Quantum Gravity and Cosmology

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Introduction

In these lectures we will discuss the application of various approaches to quantum gravity to the early universe. We will cover traditional approaches to quantum cosmology based on minisuperspace models, i.e., quantum theories of exactly homogeneous geometries, to which inhomogeneities are added perturbatively; such models have been studied at least since the 1960s as simple models of quantum spacetime that do not assume completion by a consistent quantum theory of gravity. This traditional approach to quantum cosmology has seen renewed interest recently, and we will cover the foundations of these models, explain some key results and list more modern developments. In the second and third lecture we will focus on the cosmological applications of loop quantum gravity (LQG) and group field theory (GFT), two background-independent approaches to quantum gravity.

Let us start by giving some motivations for including quantum gravity into early universe cosmology. After all, the standard theoretical framework underlying modern cosmology does not seem to require quantum gravity: all that is needed is quantum fields such as the inflaton, living on a semiclassical near-deSitter spacetime, whose quantum fluctuations source perturbations in the geometry that are converted into the classical pattern of inhomogeneities observed in the cosmic microwave background. The typical energy scales required for inflation are usually below the Planck scale, away from the quantum gravity regime, and in the subsequent evolution of the universe we would not expect quantum gravity to play a role.

There are some open issues with this standard approach to explaining the origin of the universe. First of all, while the energy scale of inflation usually stays away from the Planck energy, many models would predict that the physical wavelength of some modes that become relevant for the cosmic microwave background would have been sub-Planckian at the beginning of inflation; this raises the *trans-Planckian problem* of inflationary cosmology[1]. More recently similar lines of argument have led to the *Trans-Planckian Censorship Conjecture* forbidding the appearance of such modes[2]. Being able to study cosmological perturbations explicitly in a theory of quantum gravity might shine more light on the possible implications of trans-Planckian physics on the early universe.

A different type of argument for the necessity for quantum gravity is that the standard framework of theoretical cosmology based on quantum field theory on curved spacetime, while self-consistent, lacks more fundamental justification. Making predictions from a given inflationary model, for example, depends sensitively on the choice of initial conditions. Usually these are given by the Bunch–Davies vacuum, but we do not know how this initial state emerges from a presumably violent Planckian quantum gravity regime in which classical spacetime may not be a meaningful concept. The possibility of explaining the initial conditions of inflation was one of the primary motivations driving the development of quantum cosmology in the 1980s.

The existence of a (Big Bang) singularity in the past is an inevitable outcome of classical general

[1] J. Martin and R. H. Brandenberger, "Trans-Planckian problem of inflationary cosmology," Phys. Rev. D **63** (2001) 123501, `arXiv:hep-th/0005209`

[2] A. Bedroya and C. Vafa, "Trans-Planckian Censorship and the Swampland," `arXiv:1909.11063`

relativity together with the standard energy conditions, and extends to inflationary spacetimes[3]. Hence the completion of standard cosmology through quantum gravity is a highly relevant issue also for inflationary cosmology. One of the main achievements of some quantum gravity scenarios that we will discuss is to replace the Big Bang singularity by a "big bounce".

Now that we have hopefully motivated why the application of quantum gravity to cosmology is useful, let us give some disclaimers as to the content of these lectures:

- All approaches we will discuss are to some extent incomplete as fundamental theories; they either do not build on a full theory of quantum gravity, meaning they probably cannot consistently embedded into one, or rely on theories of quantum gravity (LQG and GFT) whose low-energy and continuum limits are not completely understood. This means some not fully justified steps are needed to obtain cosmological models. One should keep in mind that this is to be expected in any type of quantum gravity "phenomenology".
- We will not include or discuss string theory or supergravity approaches in these lectures, which have given their own ideas to the extension of standard early universe cosmology. In particular, a key concept coming out of them is holography, the idea that dynamics of quantum gravity can be specified on the boundary of a "bulk" spacetime. We will not discuss holography and will not consider spacetimes with nontrivial boundary dynamics. Unless specified otherwise, we think of compact spatial sections without boundary, such as a 3-sphere or a 3-torus.
- The main application of quantum gravity to cosmology, given the technical complications with describing spacetimes without a certain degree of symmetry, is to homogeneous and (often) isotropic spacetimes to which perturbations are then added later. That is, the main effect of quantum gravity is to modify the "background" geometry for perturbations.

1 Quantum Cosmology without Quantum Gravity

We will first discuss the traditional approach to quantum cosmology based on a quantisation of the standard Einstein–Hilbert action coupled to matter fields (usually taken to be one or more scalar fields, since this is most relevant for the early universe). The section is titled "without Quantum Gravity" since such a quantisation does not exist in the most literal sense: pure Einstein–Hilbert gravity does not make sense as a quantum theory (nonperturbatively, there might be hope for an *asymptotically safe* fixed point of gravity whose dynamics would however include an infinite number of higher-order terms when expressed as an action, and certainly not correspond to the pure Einstein–Hilbert action). No such higher order terms are included in traditional quantum cosmology, which is often called *minisuperspace or Wheeler–DeWitt quantum cosmology*. A common viewpoint on this is that one works in a semiclassical approximation to quantum gravity, which might be sufficient to describe a classical universe.

[3]A. Borde, A. H. Guth and A. Vilenkin, "Inflationary Spacetimes Are Incomplete in Past Directions," Phys. Rev. Lett. **90** (2003) 151301, `gr-qc/0110012`

1.1 Hamiltonian Dynamics of General Relativity

Quantum cosmology models usually start from a Hamiltonian or phase-space formulation of the gravitational and matter dynamics. For general relativity, the foundations of the Hamiltonian description were laid by Dirac[4] and then worked out in full by Arnowitt, Deser and Misner (ADM). They showed that writing the metric as

$$ds^2 = -N^2\,dt^2 + h_{ij}(dx^i + N^i\,dt)(dx^j + N^j\,dt) \tag{1.1}$$

where N is known as the *lapse* and N^i as the *shift*, the Einstein–Hilbert action takes the form

$$S_{\rm EH}[N, N^i, h_{ij}, \pi^{ij}] = \int_{\mathbb{R}} dt \int_{\Sigma} d^3x \left(\dot{h}_{ij}\pi^{ij} - N\mathcal{C}[h_{ij}, \pi^{ij}] - N^i\mathcal{D}_i[h_{ij}, \pi^{ij}]\right) \tag{1.2}$$

where π^{ij} is the conjugate momentum to the three-metric h_{ij} and $\mathcal{C}$ and $\mathcal{D}_i$ are the *Hamiltonian* and *diffeomorphism* constraints, functions of h_{ij} and π^{ij}.

