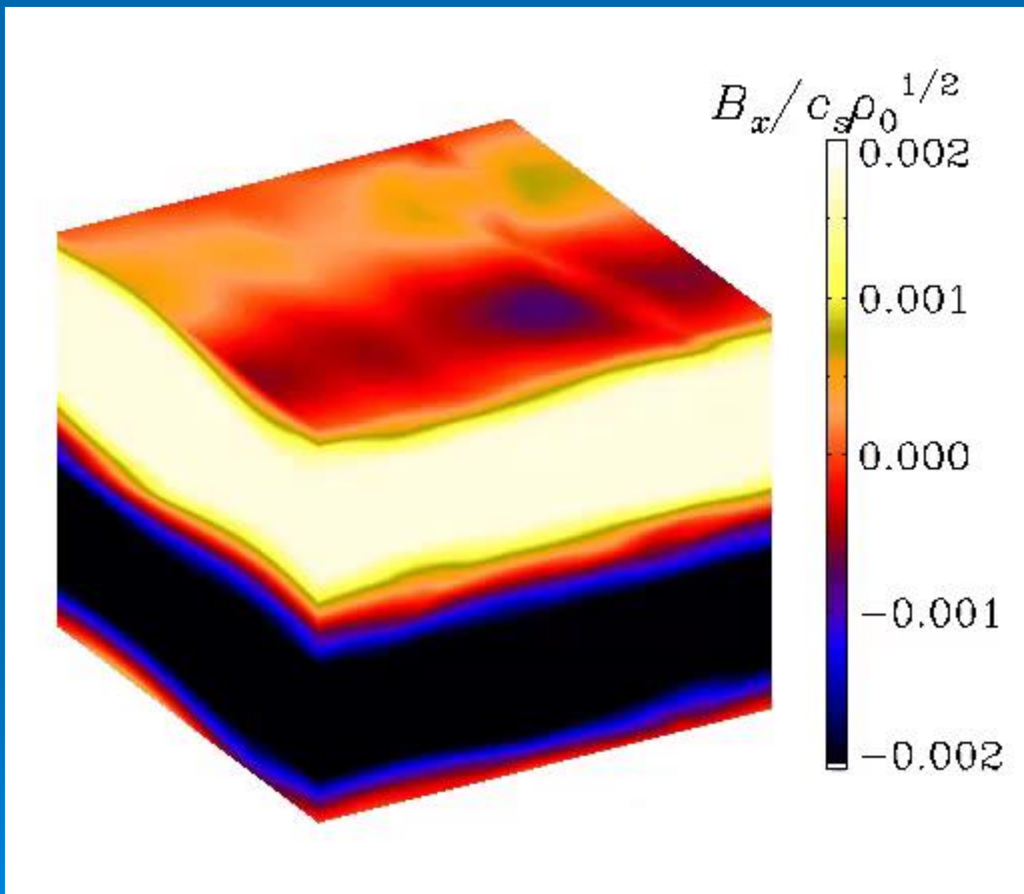
$$\lambda = 3\hbar c\left(\frac{8\alpha_{\mathrm{em}}}{k_B T}\right)^2$$

$$D/Dt \equiv \partial/\partial t + \boldsymbol{U}\cdot\boldsymbol{\nabla}, \qquad \mu \equiv (4\alpha_{\mathrm{em}}/\hbar c)\mu^{\mathrm{phys}} \qquad \mu^{\mathrm{phys}} \equiv \mu_{\mathrm{R}}^{\mathrm{phys}} + \mu_{\mathrm{L}}^{\mathrm{phys}}$$

Conservation Law: Magnetic Helicity + Chiral Chemical Potential

$$\frac{\partial}{\partial t}\left(\frac{\lambda}{2}\boldsymbol{A}\cdot\boldsymbol{B} + \mu_5\right) + \boldsymbol{\nabla}\cdot\left[\frac{\lambda}{2}\left(\boldsymbol{E} \times \boldsymbol{A} - \boldsymbol{B}\,\Phi\right) - D_5\boldsymbol{\nabla}\mu_5 + C_5\boldsymbol{B}\mu\right] = 0,$$

Chiral-Magnetically Produced Turbulence and Generation of Large-Scale Magnetic Field



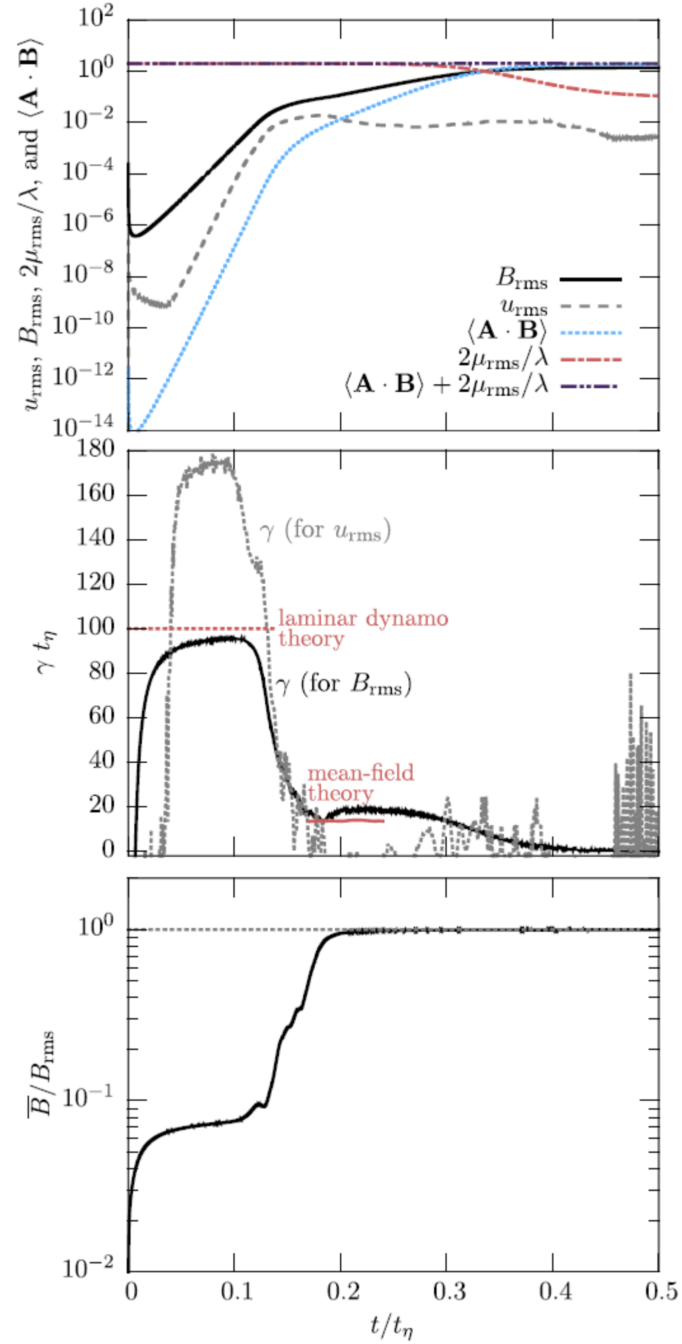


Figure 9. Chiral magnetically driven turbulence. Time evolution for different quantities.

Mean-Field Dynamo

Mean-Field Approach: $\mathrm{H} = \mathrm{B} + \mathrm{b}, \ \mathbf{v} = \mathbf{U} + \mathbf{u},$

- **Induction equation for mean magnetic field:**

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{U} \times \mathbf{B} + \boldsymbol{\varepsilon} - \eta \nabla \times \mathbf{B} \right)$$

$$\mathrm{B} = \langle \mathrm{H} \rangle, \quad \mathrm{U} = \langle \mathbf{v} \rangle$$

- **Electromotive force:**

$$\boldsymbol{\varepsilon} \equiv \left\langle \mathbf{u} \times \mathbf{b} \right\rangle = \boldsymbol{\alpha} \mathbf{B} - \boldsymbol{\eta}_{\mathrm{T}} \nabla \times \mathbf{B} + \ldots$$

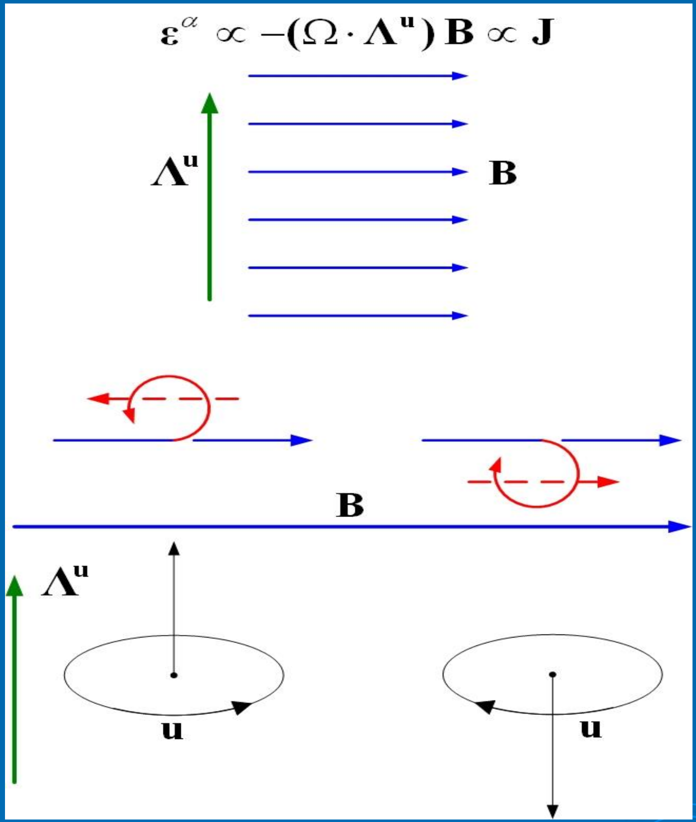
$$\boldsymbol{\alpha} = -\frac{\tau}{3} \langle \mathbf{u} \cdot \mathbf{rot}\ \mathbf{u} \rangle$$

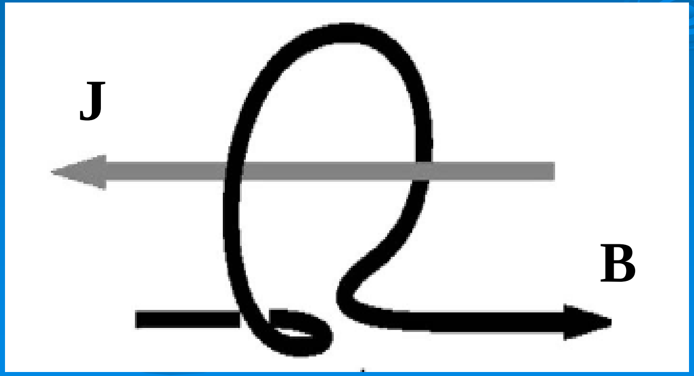
Steenbeck, Krause, Rädler (1966)

Physics of the alpha-effect

Parker (1955); Steenbeck, Krause, Rädler (1966)

- The α - effect is related to the kinetic helicity in a density stratified convective turbulence.
- The deformations of the magnetic field lines are caused by upward and downward rotating turbulent eddies.
- The stratification of turbulence breaks a symmetry between the upward and downward eddies.
- Therefore, the total effect of the upward and downward eddies on the mean magnetic field does not vanish and it creates the mean electric current parallel to the original mean magnetic field.

