Numerical simulations of gravitational waves from early universe turbulence

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A. Roper Pol *et al.*, *Geophys. Astrophys. Fluid Dyn.*, DOI:10.1080/03091929.2019.1653460, arXiv:1807.05479 (2019)

A. Roper Pol *et al.*, arXiv:1903.08585

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Introduction and Motivation

- Generation of cosmological gravitational waves (GWs) during phase transitions and inflation
  - **Electroweak phase transition** ~ 100 GeV
  - Quantum chromodynamic (QCD) phase transition ~ 100 MeV
  - Inflation
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- Magnetohydrodynamic (MHD) sources of GWs:
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- Numerical simulations using Pencil Code to solve:
  - Relativistic MHD equations
  - Gravitational waves equation

GWs equation for an expanding flat Universe

- Assumptions: isotropic and homogeneous Universe
- Friedmann–Lemaître–Robertson–Walker (FLRW) metric $\gamma_{ij} = a^2 \delta_{ij}$
- Tensor-mode perturbations above the FLRW model:

$$g_{ij} = a^2 \left(\delta_{ij} + h_{ij}^{\rm phys} \right)$$

- GWs equation is[1]

$$\left(\partial_t^2 - c^2 \nabla^2 \right) h_{ij} = \frac{16\pi G}{ac^2} T_{ij}^{\rm TT}$$

  - h_{ij} are rescaled $h_{ij} = a h_{ij}^{\rm phys}$
  - Comoving spatial coordinates $\nabla = a \nabla^{\rm phys}$
  - Conformal time $\mathrm{d}t = a \, \mathrm{d}t^{\rm phys}$
  - Comoving stress-energy tensor components $T_{ij} = a^4 T_{ij}^{\rm phys}$
  - Radiation-dominated epoch such that $a'' = 0$

[1] L. P. Grishchuk, *Sov. Phys. JETP*, 40, 409-415 (1974)

Normalized GW equation[2]

$$\left(\partial_t^2 - \nabla^2\right) h_{ij} = 6\, T_{ij}^{\mathrm{TT}} / t$$

Properties

- All variables are normalized and non-dimensional
- Conformal time is normalized with t_*
- Comoving coordinates are normalized with c/H_*
- Stress-energy tensor is normalized with $\mathcal{E}^*_{\mathrm{rad}} = 3H_*^2 c^2/(8\pi G)$
- Scale factor is $a_* = 1$, such that $a = t$

[2] A. Roper Pol *et al.*, *Geophys. Astrophys. Fluid Dyn.*, DOI:10.1080/03091929.2019.1653460, arXiv:1807.05479 (2019)

Gravitational waves equation

Properties

- Tensor-mode perturbations are gauge invariant
- h_{ij} has only two degrees of freedom: h^{+}, $h^{\times}$
- The metric tensor is traceless and transverse (TT gauge)

Contributions to the stress-energy tensor

$$T^{\mu\nu} = (p/c^2 + \rho) U^\mu U^\nu + p g^{\mu\nu} + F^{\mu\gamma} F^{\nu}{}_{\gamma} - \frac{1}{4} g^{\mu\nu} F_{\lambda\gamma} F^{\lambda\gamma}$$

- From fluid motions
  $T_{ij} = \left(p/c^2 + \rho\right) \gamma^2 u_i u_j + p\delta_{ij}$
  Relativistic equation of state:
  $p = \rho c^2/3$
- From magnetic fields:
  $T_{ij} = -B_i B_j + \delta_{ij} B^2/2$

MHD equations

Conservation laws

$$T^{\mu\nu}{}_{;\nu} = 0$$

Relativistic MHD equations are reduced to[3]

MHD equations

$$\frac{\partial \ln \rho}{\partial t} = -\frac{4}{3}\left(\nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \ln \rho\right) + \frac{1}{\rho}\left[\boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) + \eta \boldsymbol{J}^2\right)$$

$$\frac{D\boldsymbol{u}}{Dt} = \frac{4}{3}\left(\nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \ln \rho\right) - \frac{\boldsymbol{u}}{\rho}\left[\boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) + \eta J^2\right] - \frac{1}{4}\nabla \ln \rho + \frac{3}{4\rho}\boldsymbol{J} \times \boldsymbol{B} + \frac{2}{\rho}\nabla \cdot (\rho \nu \boldsymbol{S})$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{J})$$

for a flat expanding universe with comoving and normalized p, ρ, B_i, u_i, and conformal time t.