Notice that lapse and shift do not have momenta associated to them; the Einstein–Hilbert action does not contain time derivatives of these fields and they can be seen as Lagrange multipliers. Varying the action with respect to them then enforces the constraints

$$\mathcal{C} = 0\,, \qquad \mathcal{D}_i = 0\,. \tag{1.3}$$

Both the role of lapse and shift as Lagrange multipliers and the existence of constraints arise from the diffeomorphism symmetry of general relativity under arbitrary coordinate transformations

$$(x^i, t) \to (X^i(x,t), T(x,t))\,. \tag{1.4}$$

Dirac's little book gives an insightful discussion of this correspondence between gauge transformations and constraints. Here we summarise the most important points.

A theory with gauge symmetry cannot have uniquely determined time evolution: for any set of initial data, there are multiple physically equivalent solutions with this initial data, namely ones related by a gauge transformation at a later time. But this implies the Hamiltonian as the generator of time evolution cannot be uniquely defined; instead it is of the general form

$$\mathcal{H}_{\rm tot} = \mathcal{H}_0 + \lambda^m\Phi_m \tag{1.5}$$

where the λ^m are free functions. But different λ^m must give physically equivalent Hamiltonians; so the Φ_m are constrained to vanish. The time evolution of any phase-space function is then given by

$$\frac{dF}{dt} = \{F, \mathcal{H}_0\} + \lambda^m\{F, \Phi_m\} \tag{1.6}$$

(we can ignore $\{F, \lambda^m\}$ which is multiplied by a constraint). Notice that $\Phi_m = 0$ does not imply $\{F, \Phi_m\} = 0$ since the Poisson bracket involves derivatives of Φ_m, which need not vanish. The second term in (1.6) can be seen as a gauge transformation generated by the Φ_m while the first term is the "true dynamics" generated by $\mathcal{H}_0$.

[4] P. A. M. Dirac, "Lectures on Quantum Mechanics" (Dover Publications, 2001)

Exercise 1.1 *As a concrete example, derive the Hamiltonian formalism for electromagnetism starting from the action*

$$S[A] = -\frac{1}{4}\int \mathrm{d}^4x\; F_{\mu\nu}F^{\mu\nu} \tag{1.7}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ *is the field strength for* A_μ*. You should find that* A_0 *appears as a Lagrange multiplier for the* Gauss constraint $\vec{\nabla}\cdot\vec{E} = 0$*. This constraint is the generator of* U(1) *gauge transformations (which you could show explicitly by working out Poisson brackets).*

Any (non-pathological) action invariant under general coordinate transformations can be brought into the form (1.2) consisting of a term $\dot{q}^i p_i$ and a linear combination of constraints. If matter fields are added to the Einstein–Hilbert action, they will contribute to $\mathcal{C}$ and $\mathcal{D}_i$. In the notation of (1.5), a fully diffeomorphism invariant theory always has $\mathcal{H}_0 = 0$; the total Hamiltonian is constrained to vanish. There is no preferred time coordinate with respect to which there could be "true dynamics".

1.2 Minisuperspace and Wheeler–DeWitt Equation

So far we have been discussing canonical general relativity in full generality. One could try and proceed with canonical quantisation of this theory taking into account the constraints $\mathcal{C} = 0$ and $\mathcal{D}_i = 0$, however this has not been rigorously completed due to various technical complications. (In the much simpler case of electromagnetism, this programme can be successfully implemented[5].)

We now restrict to the much simpler case of homogeneous isotropic cosmology.

Exercise 1.2 *Assume the metric is of FLRW form*

$$\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + a^2(t)\,h_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j \tag{1.8}$$

where h_{ij} *is a 3-metric of constant curvature* k*. Add a homogeneous scalar field* $\phi(t)$*. Show that the action for gravity plus matter*

$$S[g,\phi] = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{1}{16\pi G}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) \tag{1.9}$$

then takes the form (up to total derivatives)

$$S[N,a,\phi] = \int_{\mathbb{R}} \mathrm{d}t \int_{\Sigma} \mathrm{d}^3x\sqrt{h}\left(-\frac{3}{8\pi G N}a\dot{a}^2 + \frac{3}{8\pi G}kaN + \frac{a^3}{2N}\dot{\phi}^2 - Na^3V(\phi)\right). \tag{1.10}$$

Completing the Legendre transform we find

$$S[N,a,p_a,\phi,p_\phi] = \int_{\mathbb{R}} \mathrm{d}t\,\left(\dot{a}p_a + \dot{\phi}p_\phi - N\mathcal{C}\right) \tag{1.11}$$

with the *Hamiltonian constraint*

$$\mathcal{C} = -\frac{2\pi G}{3}\frac{p_a^2}{a} - \frac{3}{8\pi G}ka + \frac{p_\phi^2}{2a^3} + a^3V(\phi)\,. \tag{1.12}$$

[5]H. J. Matschull, "Dirac's canonical quantization program," `quant-ph/9606031`

The action (1.10) contains an integral $\int_\Sigma \mathrm{d}^3x\sqrt{h}$ which will just give a number corresponding to the total coordinate volume in our 3-dimensional spatial slice Σ, since all variables are only functions of t by assumption. We assumed that this number is one and hence discarded the Σ integral, although it is often kept for reference since one may want to be able to rescale coordinates later. One needs to assume that Σ is compact so that this number is in any case finite, and one can then set it to one by a suitable choice of coordinates.

Exercise 1.3 *Show that the Hamiltonian constraint $\mathcal{C} = 0$ is nothing but the standard Friedmann constraint equation of an FLRW universe filled with a scalar field. (Hint: use the definitions of the canonical momenta p_a and p_ϕ)*

We now only have one constraint: there is no nontrivial action of spatial diffeomorphisms any more since all fields only depend on t. The dynamical structure of (1.11) is similar to that of a relativistic particle moving in a (1+1)-dimensional curved spacetime, as will become clear soon.

Standard rules of canonical quantisation would now suggest to define a wavefunction $\psi(a, \phi, t)$ subject to the *Wheeler–DeWitt equation*

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(a, \phi, t) = \hat{\mathcal{C}}\psi(a, \phi, t) = 0 \tag{1.13}$$

where $\hat{\mathcal{C}}$ corresponds to a differential operator, a quantum version of the constraint $\mathcal{C}$ obtained by a choice of operator ordering. Since $\mathcal{C}$ is constrained to vanish, the wavefunction ψ is in fact independent of t! This is again a reflection of the fact that the model is invariant under arbitrary redefinitions of the time variable t, which therefore cannot have physical significance.