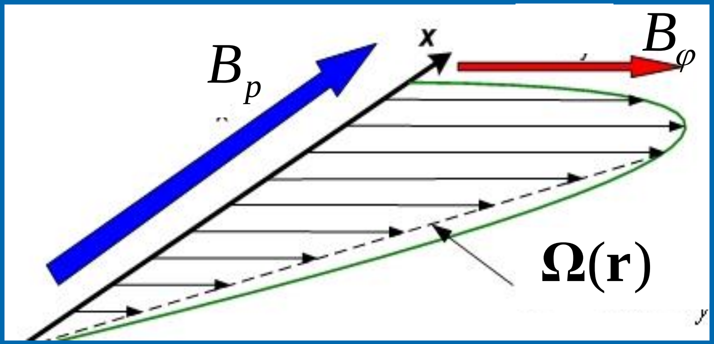


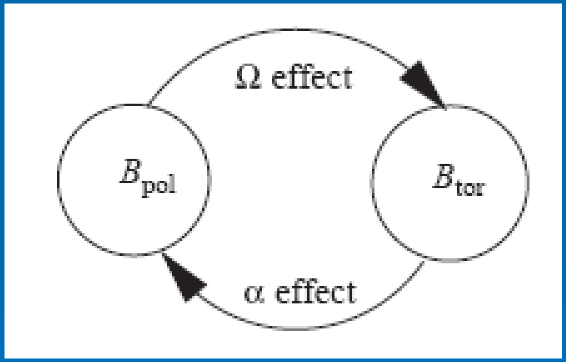


Generation of the mean magnetic field due to the $\alpha\Omega$ dynamo

Mean magnetic field:

$$\mathbf{B} = B_\varphi\,\mathbf{e}_\varphi + \nabla \times [A\,\mathbf{e}_\varphi]$$





$$\frac{\partial B_\varphi}{\partial t} = D\,[\nabla\Omega \times \nabla A]_\varphi + \Delta_s B_\varphi$$

$$\frac{\partial A}{\partial t} = \alpha B_\varphi + \Delta_s A$$

Dynamo number:

$$\Delta_s = \Delta - \frac{1}{r^2 \sin^2\theta}$$

$$D = R_\alpha R_\Omega; \quad R_\alpha = \frac{\alpha R_\odot}{\eta_T}; R_\Omega = \frac{\delta\Omega R_\odot^2}{\eta_T};$$

Methods for Derivation of EMF

- Quasi-Linear Approach or Second-Order Correlation Approximation (SOCA) or First-Order Smoothing Approximation (FOSA)

  Rm << 1 , Re << 1

  Steenbeck, Krause, Rädler (1966); Roberts, Soward (1975); Moffatt (1978)

- Path-Integral Approach (delta-correlated in time random velocity field or short yet finite correlation time)

  Zeldovich, Molchanov, Ruzmaikin, Sokoloff (1988)
  Rogachevskii, Kleeorin (1997)

  $$\mathrm{St} = \frac{\tau}{\ell/u} \ll 1$$

- Tau-approaches (spectral tau-approximation, minimal tau-approximation) – third-order or high-order closure

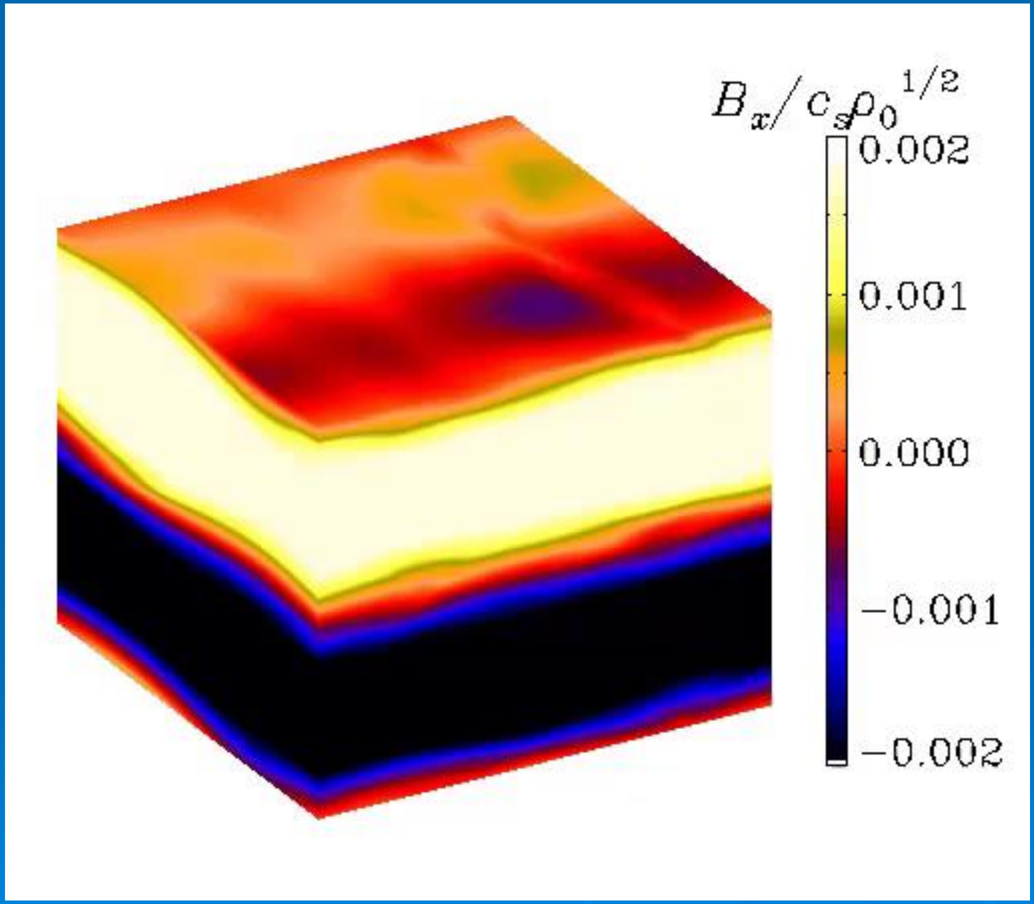
  Re >> 1 and Rm >> 1

  Pouquet, Frisch, Leorat (1976);
  Rogachevskii, Kleeorin (2000; 2001; 2003); Blackman, Field (2002);
  Rädler, Kleeorin, Rogachevskii (2003)

- Renormalization Procedure (renormalization of viscosity, diffusion, electromotive force and other turbulent transport coefficients) - there is no separation of scales

  Moffatt (1981; 1983); Kleeorin, Rogachevskii (1994)

Chiral-Magnetically Produced Turbulence and Generation of Large-Scale Magnetic Field



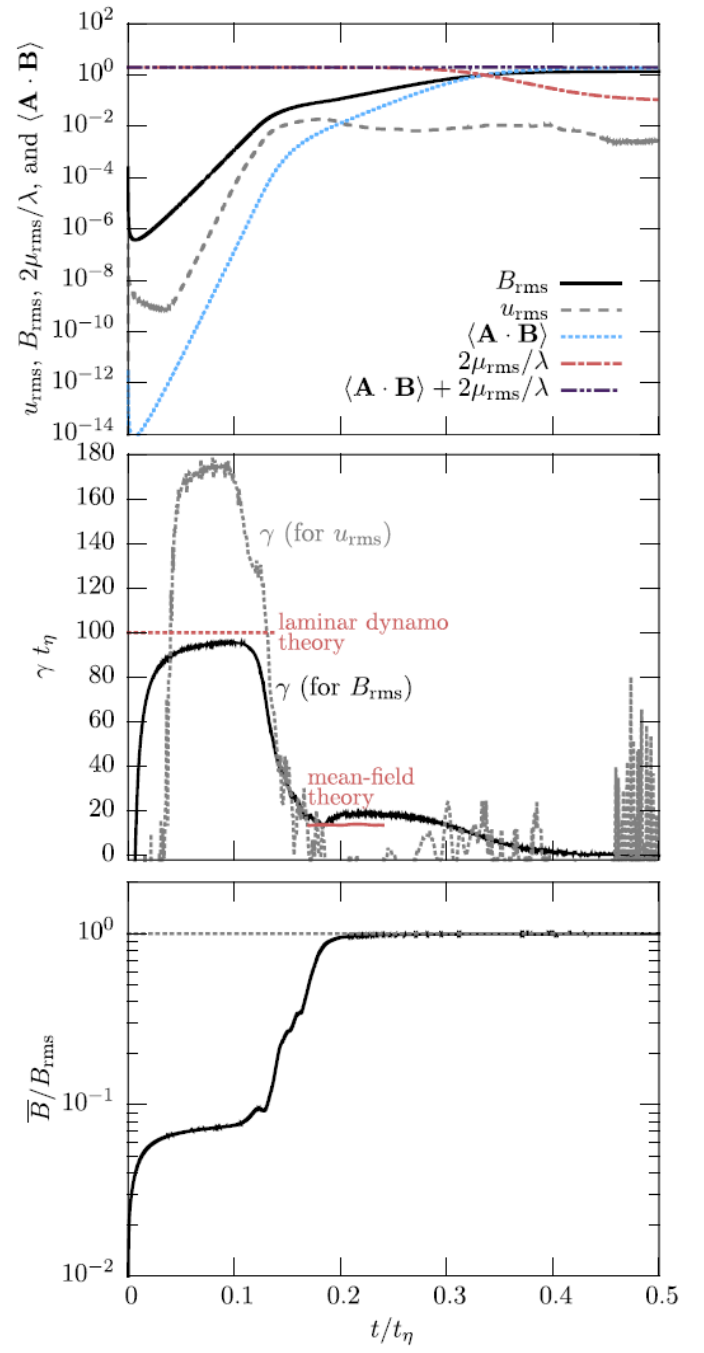


Figure 9. Chiral magnetically driven turbulence. Time evolution for different quantities.