[3] A. Brandenburg, K. Enqvist, and P. Olesen, *Phys. Rev. D*, 54(2):12911300, 1996.

Traceless and transverse projection

- CFL condition for stability:

$$\delta_t \leq C_{\rm CFL} \delta x / U_{\rm eff},$$

$U_{\rm eff} = |\boldsymbol{u}| + (c_{\rm s}^2 + v_{\rm A}^2)^{1/2}$, $c_{\rm s}^2 = 1/3$, $v_{\rm A}^2 = B^2/\rho$.

- Projection of $T_{ij}^{\rm TT}$ requires non-local Fourier transform $\tilde{T}_{ij}$ (computationally expensive):

$$\tilde{T}_{ij}^{\rm TT} = \left(P_{il} P_{jm} - \frac{1}{2} P_{ij} P_{lm} \right) \tilde{T}_{lm}$$

where $P_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j$

Linear polarization modes h^+ and $h^\times$

Linear polarization basis (defined in Fourier space)

$$e_{ij}^{+} = (\boldsymbol{e}_1 \times \boldsymbol{e}_1 - \boldsymbol{e}_2 \times \boldsymbol{e}_2)_{ij}$$

$$e_{ij}^{\times} = (\boldsymbol{e}_1 \times \boldsymbol{e}_2 + \boldsymbol{e}_2 \times \boldsymbol{e}_1)_{ij}$$

Orthogonality property

$e_{ij}^A e_{ij}^B = 2\delta_{AB}$, where $A, B = +, \times$

h^+ and $h^\times$ modes

$$\tilde{h}^+ = \frac{1}{2} e_{ij}^{+} \tilde{h}_{ij}^{\mathrm{TT}}$$

$$\tilde{h}^\times = \frac{1}{2} e_{ij}^{\times} \tilde{h}_{ij}^{\mathrm{TT}}$$

Solution 1

- Solve the GWs equation sourced by the stress-energy tensor[4]

$$\left(\partial_t^2 - \nabla^2\right) h_{ij} = 6\, T_{ij}/t$$

- Project h_{ij}^{TT} only when we are interested in spectra

$$\tilde{h}_{ij}^{\mathrm{TT}} = \left(P_{il} P_{jm} - \frac{1}{2} P_{ij} P_{lm} \right) \tilde{h}_{lm}$$

- Compute $\tilde{h}^{+}$, $\tilde{h}^{\times}$ modes

[4] A. Roper Pol *et al.*, *Geophys. Astrophys. Fluid Dyn.*, DOI:10.1080/03091929.2019.1653460, arXiv:1807.05479 (2019)

Gravitational waves energy density

GWs energy density:

$$\Omega_{\rm GW}(t) = \mathcal{E}_{\rm GW}/\mathcal{E}_{\rm rad}^*, \quad \mathcal{E}_{\rm rad}^* = \frac{3H_*^2 c^2}{8\pi G}$$

$$\Omega_{\rm GW}(t) = \int_{-\infty}^{\infty} \Omega_{\rm GW}(k,t)\, {\rm d} \ln k$$

$$\boldsymbol{\Omega_{\rm GW}(k,t)} = \frac{k}{6H_*^2} \int_{4\pi} \left(\left| \dot{\tilde{h}}_+^{\rm phys} \right|^2 + \left| \dot{\tilde{h}}_\times^{\rm phys} \right|^2 \right) k^2 \, {\rm d}\Omega_k$$

Antisymmetric GWs energy density:

$$\Xi_{\rm GW}(t) = \int_{-\infty}^{\infty} \Xi_{\rm GW}(k,t)\, {\rm d} \ln k$$

$$\boldsymbol{\Xi_{\rm GW}(k,t)} = \frac{k}{6H_*^2} \int_{4\pi} 2{\rm Im} \left(\dot{\tilde{h}}_+^{\rm phys} \dot{\tilde{h}}_\times^{\rm phys,*} \right) k^2 \, {\rm d}\Omega_k$$

$$H_* \approx 2.066 \cdot 10^{-11} \ {\rm s}^{-1} \left(\frac{T_*}{100 \ {\rm GeV}} \right)^2 \left(\frac{g_*}{100} \right)^{1/2}$$