This basic property of gravitational systems leads to the infamous *problem of time*, which is that there cannot be evolution with respect to any standard time coordinate as in normal quantum mechanics. A common strategy which we will employ in the following is to instead use a physical clock given by one of the dynamical degrees of freedom in the theory.

1.3 Universe as a Relativistic Particle

To further illustrate the formalism let us focus on the simplest case in which spatial curvature and the potential $V(\phi)$ both vanish. In this case, the scalar field can be used as a clock: its equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\frac{\dot{\phi}}{N}\right) = 0 \tag{1.14}$$

and since N and a^3 are always positive, $\dot{\phi}$ can never change sign. The function $\phi(t)$ is hence strictly monotonic (excluding the special case in which $\phi(t) = \mathrm{const}$) and ϕ can serve as a global time coordinate. We will use this in interpreting the "timeless" quantum theory.

The classical constraint is now simply (multiplying by $2a^3$)

$$\frac{4\pi G}{3}p_a^2 a^2 = p_\phi^2 \tag{1.15}$$

Now notice that the combination ap_a is canonically conjugate to $\alpha \equiv \log a$. With $p_\alpha \equiv ap_a$, the constraint takes the form of a relativistic mass-shell relation in 1+1 dimensions,

$$\frac{4\pi G}{3}p_\alpha^2 = p_\phi^2 \tag{1.16}$$

and the corresponding Wheeler–DeWitt equation is

$$-\frac{4\pi G}{3}\frac{\partial^2}{\partial\alpha^2}\psi(\alpha,\phi) = -\frac{\partial^2}{\partial\phi^2}\psi(\alpha,\phi)\,. \tag{1.17}$$

Of course, as everywhere in quantum mechanics, there is no unique choice of operator ordering. A "natural" ordering is obtained by demanding covariance under redefinitions of dynamical variables, as in $\alpha = \log a$. This ordering is in general obtained by writing the constraint in the form

$$\mathcal{C} = g^{AB}(q)p_A p_B + V(q) \tag{1.18}$$

and quantising it as $\hat{\mathcal{C}} = -\Box_g + V(q)$ where $\Box_g$ is the Laplace–Beltrami ("Box") operator for the metric g_{AB}. In our case this procedure leads to (1.17).

(1.17) is just the massless wave equation in 1+1 dimensions, corresponding to the fact that the classical dynamics of this universe is equivalent to a particle moving in 1+1 dimensional flat space. We can write down its general solution

$$\psi(\alpha,\phi) = \psi_+\left(\alpha - \sqrt{\frac{4\pi G}{3}}\phi\right) + \psi_-\left(\alpha + \sqrt{\frac{4\pi G}{3}}\phi\right) \tag{1.19}$$

where ψ_+ and ψ_- are arbitrary. To interpret these solutions we now also need an inner product or probability interpretation. For this model, let us use ϕ as a clock and demand that the inner product is preserved under ϕ evolution. A natural candidate is the Klein–Gordon inner product

$$\langle\psi|\chi\rangle_{\rm KG} \equiv {\rm i}\int {\rm d}\alpha\left(\bar\psi\frac{\partial\chi}{\partial\phi} - \frac{\partial\bar\psi}{\partial\phi}\chi\right) \tag{1.20}$$

which, as usual, is positive for "positive frequency" and negative for "negative frequency" solutions. To get a positive definite inner product, one can either exclude the second half of modes or define the inner product with an overall minus sign for these.
An obvious observable to start with would be the expectation value $\langle\alpha(\phi)\rangle$ corresponding to the average evolution of the universe as parametrised by the matter clock. We find

$$\langle\alpha(\phi)\rangle = \frac{\int {\rm d}\alpha\,\alpha\left(\bar\psi\frac{\partial\psi}{\partial\phi} - \frac{\partial\bar\psi}{\partial\phi}\psi\right)}{\int {\rm d}\alpha\left(\bar\psi\frac{\partial\psi}{\partial\phi} - \frac{\partial\bar\psi}{\partial\phi}\psi\right)} \tag{1.21}$$

which of course depends on the details of the state chosen. However, one can easily show that

Exercise 1.4 *Assume that the wavefunction is a pure "right moving" solution to the Wheeler–DeWitt equation, i.e., of the general form* $\psi(\alpha,\phi) = \psi_+\left(\alpha - \sqrt{\frac{4\pi G}{3}}\phi\right)$ *for some function* ψ_+ *of a single variable. Show that for any such a state*

$$\langle\alpha(\phi)\rangle = \sqrt{\frac{4\pi G}{3}}\,(\phi - \phi_0) \tag{1.22}$$

where ϕ_0 *is a constant depending on the state. (Hint: shift the argument of the integral)*

The expectation value follows exactly the classical solution $a(\phi) = \exp(\sqrt{4\pi G/3}(\phi - \phi_0))$ corresponding to an expanding universe. Similarly, left movers follow exactly the contracting solution $a(\phi) = \exp(-\sqrt{4\pi G/3}(\phi - \phi_0))$. These expectation values approach zero at infinite $|\phi|$, corresponding to a finite proper time, which is just the Big Bang/Big Crunch singularity of the classical cosmological model. They therefore do not resolve the singularity.
States including superpositions of left and right movers will in general have some lower bound $\langle \alpha(\phi) \rangle > C > 0$. However, these states are macroscopic superpositions of expanding and collapsing universes which may not admit a clear semiclassical interpretation.

The failures of Wheeler–DeWitt quantum cosmology to resolve singularities provide a main motivation for considering input from loop quantum gravity (LQG) as we will discuss in the next lecture. In general, one may find singularity resolution in more complicated models, but this is often dependent on the chosen details of the quantisation (e.g., a certain operator ordering).

1.4 Semiclassical Quantum Cosmology

The simple model of a free, massless scalar in a flat FLRW universe could be solved exactly and we were able to derive general properties of simple expectation values. More complicated models involving a potential $V(\phi)$ or spatial curvature often no longer admit exact solutions. We also saw that both the choice of inner product and of initial state are additional inputs.