New Kind of Chiral Alpha Effect (dimensional analysis)

- **Magnetic fluctuations:**

$$\frac{\partial b}{\partial t} = (\overline{B} \cdot \nabla) u + \overline{v}_\mu \nabla \times b + \ldots,$$

$$b_{\rm tang} = \tau \, (\overline{B} \cdot \nabla) u,$$

- **Chiral magnetic fluctuations:**

$$b_\mu = \tau \, \overline{v}_\mu \nabla \times b_{\rm tang} = \tau^2 \, \overline{v}_\mu \nabla \times \left[(\overline{B} \cdot \nabla) u \right],$$

- **Chiral electromotive force** $\overline{\mathcal{E}}^\mu \equiv \overline{u \times b_\mu}$ **:**

$$\overline{\mathcal{E}}_i^\mu = \left(\tau^2 \, \overline{v}_\mu \, \overline{u_n \nabla_i \nabla_j u_n} \right) \overline{B}_j \equiv \alpha_{ij}^\mu \, \overline{B}_j .$$

- **Chiral alpha effect:**

$$\alpha_{ij}^\mu = \overline{v}_\mu \, \tau^2 \, \overline{u_n \nabla_i \nabla_j u_n} = -\overline{v}_\mu \int \tau^2(k) \, k_i k_j \, \langle u^2 \rangle_{\boldsymbol{k}} \, dk,$$

$$\alpha_\mu = -\frac{2}{3} \overline{v}_\mu \ln {\rm Re}_{_M} .$$

New Kind of Chiral Alpha Effect (quasi-linear approach and tau approach)

- **Magnetic fluctuations:**

$$\frac{\partial b}{\partial t} = (\overline{B} \cdot \nabla) u + \overline{v}_\mu \nabla \times b + ...,$$

- **Chiral magnetic fluctuations:**

$$b_\mu = \tau\, \overline{v}_\mu \nabla \times b_{\rm tang} = \tau^2\, \overline{v}_\mu \nabla \times \left[(\overline{B} \cdot \nabla) u \right],$$

- **Chiral electromotive force** $\overline{\mathcal{E}}^\mu \equiv \overline{u \times b_\mu}$ **:**

$$\overline{\mathcal{E}}_i^\mu = \left(\tau^2\, \overline{v}_\mu\, \overline{u_n \nabla_i \nabla_j u_n} \right) \overline{B}_j \equiv \alpha_{ij}^\mu\, \overline{B}_j .$$

- **Chiral alpha effect:**

$$\alpha_{ij}^\mu = \overline{v}_\mu\, \tau^2\, \overline{u_n \nabla_i \nabla_j u_n} = -\overline{v}_\mu \int \tau^2(k)\, k_i k_j\, \langle u^2 \rangle_{\boldsymbol{k}}\, dk,$$

Quasi-linear approach: Rm<<1

$$\alpha_\mu = -\frac{(q-1)}{3(q+1)}\, {\rm Re}_{_M}^2\, \overline{v}_\mu .$$

Tau-approach: Rm>>1

$$\alpha_\mu = -\frac{2}{3} \overline{v}_\mu \ln {\rm Re}_{_M} .$$

New Kind of Chiral Alpha Effect (tau-approach)

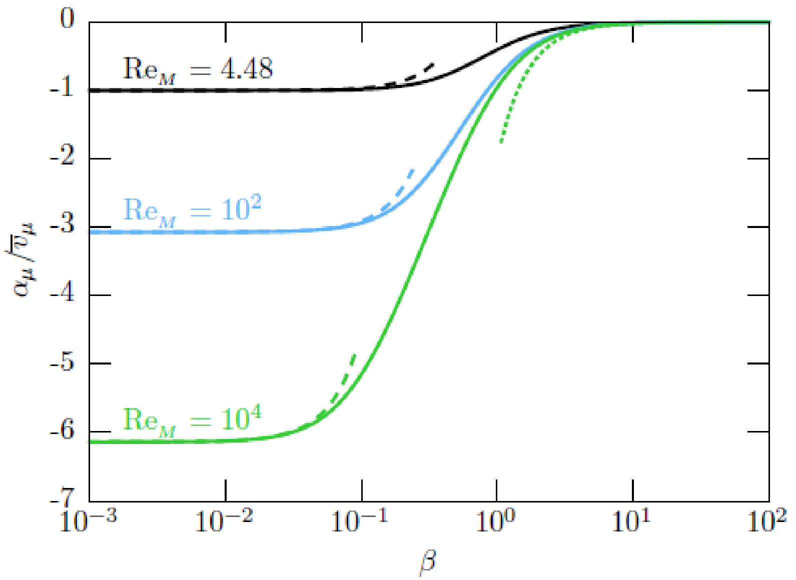


FIG. 3.— The α_μ effect as a function of $\beta = \sqrt{8}\,\overline{B}/B_{\rm eq}$ with $B_{\rm eq} = \left(\rho\,\overline{u^2}\right)^{1/2}$. Solid lines represent the expression given by Equation (105) normalized by $\overline{v}_\mu$, while the dashed lines show the asymptotics at low β as given by (106) and high β as given by (104). The black lines correspond to a fluid Reynolds number of $\mathrm{Re} = 4.48$, the blue lines to $\mathrm{Re} = 10^2$, and the green lines to $\mathrm{Re} = 10^4$.

$$\alpha_\mu = \frac{4}{3}\overline{v}_\mu \left[\ln\left(\frac{1+2\beta^2\mathrm{Re}_M^{1/2}}{(1+2\beta^2)\mathrm{Re}_M^{1/2}}\right) + \frac{1}{\beta^2}\left(\frac{\arctan(\sqrt{2}\beta)}{\sqrt{2}\beta} - 1\right) - \frac{1}{\beta^2\mathrm{Re}_M^{1/2}}\left(\frac{\arctan(\sqrt{2}\beta\mathrm{Re}_M^{1/4})}{\sqrt{2}\beta\mathrm{Re}_M^{1/4}} - 1\right)\right]. \quad (105)$$

$$\beta = \sqrt{8}\,\overline{B}/B_{\rm eq} \qquad B_{\rm eq} = \left(\rho\,\overline{\boldsymbol{u}^2}\right)^{1/2}$$

Very weak mean magnetic field:

$$\beta \ll \mathrm{Re}_M^{-1/4} \ll 1,$$

$$\alpha_\mu = -\frac{2}{3}\overline{v}_\mu \ln \mathrm{Re}_M \left[1 - \frac{12\beta^2\mathrm{Re}_M^{1/2}}{5\ln \mathrm{Re}_M}\right]$$

Weak mean magnetic field:

$$\mathrm{Re}_M^{-1/4} \ll \beta \ll 1$$

$$\alpha_\mu = -\frac{4}{3}\overline{v}_\mu |\ln(2\beta^2)| \left[1 + \frac{2}{3|\ln(2\beta^2)|}\right]$$

Strong mean magnetic field:

$$\beta \gg 1 \qquad \alpha_\mu = -\frac{2}{\beta^2}\overline{v}_\mu .$$

Large-Scale Chiral Dynamo

$$\overline{B}(t,x,z) = \overline{B}_y(t,x,z)e_y + \nabla \times [\overline{A}(t,x,z)e_y],$$

Chiral α_μ^2-dynamo:

$$\frac{\partial \overline{A}(t,x,z)}{\partial t} = (\overline{v}_\mu + \alpha_\mu)\,\overline{B}_y + (\eta + \eta_{_T})\,\Delta \overline{A},$$

$$\frac{\partial \overline{B}_y(t,x,z)}{\partial t} = -(\overline{v}_\mu + \alpha_\mu)\,\Delta \overline{A} + (\eta + \eta_{_T})\,\Delta \overline{B}_y,$$

$$\gamma = |(\overline{v}_\mu + \alpha_\mu)\,k| - (\eta + \eta_{_T})\,k^2,$$

$$\gamma^{\max} = \frac{(\overline{v}_\mu + \alpha_\mu)^2}{4(\eta + \eta_{_T})} = \frac{(\overline{v}_\mu + \alpha_\mu)^2}{4\eta\,(1 + \mathrm{Re}_{_M}/3)}.$$

Chiral alpha-shear dynamo:

$$\frac{\partial \overline{A}(t,x,z)}{\partial t} = (\overline{v}_\mu + \alpha_\mu)\,\overline{B}_y + (\eta + \eta_{_T})\,\Delta \overline{A},$$

$$\frac{\partial \overline{B}_y(t,x,z)}{\partial t} = -\overline{S}\,\nabla_z \overline{A} - (\overline{v}_\mu + \alpha_\mu)\,\Delta \overline{A} + (\eta + \eta_{_T})\,\Delta \overline{B}_y.$$

$$\gamma = \left(\frac{|(\overline{v}_\mu + \alpha_\mu)\,\overline{S}\,k_z|}{2}\right)^{1/2} - (\eta + \eta_{_T})\,k^2,$$

$$\omega = \mathrm{sgn}\,[(\overline{v}_\mu + \alpha_\mu)\,k_z]\,\left(\frac{|(\overline{v}_\mu + \alpha_\mu)\,\overline{S}\,k_z|}{2}\right)^{1/2}$$

$$\gamma^{\max} = \frac{3}{8}\left(\frac{\overline{S}^2\,(\overline{v}_\mu + \alpha_\mu)^2}{2(\eta + \eta_{_T})}\right)^{1/3},$$

$$\omega^{\max} = \frac{\mathrm{sgn}\,[(\overline{v}_\mu + \alpha_\mu)\,k_z]}{2\eta}\left[\frac{\overline{S}^2\,(\overline{v}_\mu + \alpha_\mu)^2}{2(\eta + \eta_{_T})}\right]^{1/3}$$