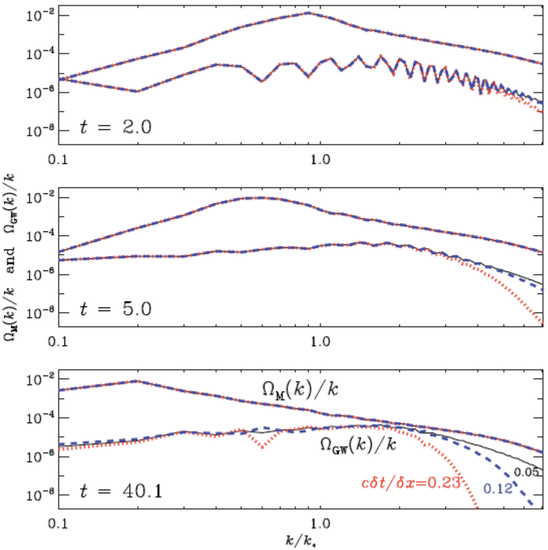
Characteristic amplitude of gravitational waves

GWs amplitude:

$$h_{\rm c}^2(t) = \int_{-\infty}^{\infty} h_{\rm c}^2(k,t)\, {\rm d} \ln k$$

$$\mathbf{h_{\rm c}^2(k,t)} = \int_{4\pi} \left(\left| \tilde{h}_+^{\rm phys} \right|^2 + \left| \tilde{h}_\times^{\rm phys} \right|^2 \right) k^2 \, {\rm d}\Omega_k$$

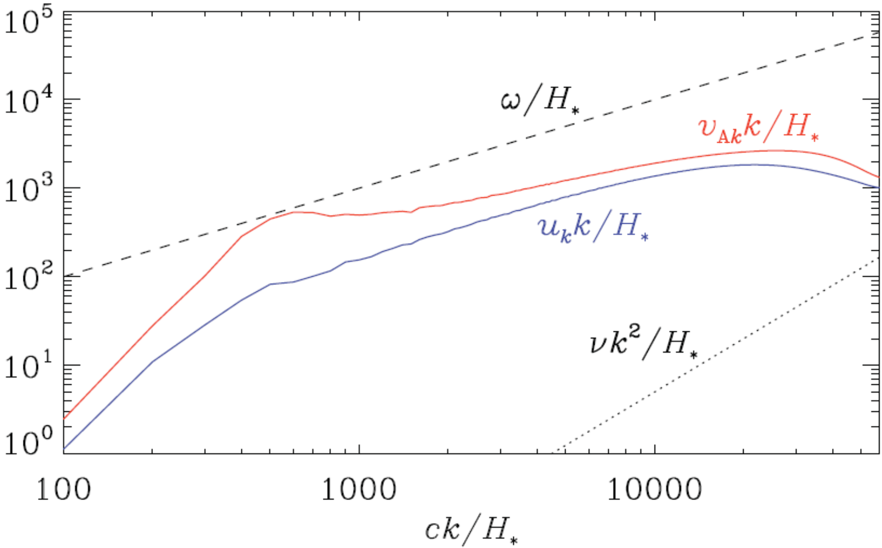
Numerical accuracy[5]



- CFL condition is not enough for GW solution to be numerically accurate
- $c\delta t/\delta x \sim 0.05 \ll 1$
- Higher resolution is required
- Hydromagnetic turbulence does not seemed to be affected

[5] A. Roper Pol *et al.*, *Geophys. Astrophys. Fluid Dyn.*, DOI:10.1080/03091929.2019.1653460, arXiv:1807.05479 (2019)

Frequency of oscillations of GWs vs MHD waves



Solution 2

- Compute Fourier transform of stress-energy tensor $\tilde{T}_{ij}$
- Project into TT gauge $\tilde{T}_{ij}^{\mathrm{TT}}$
- Compute $\tilde{T}^{+}$ and $\tilde{T}^{\times}$ modes
- Discretize time using δt from MHD simulations
- Assume $\tilde{T}^{+,\times}/t$ to be constant between subsequent timesteps (robust as $\delta t \to 0$)
- GW equation solved analytically between subsequent timesteps in Fourier space[6]

$$\begin{pmatrix} \omega \tilde{h} - 6\omega^{-1}\tilde{T}/t \\ \tilde{h}' \end{pmatrix}_{+,\times}^{t+\delta t} = \begin{pmatrix} \cos\omega\delta t & \sin\omega\delta t \\ -\sin\omega\delta t & \cos\omega\delta t \end{pmatrix} \begin{pmatrix} \omega \tilde{h} - 6\omega^{-1}\tilde{T}/t \\ \tilde{h}' \end{pmatrix}_{+,\times}^{t}$$