In light of these issues one often assumes the validity of a semiclassical WKB approximation, that is an expansion of the form

$$\psi(a, \phi) = \exp(\mathrm{i}S(a, \phi)/\kappa) \tag{1.23}$$

in leading powers for small κ (often associated with Planck's constant $\hbar$ which we however set to one). At leading order, $S(a, \phi)$ will be a solution to the Hamilton–Jacobi equation associated to the Hamiltonian constraint $\mathcal{C}$, i.e., it will satisfy

$$\mathcal{C}\left[a, p_a = \frac{\partial S}{\partial a}, \phi, p_\phi = \frac{\partial S}{\partial \phi}\right] = 0\,. \tag{1.24}$$

$S(a, \phi)$ is the action along a classical solution whose final values for scale factor and scalar field are a and ϕ. This classical trajectory is of course not unique, as one did not specify the initial values for a and ϕ (or other additional data, such as velocities or momenta).

If there is a classical solution for given initial and final values, there will be many different trajectories all representing the same physical solution; these correspond to the gauge freedom of arbitrary redefinitions of the time coordinate (keeping the endpoints fixed), as in Dirac's general discussion of gauge symmetry. After taking this gauge freedom into account, there may be one, multiple or no inequivalent classical solutions depending on the details of the model. Moreover the classical solutions should in general be considered to be complex, corresponding to complex saddle points of the action functional – the time reparametrisations that can be used to obtain equivalent representations of the same solutions can also in general be complex.

Exercise 1.5 *Consider positive spatial curvature which leads to a recollapse of the universe: in (1.12) set $V(\phi) = 0$ but leave $k > 0$ general. One useful gauge choice is $N = a^3$ which corresponds again to using the scalar ϕ as clock (see Exercise 2.3 for a derivation). The Hamiltonian is then*

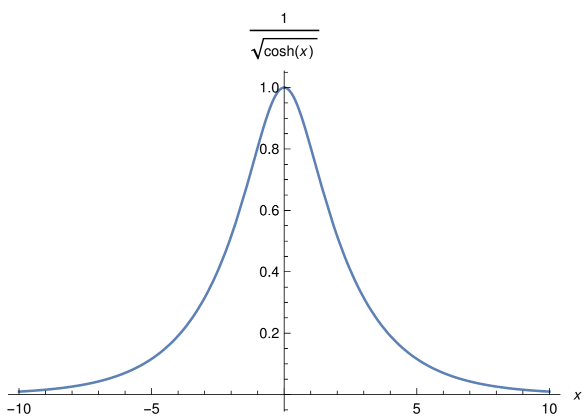
$$a^3\mathcal{C} = -\frac{2\pi G}{3}p_a^2 a^2 - \frac{3}{8\pi G}ka^4 + \frac{p_\phi^2}{2}\,. \tag{1.25}$$

By solving Hamilton's equations and using $\mathcal{C} = 0$ to fix an integration constant, show that the classical solutions take the form

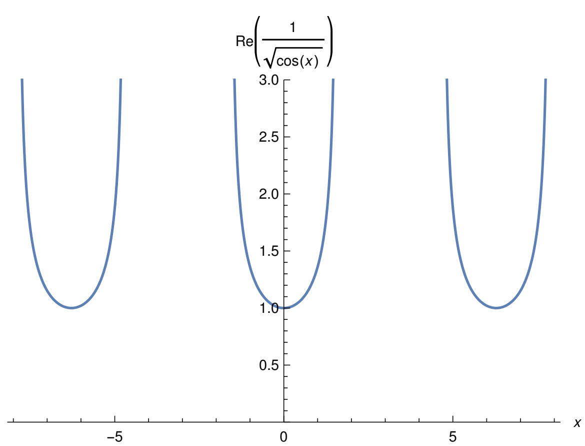
$$a(t) = \left(\frac{4\pi G p_\phi^2}{3k}\right)^{1/4} \cosh\left(4\sqrt{\frac{\pi G}{3}}p_\phi(t-t_0)\right)^{-1/2} \tag{1.26}$$

and $\phi(t) = \phi_0 + p_\phi t$ (remember that p_ϕ is still a constant of motion here). Notice that fixing the lapse removes the gauge freedom of time redefinitions, so that each solution takes a unique form.

For real arguments, $\cosh(x)^{-1/2}$ is always between 0 and 1, taking its maximum when $x = 0$.



Since each value between zero and one is taken twice, there are always two solutions for any given initial and final value of a – one that stays on one side of the recollapse and is either purely expanding or purely collapsing, and one that takes longer time including the recollapse point.
Recalling that $\cosh(\mathrm{i}x) = \cos(x)$, the situation is different if we consider the time $t - t_0$ to be purely imaginary:



Now all positive values greater than one are taken, in fact an infinite number of times. Moreover the function $\cosh(x)^{-1/2}$ is $2\pi\mathrm{i}$ periodic so it actually takes all positive values an infinite number of times in the complex plane.

We see that already in this simple example, the question of how many classical solutions exist for given boundary conditions is far from straightforward to answer; there will be in general infinitely many (complex) solutions. The use of complex trajectories may appear puzzling at first, but is rather standard in the description e.g. of tunnelling phenomena. A classically forbidden path through a potential barrier becomes allowed if the solution is allowed to venture into the complex plane. In general this will imply a complex action, so that the exponential $\exp(\mathrm{i}S(a,\phi)/\kappa)$ picks out an exponentially growing or decaying piece. This is indeed how one computes tunnelling amplitudes most easily. In our case where the infinitely many solutions require longer and longer periods of imaginary time, this would presumably result in exponential suppression of these trajectories.

An immediate conclusion from this discussion is that the ansatz (1.23) must be extended to a more general form

$$\psi(a,\phi) = \sum_I \lambda_I \exp(\mathrm{i}S_I(a,\phi)/\kappa) \tag{1.27}$$

summing over all the saddles or complex solutions for given boundary data. The different approaches and prescriptions that exist in the literature differ in their choice of boundary data in the past (here the most famous approach is the *no-boundary proposal* which posits that the universe had no boundary in the past, corresponding to a closed universe with $a = 0$) and in the selection of saddle point solutions and/or coefficients λ_I. In particular, saddle points can arise as semiclassical approximations to a path integral with given boundary conditions. These choices have been the focus of active debate in recent years: the use of new techniques, including Picard–Lefschetz theory in a path integral setting, provides mathematical criteria that select some saddle points over others[6] whereas the more traditional perspective seems to be that physical criteria, in particular normalisability of the resulting wavefunction, need to be added to choose saddle points[7]. Without going too much into the details of this debate, it might be helpful to say a bit more about what one might hope for regarding physical predictions of this approach.