Growth rates of Magnetic Field

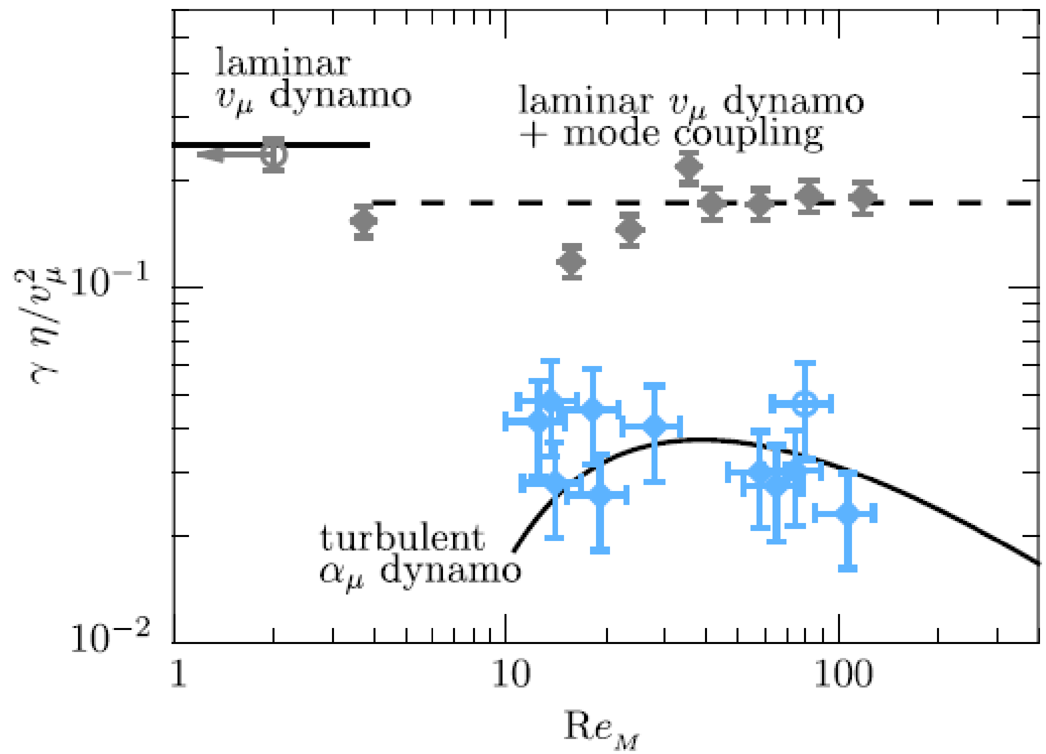


Figure 17. Externally forced turbulence and chiral magnetically driven turbulence. The normalized growth rate $\gamma\ \eta/v_\mu^2$ of the magnetic field as a function of the magnetic Reynolds number Re_M. The gray data points show the growth rate in the initial, purely kinematic phase of the simulations. The blue data points show the measured growth rate of the magnetic field on $k = 1$, when the large-scale dynamo occurs. The diamond-shaped data points represent simulations of forced turbulence, while the dot-shaped data points refer to the case of chiral magnetically driven turbulence. The growth rate observed in the initial laminar phase for the case of chiral magnetically driven turbulence is shown at $\mathrm{Re}_\mathrm{M} = 2$, with the left arrow indicating that the actual Re_M is much lower and out of the plot range at this time; see Figure 11.

Generation of Kinetic Helicity in Chiral MHD

$$\chi_K = \overline{\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u})}$$

$$\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{u}$$

$$\frac{\partial \chi_K}{\partial t} = -2\overline{\boldsymbol{w} \times (\boldsymbol{\nabla} \times \boldsymbol{b})} \cdot \overline{\boldsymbol{B}} - 2\overline{\boldsymbol{w} \times \boldsymbol{b}} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}) - \varepsilon_\chi - \boldsymbol{\nabla} \cdot \boldsymbol{F}_\chi,$$

$$[\,\overline{\boldsymbol{w} \times \boldsymbol{b}}\,]_\mu = -\frac{\alpha_\mu}{4}(\overline{\boldsymbol{B}} \times \boldsymbol{\nabla})\overline{\boldsymbol{u}^2},$$

$$[\,\overline{\boldsymbol{w} \times (\boldsymbol{\nabla} \times \boldsymbol{b})}\,]_\mu = 0.$$

$$\boldsymbol{\nabla} \to \boldsymbol{\nabla} - 2\boldsymbol{\lambda}, \text{ where } \boldsymbol{\lambda} = -\boldsymbol{\nabla}\rho/\rho.$$

$$\frac{\partial \chi_K}{\partial t} = \frac{\alpha_\mu}{2}[(\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}) \times \overline{\boldsymbol{B}}] \cdot \boldsymbol{\nabla} \ln(\rho^2 \overline{\boldsymbol{u}^2}) - \varepsilon_\chi - \boldsymbol{\nabla} \cdot \boldsymbol{F}_\chi.$$

$\varepsilon_\chi \sim \chi_K / \tau_0$ is the rate of the dissipation of χ_K,

The generation of the mean kinetic helicity by the chiral magnetic effect in nonhelical turbulence:

(i) occurs only in inhomogeneous or density-stratified turbulence;

(ii) it is a nonlinear effect (it is quadratic in the mean magnetic field).

Chiral-Magnetically Produced Turbulence; Forced Turbulence and Large-Scale Dynamo

Chiral-Produced Turbulence

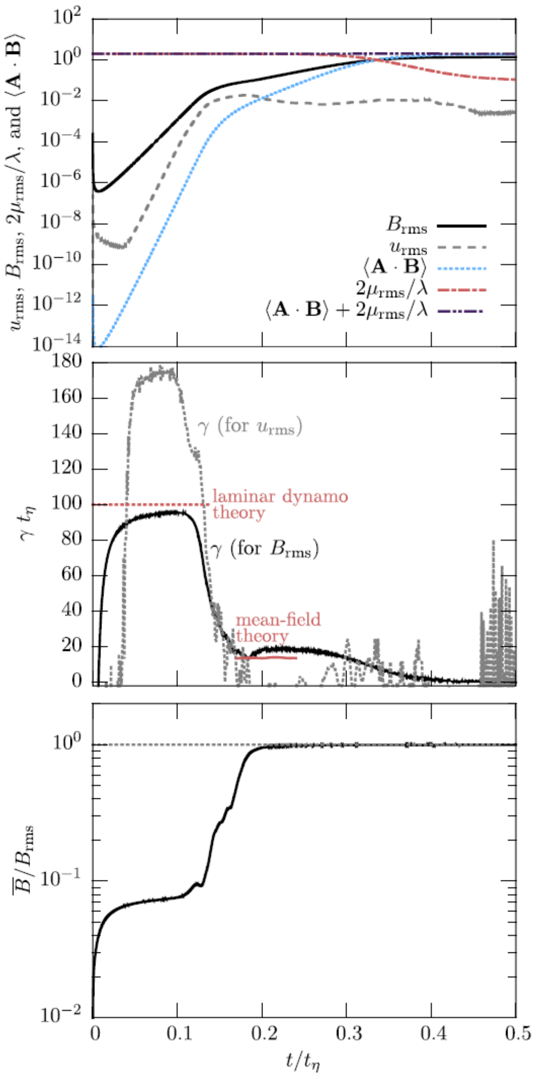
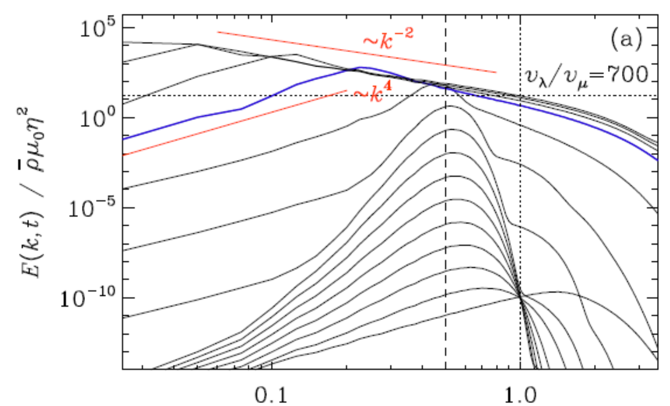


Figure 9. Chiral magnetically driven turbulence. Time evolution for different quantities.

Chiral-Produced Turbulence



Forced Turbulence

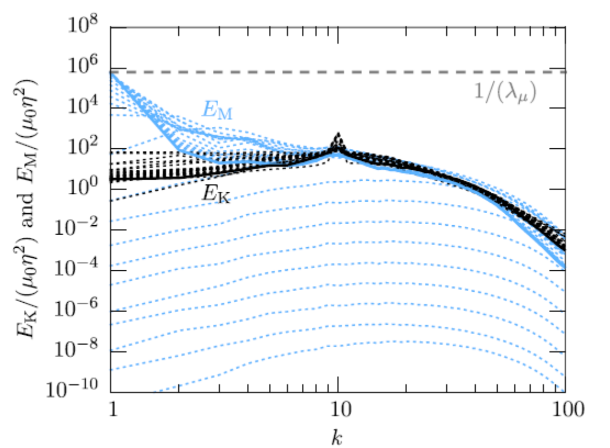


Figure 15. Externally forced turbulence. Evolution of kinetic (black lines) and magnetic energy spectra (blue lines) for the reference run Ta2-5. The ratio μ_0/λ is indicated by the horizontal dashed line.

Forced Turbulence

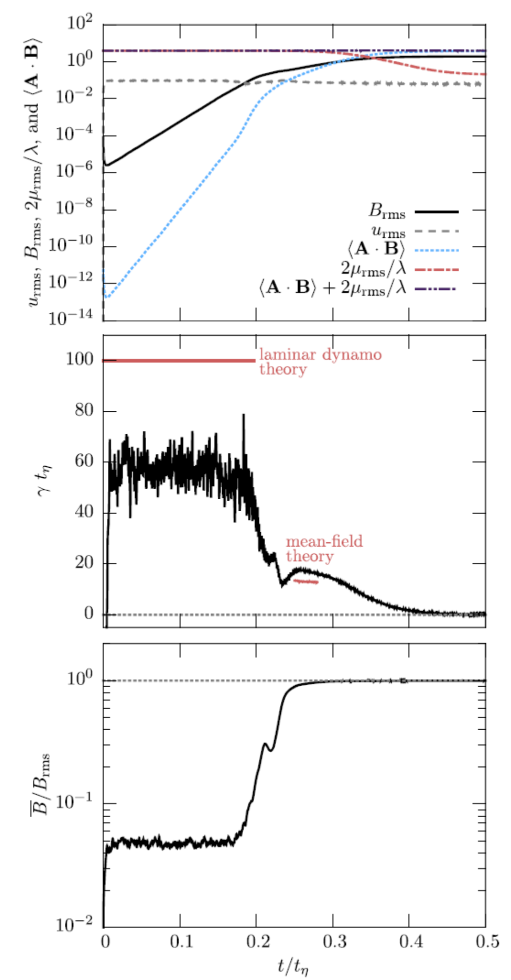
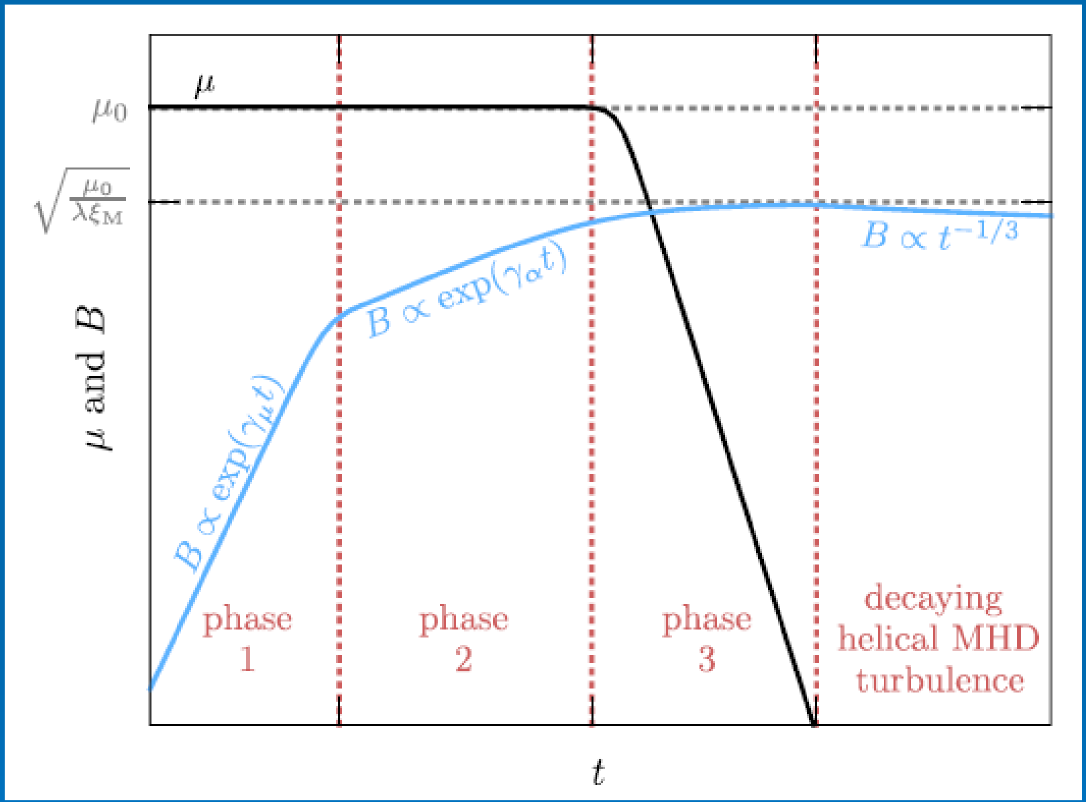