[6] A. Roper Pol *et al.*, *Geophys. Astrophys. Fluid Dyn.*, DOI:10.1080/03091929.2019.1653460, arXiv:1807.05479 (2019)

Numerical results for decaying MHD turbulence[7]

Initial conditions[8]

- Fully helical stochastic magnetic field
- Batchelor spectrum, i.e., $E_{\rm M} \propto k^4$ for small k
- Kolmogorov spectrum for inertial range, i.e., $E_{\rm M} \propto k^{-5/3}$
- Total energy density at t_* is $\sim 10\%$ to the radiation energy density
- Spectral peak at $k_{\rm M} = 100 \cdot 2\pi$, normalized with $k_H = 1/(cH)$

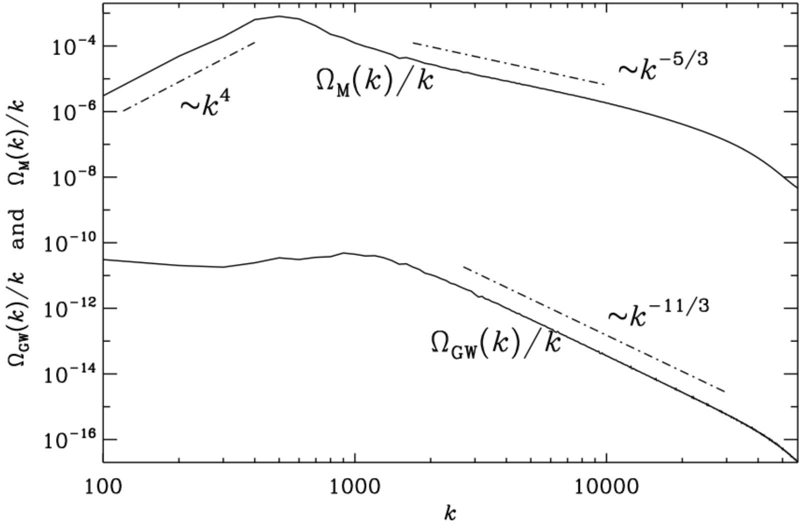
Numerical parameters

- 1152^3 mesh gridpoints
- 1152 processors
- Wall-clock time of runs is $\sim 1 - 5$ days

[7] A. Roper Pol, *et al.* arXiv:1903.08585

[8] A. Brandenburg, *et al.* *Phys. Rev. D* (2017)

Numerical results for decaying MHD turbulence



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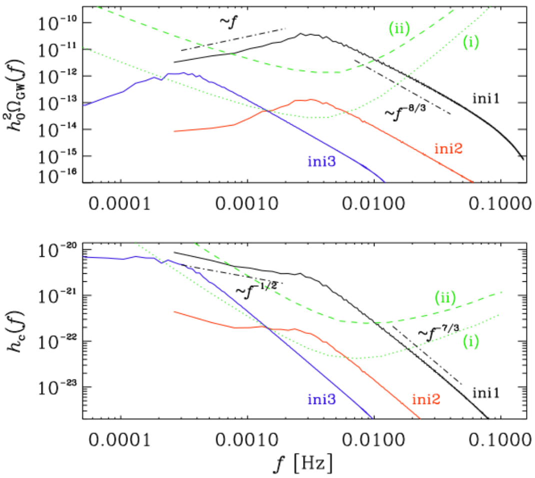
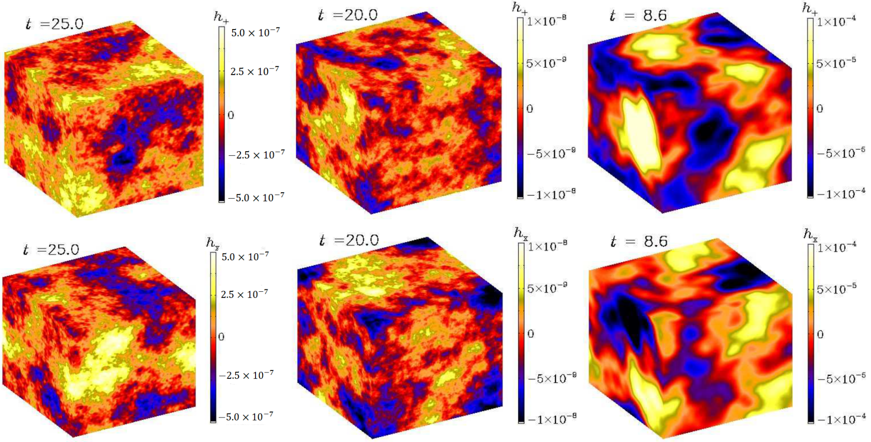