1.5 Towards Physical Predictions

Let us now specify to a case of particular interest, namely making predictions about the likelihood and initial conditions for inflation. We are now in the most general context within the class of models we have been discussing in which the scalar field has a potential and there is also spatial curvature $k > 0$. The action is

$$S[N, a, p_a, \phi, p_\phi] = \int_{\mathbb{R}} \mathrm{d}t \left(\dot{a} p_a + \dot{\phi} p_\phi - N\mathcal{C} \right) \tag{1.28}$$

[6] J. Feldbrugge, J. L. Lehners and N. Turok, "Lorentzian quantum cosmology," Phys. Rev. D **95** (2017) no.10, 103508, `arXiv:1703.02076` and many follow-up papers

[7] For a recent summary see J. J. Halliwell, J. B. Hartle and T. Hertog, "What is the no-boundary wave function of the Universe?," Phys. Rev. D **99** (2019) no.4, 043526, `arXiv:1812.01760`

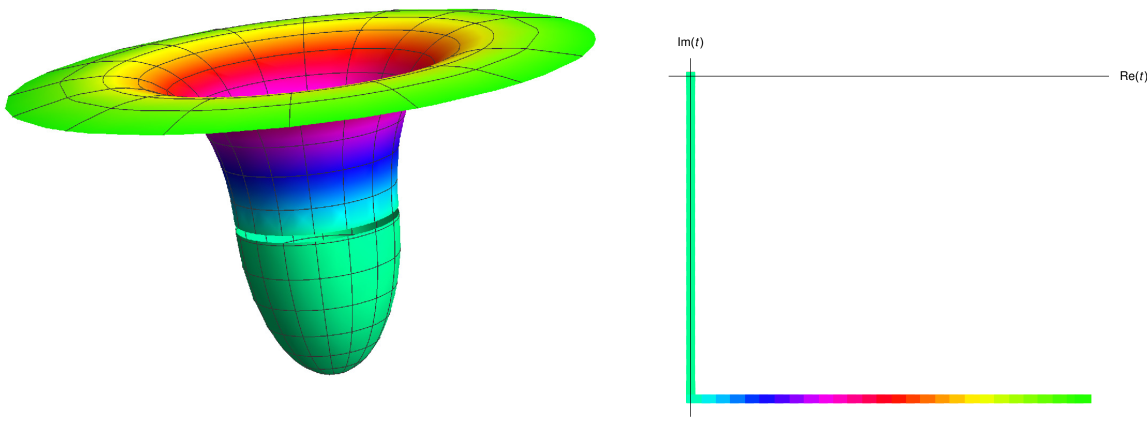
with

$$\mathcal{C} = -\frac{2\pi G}{3}\frac{p_a^2}{a} - \frac{3}{8\pi G}ka + \frac{p_\phi^2}{2a^3} + a^3 V(\phi)\,. \tag{1.29}$$

For concreteness let us now focus on no-boundary conditions in the past. Here we assume that the universe started out at $a = 0$, and demand that the metric was regular at this initial point. Then the scalar cannot have had any kinetic energy (which would diverge when $a = 0$), so must have been at some constant value ϕ. These initial conditions hence correspond to a slow-roll regime in which the scalar field is dominated by its potential energy. One then uses an approximation in which one neglects the p_ϕ^2 term and treats $V(\phi)$ as approximately constant in (1.29). The classical solution in the limit where $V(\phi)$ is exactly constant is just deSitter space, which never goes through $a = 0$ in closed slicing. We then again need complex solutions to connect $a = 0$ to final $a > 0$.

There are several proposals as to what the complex solution of choice should be, most notably the *no-boundary proposal* of Hartle and Hawking and the *tunnelling proposal* of Vilenkin. In terms of their trajectories in time they are essentially complex conjugates of each other.

These trajectories can be thought of as consisting of a Euclidean part (corresponding to a 4-sphere in the limit of constant $V(\phi)$) glued to an expanding deSitter space. The corresponding action will be purely imaginary in the Euclidean region and real in the Lorentzian region. What the two proposals differ in is the overall sign of the imaginary part, and hence whether the factor $\exp(\mathrm{i}S)$ is exponentially enhanced or suppressed.



Here we illustrate the geometry of the Hartle–Hawking no-boundary instanton. Starting from the regular "south pole" in the Euclidean region, one goes into the direction of negative imaginary time. In Vilenkin's proposal one would move into the opposite, positive direction.

Concretely, the Hartle–Hawking proposal leads to a WKB wavefunction "for the universe" of the form (this is a linear combination of two saddle point solutions)

$$\psi_{\mathrm{HH}}(a,\phi) \propto \exp\left(\frac{1}{3V(\phi)}\right)\cos\left(\frac{1}{3V(\phi)}(a^2V(\phi)-1)^{3/2} - \frac{\pi}{4}\right) \tag{1.30}$$

whereas the Vilenkin tunnelling proposal leads to

$$\psi_{\rm V}(a,\phi) \propto \exp\left(-\frac{1}{3V(\phi)}\right)\exp\left(-\frac{\rm i}{3V(\phi)}(a^2V(\phi)-1)^{3/2}+{\rm i}\frac{\pi}{4}\right)\,, \tag{1.31}$$

the potential $V(\phi)$ now expressed in Planck units (and $k=+1$).

Exercise 1.6 *Show that both of these wavefunctions follow exponentially expanding solutions for large* a*, if the potential* $V(\phi)$ *is taken to be constant. (Hint: view the oscillating parts as* $\exp({\rm i}S(a,\phi))$*, use the Hamilton–Jacobi relation* $p_a=\frac{\partial S}{\partial a}$ *and the definition of* p_a *to show that* $\dot{a}(t)\sim\sqrt{V}a(t)$ *in proper time* t*.)*

One could now view the real exponential (squared) as a probability density for the scalar field ϕ to take a certain value at the beginning of inflation. The conclusions for the likelihood of inflation then depend rather sensitively on the potential $V(\phi)$ and on the allowed range of values for ϕ, since an integral over ϕ in general needs a cutoff to converge. The Hartle–Hawking wavefunction (1.30) favours configurations for which ϕ sits in a minimum of $V(\phi)$, which can be problematic as an initial condition for inflation.