Figure 14. Externally forced turbulence. Time evolution of the magnetic field, the velocity field, and the chemical potential, as well as the mean value of the magnetic helicity (top panel). The middle panel shows the growth rate of $B_{\rm rms}$ as a function of time (solid black line). The red lines are theoretical expectations in different dynamo phases. In the bottom panel, the ratio of the mean magnetic field to the total field $B_{\rm rms}$ is presented.

Four Phases of Magnetic Field Evolution



(1) small-scale chiral dynamo instability;

(2) production of small-scale turbulence, inverse transfer of magnetic energy, and generation of a large-scale magnetic field by the chiral α_μ effect;

(3) saturation of the large-scale chiral dynamo by a decrease of the CME controlled by the conservation law for the total chirality: $\lambda \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle / 2 + \langle \mu \rangle = \mu_0$.

Correlation length and Growth rate of the Magnetic Field in Early Universe

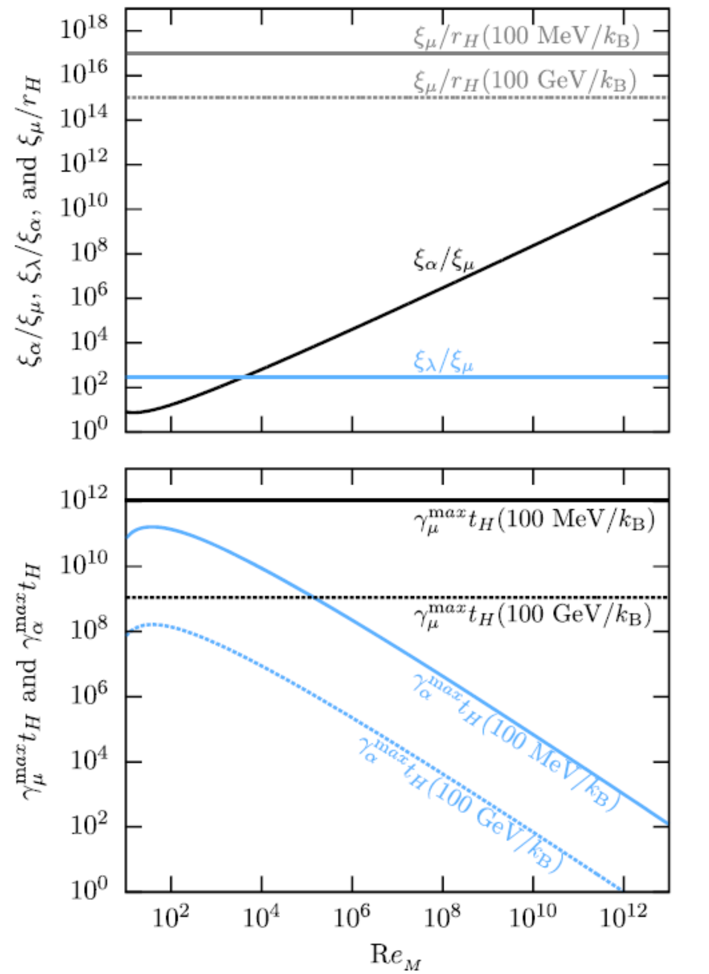


Figure 19. Chiral MHD dynamos in the early universe. The ratios between ξ_α of the turbulence-driven dynamo (Equation (45)) and scale ξ_μ (Equation (52)), as well as the ratio between ξ_μ and the Hubble radius at different temperatures. In the top panel, furthermore, the ratio ξ_μ/ξ_λ is presented. Maximum growth rates over the Hubble time for laminar ($\gamma_\mu^{\max}$) and turbulent ($\gamma_\alpha^{\max}$) regimes are shown in the bottom panel.

(1) helical magnetic fields are excited,
(2) turbulence with large Re_M is produced, and
(3) the comoving correlation scale increases.

The Growth Rate and Scale for Small-Scale Magnetic Field:

$$\gamma_\mu^{\max} = \frac{\mu_0^2\eta}{4} \approx 2.4 \times 10^{19} T_{100}^{-1}\ \mathrm{s}^{-1}.$$

$$\xi_\mu \equiv \frac{2}{|\mu_0|},$$

The Growth Rate and Scale for Large-Scale Magnetic Field:

$$\gamma_\alpha^{\max} = \gamma_\mu^{\max}\frac{4}{3}\frac{(\ln\mathrm{Re}_M)^2}{\mathrm{Re}_M},$$

$$\frac{\xi_\alpha}{\xi_\mu} = \frac{3\mathrm{Re}_M}{2\ln\mathrm{Re}_M}$$

The Hubble time:

$$T_{100} = k_B T/100\,\mathrm{GeV}$$

$$t_H = H^{-1}(T) \approx 4.8 \times 10^{-11}\, g_{100}^{-1/2} T_{100}^{-2}\ \mathrm{s}$$

$$g_{100} = g_*/100$$

g_* is the number of relativistic degrees of freedom

Estimates: Early Universe

(i) The equations are used in comoving variables. (ii) The scale factor a is normalized so that at the electroweak phase transition (EWPT) epoch: $a = 1$.

- **Upper limit:** $$|\mu| \ll 4\alpha_{\rm em}\frac{k_{\rm B}T}{\hbar c} \approx 1.5\times 10^{14}\,T_{100}\ {\rm cm}^{-1}$$

$$T_{100} = k_B T/100\,{\rm GeV}$$

$$100\,{\rm GeV} = 1.2\times 10^{15}\,{\rm K}$$

- **Magnetic Diffusion:** $$\eta = 7.3\times 10^{-4}\,\frac{\hbar c^2}{k_{\rm B}T} = 4.3\times 10^{-9} T_{100}^{-1}\,{\rm cm}^2\,{\rm s}^{-1}.$$

- **Chiral velocities:** $$v_\mu = 6.4\times 10^5\,{\rm cm\,s}^{-1}$$ $$v_\lambda \approx 1.5\times 10^9\,{\rm cm\,s}^{-1}$$

$$k_{\rm B}T \gg \max(|\mu_L|,\,|\mu_R|).$$

- **Saturation parameter:** $$\lambda = 3\hbar c\left(\frac{8\alpha_{\rm em}}{k_{\rm B}T}\right)^2 \approx 1.3\times 10^{-17}\,T_{100}^{-2}\ {\rm cm\,erg}^{-1}$$

- **Magnetic energy in unit square:** at $T = 100$ GeV

$$\langle B^2\rangle\,\xi_{\rm M} \approx \mu/\lambda \ll 1.2\times 10^{31}\,T_{100}^3\ {\rm G}^2\,{\rm cm.} = 6\times 10^{-15}\,{\rm G}^2\,{\rm cm}$$

Magnetic Field versus Correlation Length

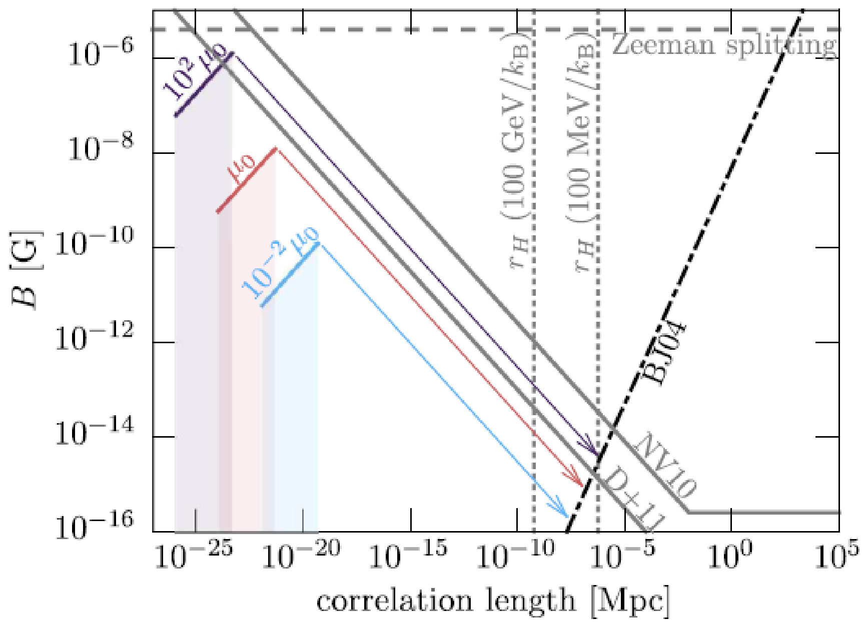


Figure 20. Chiral MHD dynamos in the early universe. The magnetic field strength resulting from a chiral dynamo as a function of correlation length in comoving units and comparison with observational constraints. The differently colored lines show the chiral magnetically produced magnetic field strength in the range between the injection length μ^{-1} and the saturation length k_λ^{-1}; see Equations (52) and (21), respectively. The colors indicate different values of the chiral chemical potential: red refers to the value of μ_0 given in Equation (53), blue to $10^{-2}\mu_0$, and purple to $10^2\mu_0$. The dashed gray line is an upper limit on the intergalactic magnetic field from Zeeman splitting. Solid gray lines refer to the lower limits reported by Neronov & Vovk (2010; NV10) and Dermer et al. (2011; D+11). The vertical dotted gray lines show the horizon at $k_\mathrm{B}T = 100$ GeV and 100 MeV correspondingly. The thin colored arrows refer to the nonlinear evolution of magnetic fields in an inverse cascade in helical turbulence up to the final value as given in Banerjee & Jedamzik (2004; line BJ04).