- ini1: $k_{\rm M} = 100$, $\Omega_{\rm M} \approx 0.1$
- ini2: $k_{\rm M} = 100$, $\Omega_{\rm M} \approx 0.01$
- ini3: $k_{\rm M} = 10$, $\Omega_{\rm M} \approx 0.01$

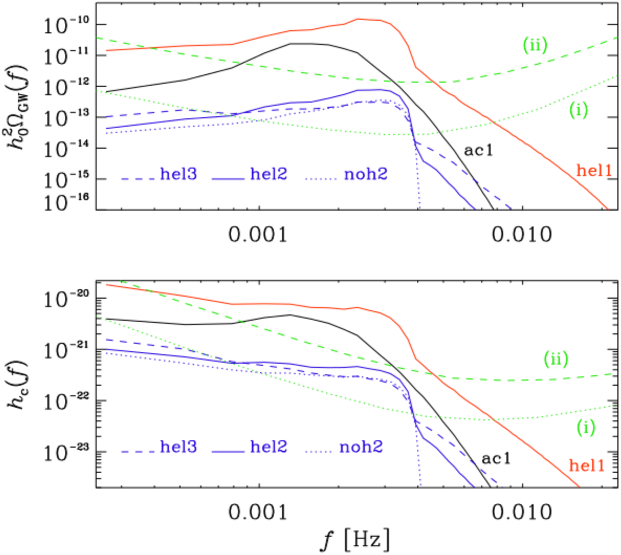
Numerical results for decaying MHD turbulence

Box results for positive initial helicity:

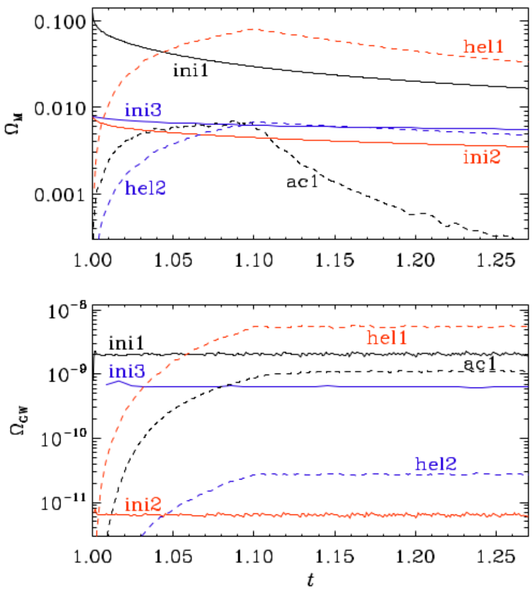


$k_M / k_H = 300$ $\quad$ $k_M / k_H = 60$ $\quad$ $k_M / k_H = 2$

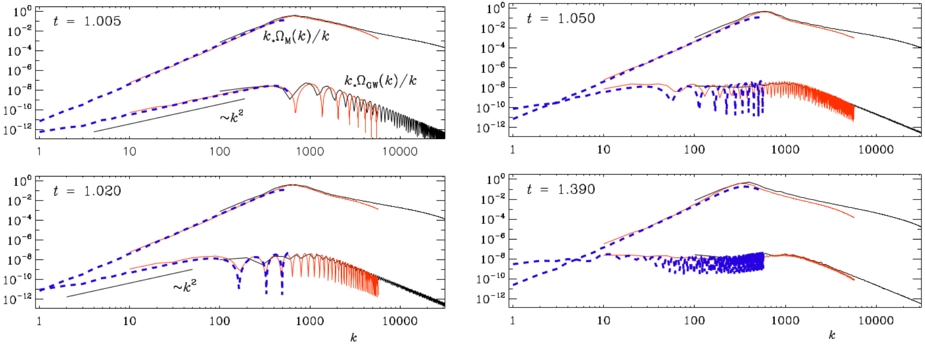
Forced turbulence (built-up primordial magnetic fields and hydrodynamic turbulence), low resolution