A second type of prediction from quantum cosmology involves inhomogeneities. These can be included perturbatively into the formalism; the leading order contribution to the action and Hamiltonian is then quadratic in the perturbations. For instance, consider scalar perturbations (appropriately defined[8]) in a closed universe, decomposed into spherical harmonics Q_{nlm} which satisfy

$$\Delta_{\rm S^3}Q_{nlm} = -(n^2-1)Q_{nlm}\,; \tag{1.32}$$

their contribution to the Hamiltonian constraint is (assuming a quadratic potential with mass m)

$$\delta\mathcal{C} = \frac{1}{2}\sum_{nml}\left[\frac{1}{a^3}p^2_{nlm}+((n^2-1)a+m^2a^3)q^2_{nlm}\right] \tag{1.33}$$

where each mode is denoted by q_{nlm} with momentum p_{nlm}. As usual, for each mode the dynamics are those of a harmonic oscillator with time-dependent frequency, with all modes decoupled.

The combined equations for background and perturbations are usually solved in an approximation similar to the Born–Oppenheimer approximation of molecular physics; one first solves for the background modes neglecting backreaction and then for the perturbation modes separately, resulting in a product wavefunction

$$\psi(a,\phi,q_{nlm}) = \psi_0(a,\phi)\prod_{nlm}\psi_{nlm}(q_{nlm};a) \tag{1.34}$$

where derivatives of the perturbation wavefunctions ψ_{nlm} with respect to a are neglected. If a semiclassical WKB-type approach is followed, one would again pick out an (approximate) classical solution for both background and perturbations which is in general complex. In particular, the

[8]The full formalism for all possible types of perturbations was developed in J. J. Halliwell and S. W. Hawking, "Origin of structure in the Universe," Phys. Rev. D **31** (1985) 1777–1791.

perturbations interact with a complex scale factor $a(t)$.

The total action for perturbation modes (using the boundary condition that these were all zero initially) is quadratic in q_{nlm}. As for the background solution, if the saddle point solution(s) used contain(s) a complex part the action for the perturbations will pick out an imaginary piece. Calculations in the no-boundary proposal (see again Halliwell and Hawking) suggest that this imaginary piece leads to an exponential suppression factor

$$\psi_{nlm}(q_{nlm}; a) \sim \exp\left(-\frac{1}{2}n\, a^2 q_{nlm}^2\right) \tag{1.35}$$

whereas choosing the opposite saddle point would give enhancement $\sim \exp(+\frac{1}{2}na^2q_{nlm}^2)$. The exponentially suppressed form corresponds to a ground-state wavefunction for the harmonic oscillator, "predicting" the appearance of the Bunch–Davies vacuum in which all perturbation modes are in the ground state, and justifying the usual choice of initial conditions for inflation. In contrast, the exponentially growing form signals disaster; it corresponds to a distribution favouring large inhomogeneites, leading to a total breakdown of perturbation theory. The choice of complex solution used to define the WKB wavefunction will again determine which of those one gets.

1.6 Summary

We saw how following Dirac's procedure for the Hamiltonian analysis of a cosmological model (obtained after symmetry reduction to FLRW universes) leads to the appearance of a Hamiltonian constraint at the classical level, or a Wheeler–DeWitt equation at the quantum level, which takes the form of a Schrödinger equation where only zero energy states are physical.

To make sense of the formalism one then needs to add a probability interpretation in the form of an inner product; one can then ask questions such as whether the expectation value of the scale factor avoids the classical singularity $a = 0$. The choice of inner product is in general not unique.

One also needs to choose a state to make predictions. Since the formalism is anyway only valid semiclassically, and we want to make predictions for a classical universe, these states are chosen to be semiclassical. In a WKB approximation they arise from summing over one or several classical complex solutions and weighting each by the WKB factor $\exp(\mathrm{i}S)$ where S is the action along each solution. The choice of state then reduces to the choice of these classical solutions. Imaginary parts of the action lead to exponentially enhanced or suppressed values for background fields and/or perturbations which one could then view as the predictions of this approach.

2 Quantum Cosmology à la Loop Quantum Gravity

Quantum cosmology in the Wheeler–DeWitt approach provides a self-consistent setting in which one can address questions about the likelihood of certain initial conditions for the universe. In the semiclassical approximation, the new input compared to classical cosmology is the use of complex tunnelling-like solutions which can lead to exponential enhancement or suppression of certain configurations. It is clear that this approach is not sensitive to other types of quantum gravity corrections such as corrections to the Einstein–Hilbert action; moreover it relies on a Hilbert space structure (inner product, etc) that is not related to a Hilbert space of full quantum gravity.

As we mentioned earlier, a quantisation based on the Wheeler–DeWitt equation has not been implemented in full general relativity. Substantial progress was only made in the 1990s within loop quantum gravity (LQG): using a new set of variables and techniques similar to those of lattice Yang–Mills theory, some steps of the canonical quantisation programme could be completed while there are proposals for the main missing step, the implementation of a Hamiltonian constraint. This additional structure compared to Wheeler–DeWitt theory can now be used for the construction of improved minisuperspace models which use a different Hilbert space and different dynamics, known as loop quantum cosmology (LQC). These models contain the main physical insight of LQG, namely an underlying discrete structure of spacetime, which was not present in the traditional approach.

2.1 Elements of Loop Quantum Gravity

We saw above that standard Hamiltonian analysis of the Einstein–Hilbert action leads to a gravitational phase space in which the main variables are the spatial metric h_{ij} and π^{ij}; these variables are subject to the Hamiltonian and diffeomorphism constraints. Quantisation of this structure poses various difficulties. For example, it is difficult to construct well-defined observables out of the metric and extrinsic curvature that can then be made into operators for the quantum theory (the metric at a point has no observable content, for example).

It turns out that a classically equivalent formulation of general relativity can be found whose structure is much closer to that of Yang–Mills theory. In this formulation, the main variables are an SU(2) connection A_i^a and a "densitised triad" E_a^i. These are canonically conjugate,

$$\{A_i^a(x), E_b^j(y)\} = 8\pi G\gamma\delta_i^j\delta_b^a\,\delta^3(x,y) \tag{2.1}$$

where γ is the Barbero–Immirzi parameter, a free parameter appearing in the definition of A_i^a. The densitised triad is a vector density, which can be seen as dual to a 2-form $\epsilon_{ijk}E_a^k$. These variables are known as *Ashtekar* or *Ashtekar–Barbero variables* (Barbero found a real formulation of the originally complex Ashtekar formalism). As in the ADM formalism, these variables live on a 3-dimensional spatial slice Σ and are subject to constraints which correspond to the gauge freedom under spacetime diffeomorphisms and now also under local SU(2) gauge transformations.