$$\mu_0 \approx 4\alpha_\mathrm{em}\frac{\mu_5}{\hbar c} \approx 1.5 \times 10^{14}\ \mathrm{cm}^{-1}\frac{\mu_5}{100\ \mathrm{GeV}}.$$

$$\text{at } T = 100\ \mathrm{GeV}$$

$$\langle B^2\rangle\, \xi_\mathrm{M} \approx \mu/\lambda \ll 1.2 \times 10^{31}\, T_{100}^3\ \mathrm{G}^2\,\mathrm{cm}.$$

$$= 6 \times 10^{-15}\ \mathrm{G}^2\,\mathrm{cm}$$

$$\mathrm{erg\ cm}^{-3} = \mathrm{G}^2/4\pi$$

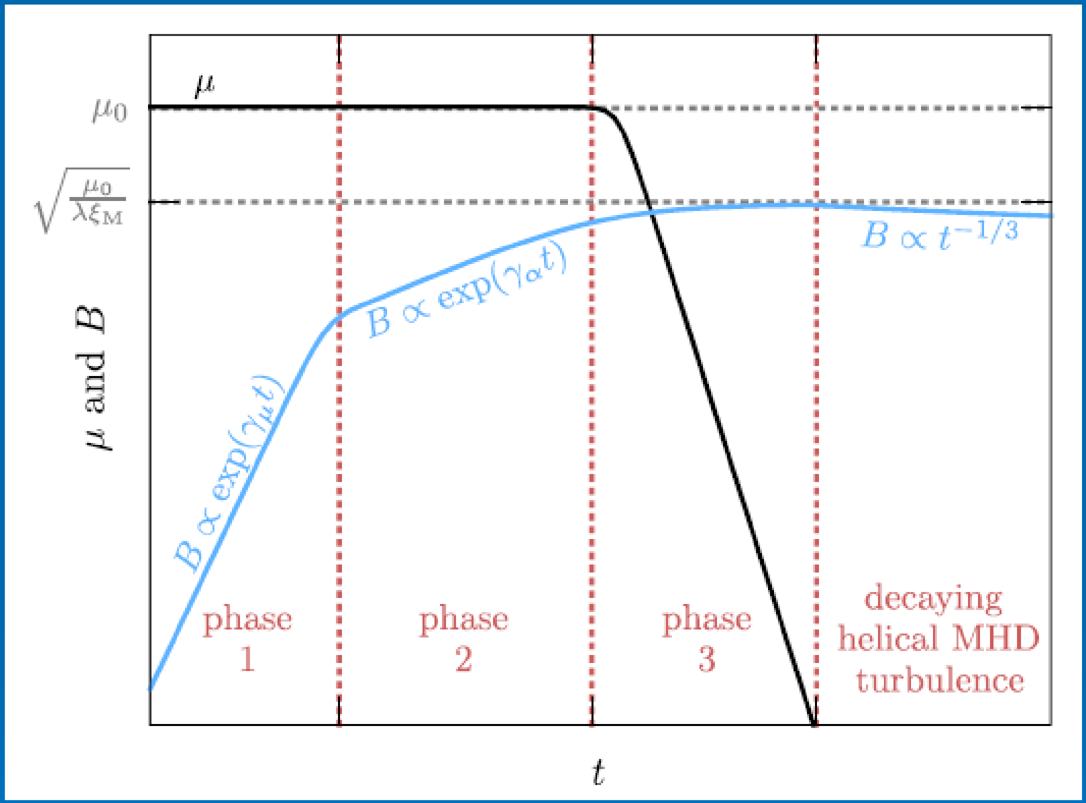
at $\mu_5 \sim k_\mathrm{B}T$

$$\langle B^2\rangle\, \xi_\mathrm{M} \simeq \frac{\hbar c}{4\alpha_\mathrm{em}}\frac{g_0}{g_*}n_\gamma^{(0)} \simeq 6 \times 10^{-38}\ \mathrm{G}^2\,\mathrm{Mpc}.$$

$$g_0/g_* \approx 3.36/106.75$$

$$n_\gamma^{(0)} = 411\ \mathrm{cm}^{-3}.$$

Four Phases of Magnetic Field Evolution



(1) small-scale chiral dynamo instability;

(2) production of small-scale turbulence, inverse transfer of magnetic energy, and generation of a large-scale magnetic field by the chiral α_μ effect;

(3) saturation of the large-scale chiral dynamo by a decrease of the CME controlled by the conservation law for the total chirality: $\lambda \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle / 2 + \langle \mu \rangle = \mu_0$.

Cosmic Ray Current-Driven Turbulence (Bell Turbulence) and Mean-Field Dynamo Effect

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Bell Instability in Turbulence with Cosmic Rays

A. Bell, MNRAS 353, 550 (2004)

- **Equation of plasma motion:**

$$\rho \frac{\mathrm{D}U}{\mathrm{D}t} = -\nabla P + \frac{1}{4\pi}(\nabla \times B) \times B + F_{\nu} - \frac{1}{c} J^{\mathrm{cr}} \times B$$

- **Induction and continuity equations:**

$$\frac{\partial B}{\partial t} = \nabla \times (U \times B - \eta \nabla \times B + c\, J^{\mathrm{cr}}/\sigma),$$

- **Growth rate of the Bell Instability:**

$$\gamma_B = \left[\frac{|\omega_{\mathrm{A}}\, \omega^{\mathrm{cr}}\, k_z|}{k} - \omega_{\mathrm{A}}^2 \right]^{1/2} - \eta k^2,$$

$$\omega^{\mathrm{cr}} = c^{-1}\, J^{\mathrm{cr}}\, (4\pi/\rho)^{1/2}$$

$$\omega_{\mathrm{A}} = k \cdot v_{\mathrm{A}}$$

$$v_{\mathrm{A}} = B_{*}/\,(4\pi\rho)^{1/2}$$

$$J^{\mathrm{eq}} + J^{\mathrm{cr}} = 0,\ B^{\mathrm{eq}} = B_{*} = \mathrm{const},\ U^{\mathrm{eq}} = 0, \text{ and } \rho^{\mathrm{eq}} = \rho_{*}$$

Kinetic Helicity for the Bell Mode

Rogachevskii I., Kleeorin N., Brandenburg A., Eichler D., Astrophys. J.753, 6 (2012)

- **Velocity fluctuations:**

$$\frac{\partial \boldsymbol{u}^{(0)}}{\partial t} \propto -\frac{1}{c\,\overline{\rho}}\overline{\boldsymbol{J}^{\rm cr}}\times \boldsymbol{b}^{(0)}.$$

- **Vorticity fluctuations:**

$$\frac{\partial}{\partial t}\boldsymbol{\nabla}\times \boldsymbol{u}^{(0)} \propto \frac{1}{c\,\overline{\rho}}(\overline{\boldsymbol{J}^{\rm cr}}\cdot\boldsymbol{\nabla})\boldsymbol{b}^{(0)}.$$

- **Kinetic helicity equation:**

$$\frac{\partial}{\partial t}\overline{\boldsymbol{u}^{(0)}\cdot(\boldsymbol{\nabla}\times \boldsymbol{u}^{(0)})} \propto -\frac{\overline{J_j^{\rm cr}}}{c\,\overline{\rho}}\left(\overline{\boldsymbol{b}^{(0)}\times(\boldsymbol{\nabla}\times \boldsymbol{u}^{(0)})_j} - \overline{u_n^{(0)}\nabla_j b_n^{(0)}}\right).$$

- **Kinetic helicity:**

$$\overline{\boldsymbol{u}^{(0)}\cdot(\boldsymbol{\nabla}\times \boldsymbol{u}^{(0)})} \propto \left(\frac{\tau^2\,\overline{J_j^{\rm cr}}\,\overline{J_i^{\rm cr}}}{c^2\,\overline{\rho}^2}\right)\varepsilon_{jnm}\,\overline{b_m^{(0)}\nabla_i b_n^{(0)}} + \frac{\tau\,\overline{J_j^{\rm cr}}}{c\,\overline{\rho}}\,\overline{u_n^{(0)}\nabla_j b_n^{(0)}},$$

- **Alpha effect:**

$$\alpha_{ij}^{(\rm I)} \propto \tau\left(\varepsilon_{inm}\,\overline{u_m^{(0)}\nabla_j u_n^{(0)}} + \varepsilon_{jnm}\,\overline{u_m^{(0)}\nabla_i u_n^{(0)}}\right),$$

$$\alpha_1^{\rm cr} = -C_1\,\tau_0\,\overline{\boldsymbol{u}^{(0)}\cdot(\boldsymbol{\nabla}\times \boldsymbol{u}^{(0)})},$$

Mechanism for the New Kind of Alpha Effect

- **Magnetic fluctuations:**

$$\frac{\partial \boldsymbol{b}^{(1)}}{\partial t} \propto (\tilde{\boldsymbol{B}} \cdot \boldsymbol{\nabla}) \boldsymbol{u}^{(0)}.$$

- **Velocity fluctuations:**

$$\frac{\partial \boldsymbol{u}^{(1)}}{\partial t} \propto -\frac{1}{c\,\overline{\rho}} \overline{\boldsymbol{J}^{\mathrm{cr}}} \times \boldsymbol{b}^{(1)}.$$

$$\boldsymbol{u}^{(1)} \propto -\frac{\tau}{c\,\overline{\rho}} \overline{\boldsymbol{J}^{\mathrm{cr}}} \times \boldsymbol{b}^{(1)} \propto -\frac{\tau^2}{c\,\overline{\rho}} \overline{\boldsymbol{J}^{\mathrm{cr}}} \times (\tilde{\boldsymbol{B}} \cdot \boldsymbol{\nabla}) \boldsymbol{u}^{(0)},$$

- **Mean electromotive force:**

$$\boldsymbol{\mathcal{E}}^{(\mathrm{II})} = \overline{\boldsymbol{u}^{(1)} \times \boldsymbol{b}^{(0)}}.$$

$$\mathcal{E}_i^{(\mathrm{II})} \propto \frac{\tau^2}{c\,\overline{\rho}} \left(\overline{J_i^{\mathrm{cr}}}\ \overline{b_n^{(0)} \nabla_j u_n^{(0)}} - \overline{J_n^{\mathrm{cr}}}\ \overline{b_n^{(0)} \nabla_j u_i^{(0)}} \right) \tilde{B}_j.$$