Time evolution of GW energy density



Early time evolution of GW energy density spectral slope



Detectability with LISA

LISA

- Laser Interferometer Space Antenna (LISA) is a space based GWs detector
- LISA is planned for 2034
- LISA was approved in 2017 as one of the main research missions of ESA
- LISA is composed by three spacecrafts in a distance of $\sim$2M km

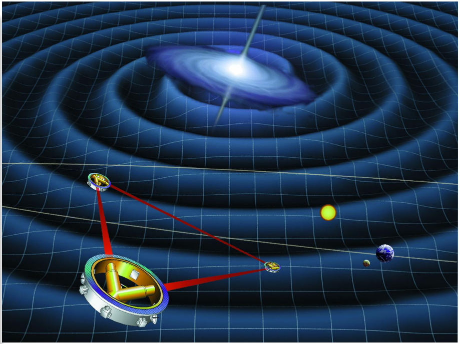


Figure: Artist's impression of LISA from Wikipedia

Detectability with LISA

- LISA sensitivity is usually expressed as $h_0^2 \Omega_{\rm GW}$
- $\Omega_{\rm GW}$ is the ratio of GWs energy density to critical energy density
- Critical energy density is

$$\mathcal{E}_{\rm crit} = \frac{3 H_0^2 c^2}{8\pi G}$$

- Current Hubble parameter is usually expressed as

$$H_0 = 100\, h_0 \ {\rm km\, s^{-1} Mpc^{-1}}$$

  where h_0 represents the uncertainties in the actual value of H_0
- We consider two different LISA configurations [7]
  - 4-link configuration with 2×10^9 m arm length after 5 years of duration
  - 6-link configuration with 6×10^9 m arm length after 5 years of duration

[7] C. Caprini et al., JCAP, 2016(04): 001001 (2016)

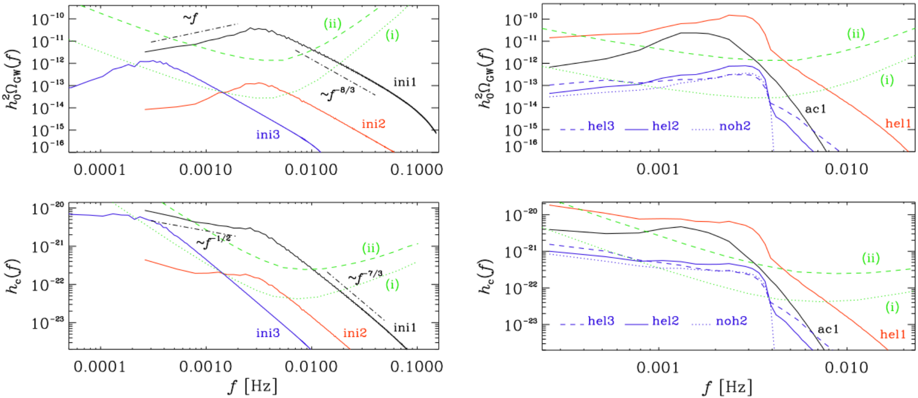
Detectability with LISA

GW energy density and characteristic amplitude

- Shifting due to the expansion of the universe:
  - $\Omega^0_{\rm GW}(k) = a_0^{-4}(H_*/H_0)^2\Omega_{\rm GW}(k, t_{\rm end})$
  - $h^0_{\rm c}(k) = a_0^{-1}h_{\rm c}(k, t_{\rm end})$
  - $f^0 = a_0^{-1}f$

$$a_0 \approx 1.254 \cdot 10^{15}\,(T_*/100\ {\rm GeV})\,(g_{\rm S}/100)$$

Detectability with LISA



Conclusions

- We have implemented a module within the PENCIL CODE that allows to obtain background stochastic GW spectra from primordial magnetic fields and hydrodynamic turbulence
- For some of our simulations we obtain a detectable signal by future mission LISA
- GW equation is normalized such that it can be easily scaled for different moments within the radiation-dominated epoch
- Novel f spectrum obtained for GWs in high frequencies range vs f^3 obtained from analytical estimates
- Bubble nucleation and magnetogenesis physics can be coupled to our equations for more realistic production analysis

The End
Thank You!