Ashtekar–Barbero variables are closely related to the perhaps more widely known spin connection ${\omega_\mu}^I{}_J$ and tetrad e_μ^I that one uses to give general relativity a local Lorentz invariance; the spin connection encodes parallel transport while the tetrad can be seen as the square root of the spacetime metric $g_{\mu\nu}$ via

$$e_\mu^I e_\nu^J \eta_{IJ} = g_{\mu\nu}\,. \tag{2.2}$$

This local Lorentz group $\mathrm{SL}(2,\mathbb{C})$ can be partially gauge-fixed to SU(2) leading to A_i^a and E_a^i.

The key idea is now to replace the distributional Poisson algebra of continuum fields A_i^a and E_b^j by an algebra of regularised objects. As n-forms are naturally smeared over n-dimensional manifolds to give a coordinate-independent quantity, this suggests defining *holonomies* of the connection and *fluxes* of the densitised triad,

$$h_A(e) := \mathcal{P}\exp\left(\int_e A\right), \quad E_a(S) := \int_S (\star E)_a\,. \tag{2.3}$$

Here $\mathcal{P}\exp$ is the "path-ordered exponential" (needed because the connection A is non-Abelian) of an integral along 1-dimensional curve e, and $(\star E)_a$ denotes the Lie algebra valued 2-form $\epsilon_{ijk}E^k_a$ defined before, which is integrated over a two-dimensional surface S. Holonomies are natural objects in lattice Yang–Mills theory; they have nice gauge covariance properties, with gauge transformations only acting on the endpoints of e. In particular, for closed e $h_A(e)$ is invariant under SU(2) gauge transformations. These objects now have a nice regular Poisson algebra with the delta distribution replaced by a regular function on the right-hand side.

The quantum theory of LQG in terms of a Hilbert space and operators acting on states is built out of holonomies and fluxes. The continuum fields A and E are not well-defined in this theory. (This is anologous to a form of quantum mechanics in which only exponentials $\exp(\mathrm{i}\lambda x)$ but not x itself exist as operators; this is commonly known as *polymer* quantum mechanics.)

Typical states in the LQG Hilbert space live on a *spin network*, a graph consisting of a number of edges and vertices, so that each edge starts and ends in a vertex. Given such a graph Γ one can define wavefunctions by

$$\psi_\Gamma[A] = f(h_{e_1}(A), \ldots, h_{e_n}(A))\,; \tag{2.4}$$

such wavefunctions depend on the (continuum) connection A but only through its holonomies along the edges $e_1, \ldots, e_n$. The state is gauge-invariant if it is invariant under the action of gauge transformations on the vertices which transform each holonomy as $h_{e_i}(A) \to g^{-1}_{s(e_i)}h_{e_i}(A)g_{t(e_i)}$ where $s(e_i)$ is the source and $t(e_i)$ the target vertex of the edge e_i. The inner product for such wavefunctions (on the same graph Γ) is derived from the normalised Haar measure on n copies of SU(2).

Dynamics for such states can be defined through a Hamiltonian constraint, as in the Wheeler–DeWitt approach. One writes the Hamiltonian constraint of general relativity (in Ashtekar–Barbero variables) in terms of well-defined (regularised) operators, such that in the limit of the regularisation being removed it reduces to the continuum expression. This is not a unique procedure.

Exercise 2.1 *Show that if the only well-defined operators are* $\exp(\mathrm{i}\lambda X)$ *rather than* X*, there would be different ways of replacing a polynomial function* $f(X)$ *by a regularised function* $f_{\rm reg}(\exp(\mathrm{i}\lambda X))$ *built out of polynomials in* $\exp(\mathrm{i}\lambda X)$ *and* $\exp(-\mathrm{i}\lambda X)$*.*

2.2 Loop Quantum Cosmology

Without exploring further the details of full LQG, let us focus on flat FLRW universes and see how various ingredients from LQG appear leading to a different theory than Wheeler–DeWitt quantum cosmology. Starting again at the classical continuum level, FLRW symmetry implies that the

Ashtekar–Barbero connection and densitised triad can be parametrised as

$$A_i^a = c\,\delta_i^a\,, \quad E_a^i = p\,\delta_a^i \tag{2.5}$$

where c and p are now functions of time only. The notation c and p is not very intuitive but we will use it for ease of comparison with LQC literature. In general, these definitions also involve the coordinate volume $V_0 = \int_\Sigma \mathrm{d}^3x\,\sqrt{h}$ that we set to one above[9]. The variable c is proportional to $\dot{a}/N$ in a flat FLRW universe and the variable p is proportional to a^2.

The Hamiltonian constraint for general relativity coupled to a free, massless scalar is then

$$\mathcal{C} = -\frac{3}{8\pi G\gamma^2}c^2\sqrt{|p|} + \frac{p_\phi^2}{2|p|^{3/2}}\,. \tag{2.6}$$

As in the example in Wheeler–DeWitt quantum cosmology, we again consider a free scalar which can be used as a matter clock.

Recall that in LQG the continuum connection A_i^a does not exist as an operator. In this cosmological model, we similarly have to replace the variable c with regularised quantities of the form $\exp(\mathrm{i}\mu c)$. The usual procedure to do this is the following. In the Hamiltonian constraint of the full theory, it is not A_i^a which appears (since this is not gauge-invariant) but the field strength F_{ij}^a, which needs to be regularised: consider the expansion of the holonomy of A_i^a around a small square loop in the i-j-plane,

$$h_{ij}^\mu(A) = \mathbf{1} + \frac{1}{2}\mu^2 F_{ij}^a \tau_a + \ldots \tag{2.7}$$

where μ is the coordinate length of the sides of this loop. For small enough μ we can approximate the first two terms on the right-hand side by the left-hand side, thus replacing the field strength by a holonomy. When applied to our cosmological model this procedure suggests the replacement

$$c \to \frac{\sin(\mu c)}{\mu}\,. \tag{2.8}$$

Here μ can be a constant or itself be a function of the dynamical variables p and c. In general the form of μ is a choice, but physical considerations lead to the *improved dynamics* prescription in which $\mu \sim |p|^{-1/2}$, i.e., the coordinate length of sides of the "elementary loop" scales as $1/a$, so that the *physical length* is a constant which one may take to be Planckian.

2.3 Singularity Resolution

Still at the classical, LQG-regularised level, we see that the Hamiltonian constraint takes the form

$$\frac{3}{8\pi G\gamma^2}\frac{\sin^2(\mu c)}{\mu^2}\sqrt{|p|} = \frac{p_\phi^2}{2|p|^{3/2}} \tag{2.9}$$

[9]For this and other technical issues in LQC see e.g. the review K. Banerjee, G. Calcagni and M. Martin-Benito, "Introduction to Loop Quantum Cosmology," SIGMA **8** (2012) 016, `arXiv:1109.6801`.