- **New kind of alpha effect:**

$$\alpha_2^{\mathrm{cr}} = C_2 \left(\frac{4\pi}{c} \frac{\overline{J^{\mathrm{cr}}}\,\ell_0}{\overline{B}_*} \right)^{1/2} \overline{V}_{\mathrm{A}}\ \mathrm{sgn}\,(\overline{\boldsymbol{J}^{\mathrm{cr}}} \cdot \boldsymbol{B}_*),$$

Mean-Field Dynamo

- **Toroidal magnetic field:**

$$\frac{\partial \tilde{B}_z}{\partial t} = -\alpha^{\rm cr}_{yy} \nabla^2_x \tilde{A} - \alpha^{\rm cr}_{xx} \nabla^2_y \tilde{A} + \eta_t \Delta_{\rm H} \tilde{B}_z ,$$

- **Poloidal magnetic field:**

$$\frac{\partial \tilde{A}}{\partial t} = \alpha^{\rm cr}_{zz} \tilde{B}_z + \eta_t \Delta_{\rm H} \tilde{A},$$

- **Growth rate of the mean-field dynamo:**

$$\gamma_{\rm inst} = |K| \sqrt{\alpha^{\rm cr}_{yy} \alpha^{\rm cr}_{zz}} - \eta_t K^2,$$

$$\alpha^{\rm cr}_{ij} = \alpha^{\rm (I)}_{ij} + \alpha^{\rm (II)}_{ij}.$$

- **Magnetic anisotropy:**

$$\frac{\tilde{B}^2_z}{\tilde{B}^2_x + \tilde{B}^2_y} = \frac{\alpha^{\rm cr}_{xx}}{\alpha^{\rm cr}_{zz}},$$

DNS set-up for Bell Turbulence

Rogachevskii I., Kleeorin N., Brandenburg A., Eichler D., Astrophys. J.753, 6 (2012)

$$\rho \frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} = \frac{1}{4\pi}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} - c_{\mathrm{s}}^2 \boldsymbol{\nabla} \rho + \boldsymbol{\nabla} \cdot (2\nu\rho\mathbf{S}) - \frac{1}{c}\boldsymbol{J}^{\mathrm{cr}} \times \boldsymbol{B},$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{U} \times \boldsymbol{B} + \eta \nabla^2 \boldsymbol{A},$$

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \rho \boldsymbol{U},$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{\nabla} \times \boldsymbol{A}$$

Isothermal equation of state with constant sound speed

$$\mathrm{P}_m = 1$$

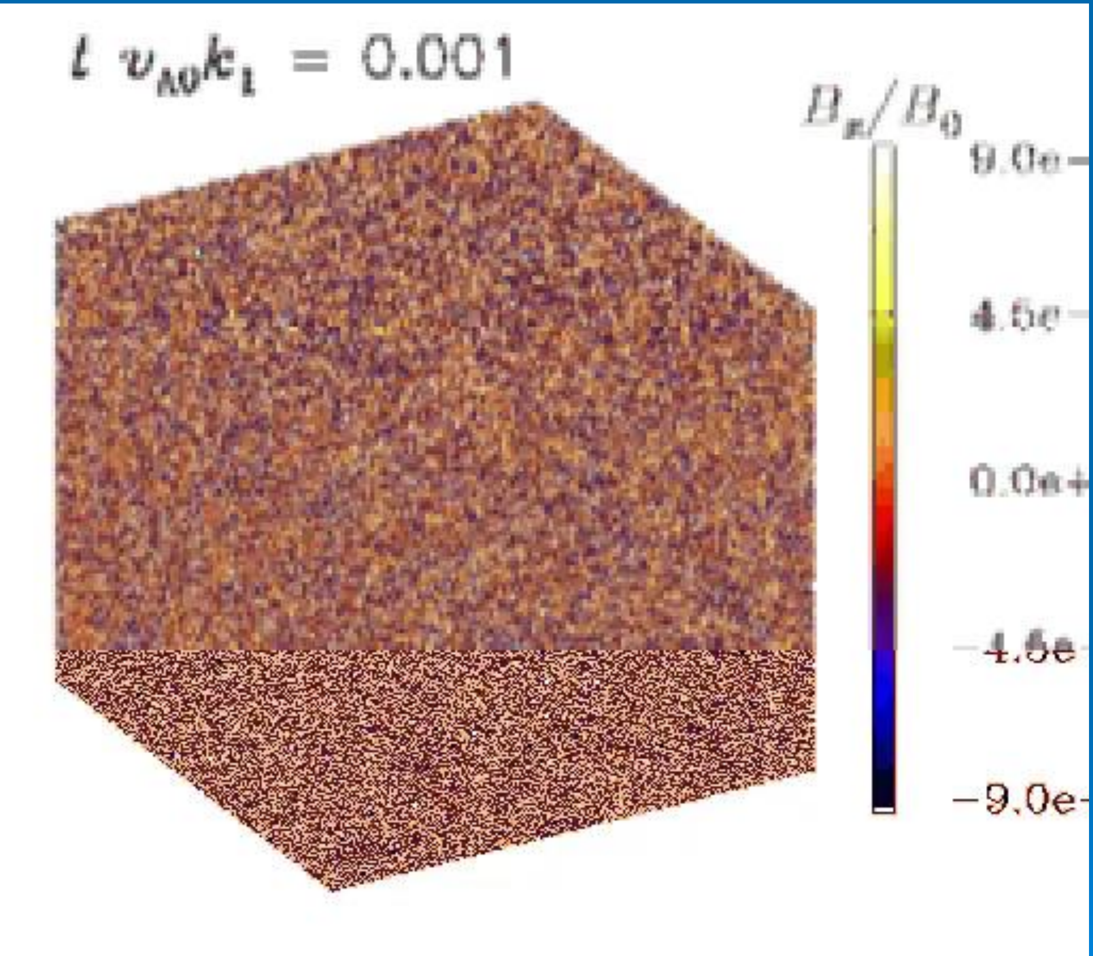
$$c_s = 10 \qquad B_0 = 10^{-2}$$

$$L = 16\pi \qquad J^{\mathrm{cr}} = 0.1$$

All simulations are performed with the PENCIL CODE, which uses sixth-order explicit finite differences in space and a third-order accurate time stepping method.

BOUNDARY CONDITIONS are periodic in 3D.

DNS for Bell Turbulence



- In the early stage there is an excitation of small-scale instability (Bell-instability).
- The instability results in a production of small-scale Bell turbulence.
- Formation of large-scale magnetic structures due to mean-field dynamo (during the interval 0.2-0.4).

DNS: Formation of Magnetic Structures

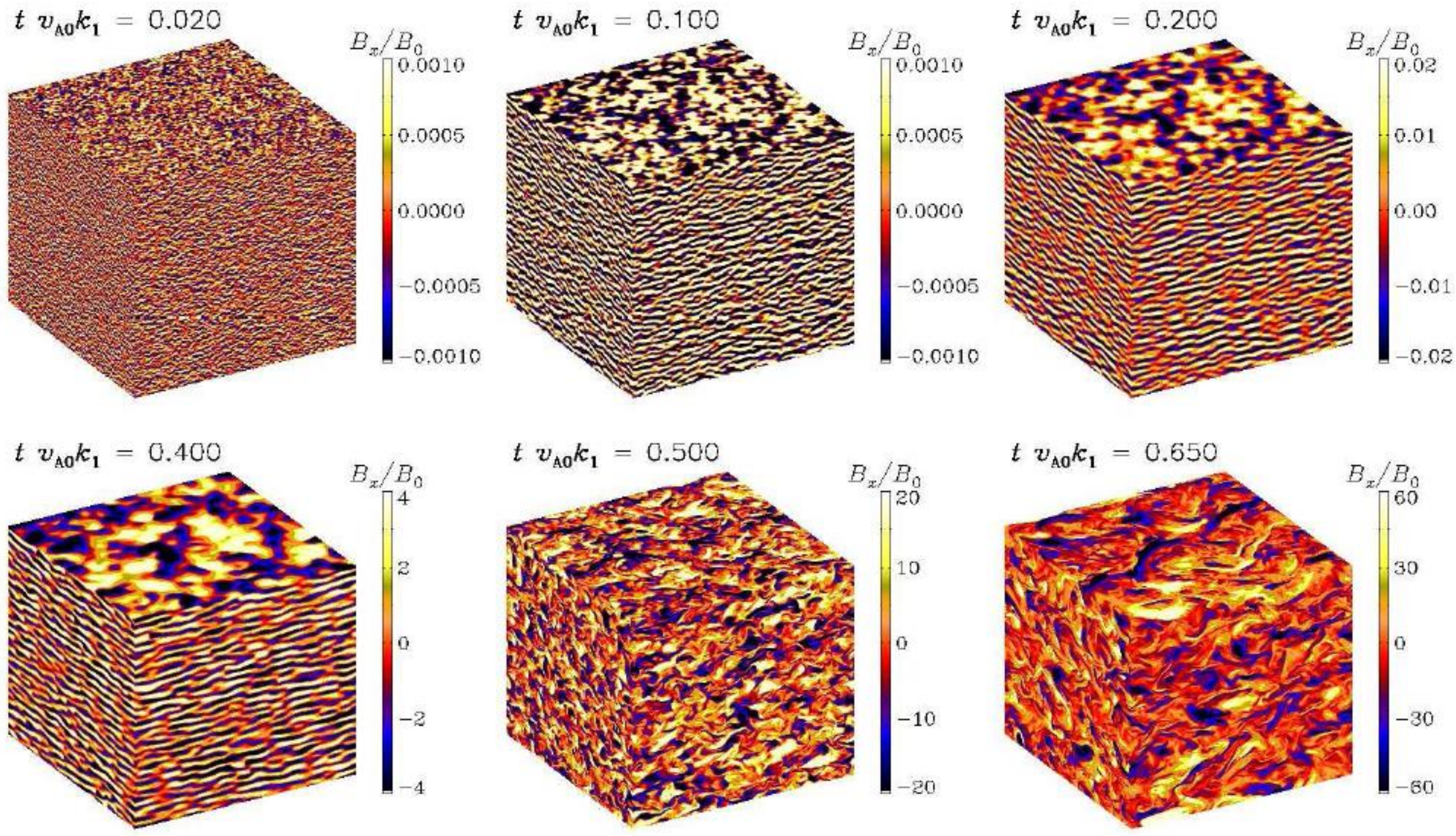
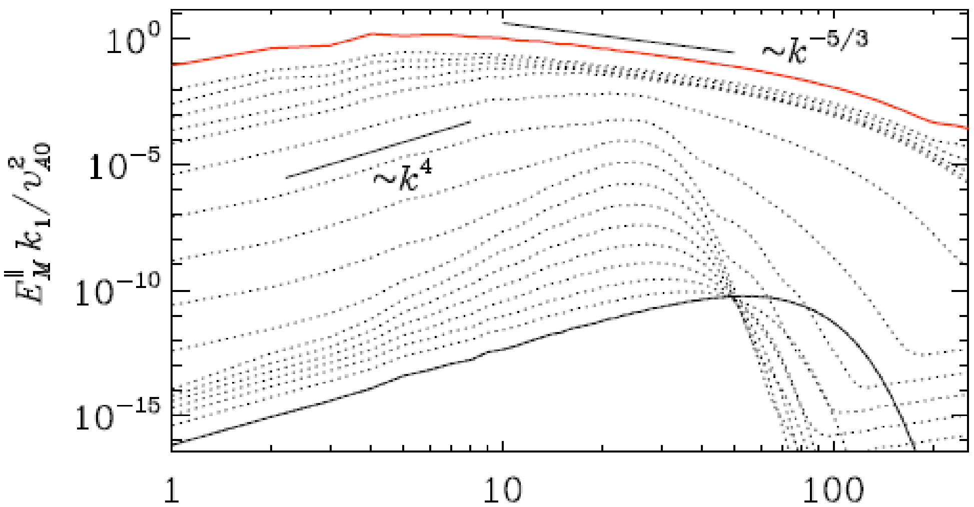
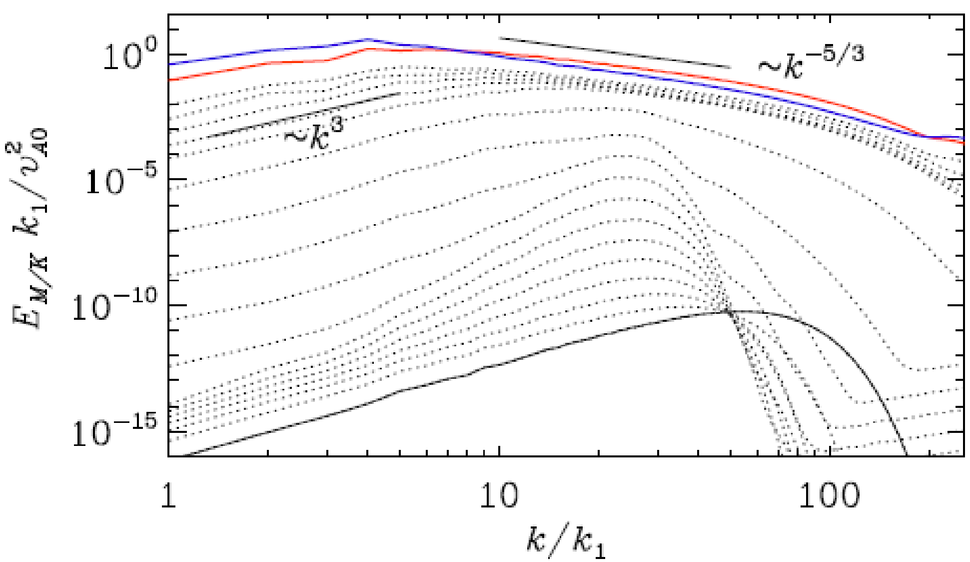


FIG. 1.— Visualization of B_x/B_0 on the periphery of the computational domain using 512^3 mesh points for $J^{\rm cr} = 0.1$, $B_0 = 0.01$, $k_1 = 1/8$ (so that $\mathcal{J} = 80$), and $\nu = \eta = 10^{-3}$ (so that the Lundquist number Lu $= 80$).

DNS: Bell Turbulence Spectra





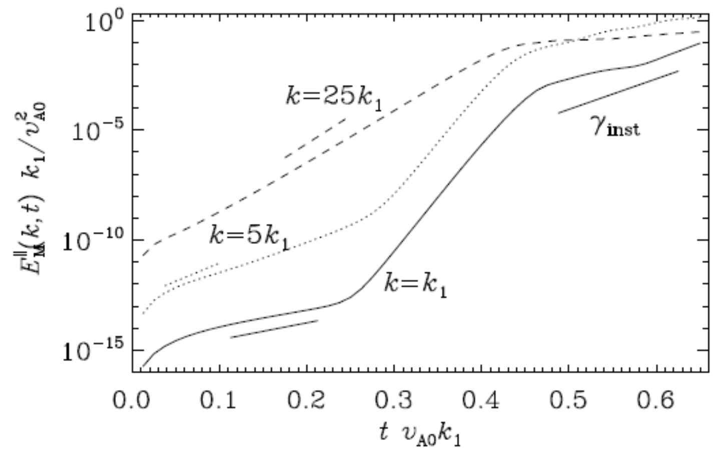


FIG. 2.— Time evolution of $E_M^{\parallel}\, k_1/v_{A0}^2$ for modes with different wavenumbers for the run with $\mathcal{J} = 80$. The short straight lines show the growth rate of the Bell instability, as given by Equation (9) for modes with three selected values of k, as well as the value of $\gamma_{\rm inst}$ as given by Equation (33).

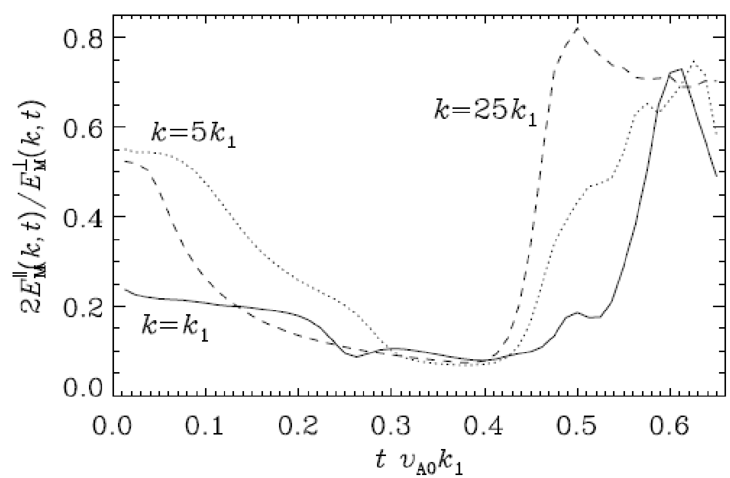


FIG. 3.— Time evolution of the ratio of the spectral vertical (along the imposed field B_0) and horizontal magnetic energies $2E_M^{\parallel}/E_M^{\perp}$ for the run with $\mathcal{J} = 80$.

Comparison Theory and DNS

Rogachevskii I., Kleeorin N., Brandenburg A., Eichler D., Astrophys. J. 753, 6 (2012)

- **Alpha effect:**

$$\alpha_{zz}^{\rm theory}\, v_{\rm A0}/u_{\rm rms}^2 \approx 0.5.$$

$$\alpha_{zz}^{\rm DNS}\, v_{\rm A0}/u_{\rm rms}^2 \approx 0.6$$

- **Dynamo growth rate:**

$$\gamma^{\rm theory}/v_{\rm A0}k_1 \approx 24$$

$$\gamma_{\rm inst}^{\rm DNS}/v_{\rm A0}k_1 \approx 20.$$

- **Anisotropy ratio:**

$$[\tilde{B}_z^2/(\tilde{B}_x^2 + \tilde{B}_y^2)]^{\rm theory} \approx 0.08,$$

$$[\tilde{B}_z^2/(\tilde{B}_x^2 + \tilde{B}_y^2)]^{\rm DNS} \approx 0.06.$$

Cosmic Rays in Supernova Remnants

- Ultra-high-energy gamma-ray emissions from supernova remnants indicate about existence of strong magnetic fields about 100-1000 μG
- **Young supernova remnants:**

$$\ell_0 = 3 \times 10^{18}\ \mathrm{cm},$$

$$n^{\mathrm{cr}} = 10^{-10} - 10^{-7}\ \mathrm{cm}^{-3},$$

$$n_i + n^{\mathrm{cr}} = n_e,$$

$$v^{\mathrm{cr}} = (0.3 - 1) \times 10^{10}\ \mathrm{cm/s}$$

$$n_i = 1\ \mathrm{cm}^{-3},$$

$$J^{\mathrm{eq}} + J^{\mathrm{cr}} = 0,\ B^{\mathrm{eq}} = B_* = \mathrm{const},\ U^{\mathrm{eq}} = 0,\ \text{and}\ \rho^{\mathrm{eq}} = \rho_*$$

$$B_* = 1\,\mu\mathrm{G},$$

$$B_{\mathrm{dynamo}} = 10^2 - 10^3\,\mu\mathrm{G},$$

Cosmic Rays in Supernova Remnants

- **Estimates for young supernova remnants:**

$$|\alpha^{\rm cr}| = C_\alpha \left(\omega_B^{\rm cr}\,\ell_0\right)^{1/2} \left(v^{\rm cr}\right)^{1/2} \left(\frac{n^{\rm cr}\,m^{\rm cr}}{n_i m_i}\right)^{1/2} \approx 10^9\text{–}10^{10}\ {\rm cm/s}$$

$$\ell_0 = 3\times 10^{18}\ {\rm cm},$$

$$\alpha^{\rm cr} \approx 3\left(\frac{J^{\rm cr}\,\ell_0}{B_*}\right)^{1/2} \frac{B_*}{\sqrt{\mu\rho}}$$

$$n^{\rm cr} = 10^{-10} - 10^{-7}\ {\rm cm}^{-3},$$

$$v^{\rm cr} = (0.3-1)\times 10^{10}\ {\rm cm/s}$$

$$B_{\rm dynamo} = 10^2 - 10^3\,\mu{\rm G},$$

$$n_i = 1\ {\rm cm}^{-3},$$

$$B_* = 1\,\mu{\rm G},$$

$$\omega_B^{\rm cr} = 10^{-2}\ {\rm s}^{-1},$$

Conclusions

- We have chiral magnetic effect which occurs due to relativistic fermions in a magnetized plasma.

- In chiral magnetohydrodynamics, magnetic field evolution proceeds in distinct stages

(i) small-scale chiral dynamo instability;

(ii) first nonlinear stage when the Lorentz force drives small-scale chiral turbulence;

(iii) development of inverse energy transfer with a k^{-2} magnetic energy spectrum;

(iv) generation of large-scale magnetic field by the new chiral α_μ - effect and saturation.

We have considered turbulent flows (chirally produced or forced turbulence) with a zero mean kinetic helicity.

- The origin of MHD turbulence and evolution of magnetic fields in the early Universe. This MHD turbulence can serve as a source of GW after EWPT.

THE END