In the improved dynamics case where $\mu = \mu_0/\sqrt{|p|}$ for some constant μ_0, we can rewrite this constraint as

$$\frac{3}{8\pi G\gamma^2}\frac{\sin^2(\mu_0 c/\sqrt{|p|})}{\mu_0^2} = \frac{p_\phi^2}{2|p|^3}\,. \tag{2.10}$$

The left-hand side is now bounded, so the right-hand side must be too; but the right-hand side is just (proportional to) the matter energy density, which therefore must have an upper bound too! This upper bound, the *critical energy density*, can be computed explicitly, and involves the Planck scale through μ_0. This replacement of unbounded by bounded functions is at the heart of singularity resolution in LQC, since it implies there is an upper bound for curvature in this model.

Depending on the form of μ it is convenient to pass to new variables before quantising. In particular, for the preferred form $\mu \sim |p|^{-1/2}$ one would like to choose $c/\sqrt{|p|}$ as one canonical variable, which is conjugate to $|p|^{3/2}$. These variables correspond to the Hubble rate $\dot{a}/(aN)$ and volume $v \sim a^3$. After this change of variables one finds the Wheeler–DeWitt equation[10]

$$-\frac{\partial^2}{\partial\phi^2}\psi(b,\phi) = -12\pi G\left(\frac{\sin(\mu_0 b)}{\mu_0}\frac{\partial}{\partial b}\right)^2\psi(b,\phi) \tag{2.11}$$

where b is proportional to the Hubble rate. This can again be brought into a more "canonical" form as in (1.17) by replacing

$$\frac{\sin(\mu_0 b)}{\mu_0}\frac{\partial}{\partial b} \equiv \left(\frac{\partial\alpha}{\partial b}\right)^{-1}\frac{\partial}{\partial b} = \frac{\partial}{\partial\alpha} \tag{2.12}$$

which holds for $\alpha = \log(\tan(\mu_0 b/2))$. In analogy with the discussion above, we would then expect "left moving" and "right moving" wavefunctions to follow the classical solutions

$$\alpha(\phi) = \pm\sqrt{12\pi G}(\phi-\phi_0) \quad \Leftrightarrow \quad b(\phi) = \arctan\exp(\pm\sqrt{12\pi G}(\phi-\phi_0))\,. \tag{2.13}$$

Recalling that b is proportional to the Hubble rate, we see again that the Hubble rate remains bounded for these solutions; they correspond to a "big bounce" connecting a collapsing and expanding universe, which never reaches the classical singularity at $v = 0$.

Exercise 2.2 *Consider the effective LQC Hamiltonian constraint*

$$\mathcal{H} = -6\pi G\left(\frac{\sin(\mu_0 b)}{\mu_0}v\right)^2 + \frac{p_\phi^2}{2} \tag{2.14}$$

where v is canonically conjugate to b. Using Hamilton's equations show that $v(t)$ satisfies the effective Friedmann-like equation

$$\left(\frac{\dot{v}(t)}{v(t)^2}\right)^2 = 12\pi G\frac{p_\phi^2}{v(t)^2}\left(1-\frac{p_\phi^2\mu_0^2}{12\pi G\,v(t)^2}\right) \tag{2.15}$$

and that the classical solutions for $v(t)$ take the form

$$v(t) = \frac{|p_\phi|\mu_0}{\sqrt{12\pi G}}\cosh\left(\sqrt{12\pi G}|p_\phi|(t-t_0)\right)\,. \tag{2.16}$$

[10]For more details see e.g. A. Ashtekar and P. Singh, "Loop quantum cosmology: a status report," Class. Quant. Grav. **28** (2011) 213001, `arXiv:1108.0893`.

These classical solutions capture the behaviour of expectation values of semiclassical states to a good approximation; they show the "big bounce" behaviour explicitly. In general, leading order LQC corrections can be captured in an effective Friedmann equation of the form[11]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right) \tag{2.17}$$

where the critical density ρ_c provides an explicit upper bound for the matter energy density ρ.

The model we have discussed here is particularly simple, and allows direct exact solution of the Wheeler–DeWitt equation. This is not true for general models of LQC, which may include additional LQG-like correction terms in particular from *inverse-triad corrections*, where inverse powers of p are replaced by regularised objects that do not have a singularity as $p = 0$. In general, these models can only be studied numerically. The general properties of upper bounds on curvature and energy density, and a corresponding singularity resolution, are general features of LQC models. There is again a question of how to choose initial states; in most of the literature semiclassical sharply peaked states are evolved through the bounce and it can be shown that these remain sharply peaked throughout the evolution including in the Planckian bounce regime.

2.4 Adding Inhomogeneities

The resolution of the classical Big Bang singularity through quantum gravity effects would signal a major conceptual shift in our understanding of the beginning of the Universe. One might however wonder whether it has any observable implications for cosmology. To study this question an extension of the standard LQC formalism to slightly inhomogeneous universes is needed[12].

In many ways, this is analogous to the formalism for inhomogeneities in traditional quantum cosmology. One works on a phase space $\Gamma_{\rm Trun} = \Gamma_0 \times \Gamma_1$ corresponding to a truncation of full general relativity at linearised order; Γ_0 denotes homogeneous degrees of freedom and Γ_1 linear perturbations. The homogeneous degrees of freedom are subject to a Hamiltonian constraint; constraints for the perturbation modes can be solved to reduce them to gauge-invariant variables. These then again contribute to the Hamiltonian constraint; for tensor perturbations one has

$$\delta\mathcal{C} = \frac{1}{2}\sum_{\vec{k}}\left[\frac{1}{a^3}P_{\vec{k}}^2 + ak^2 Q_{\vec{k}}^2\right] . \tag{2.18}$$

This is analogous to the previous expression (1.33) for scalar perturbations, except that we are now not on a closed but on a flat background universe, and there is no analogue of the potential term for tensor modes. Recall that the Hamiltonian constraint is multiplied by a lapse function to obtain the Hamiltonian.

[11] V. Taveras, "Corrections to the Friedmann equations from loop quantum gravity for a universe with a free scalar field," Phys. Rev. D **78** (2008) 064072, `arXiv:0807.3325`.

[12] I. Agullo, A. Ashtekar and W. Nelson, "Extension of the quantum theory of cosmological perturbations to the Planck era," Phys. Rev. D **87** (2013) no.4, 043507, `arXiv:1211.1354`.

Exercise 2.3 *In LQC time is measured with respect to the scalar field clock ϕ. By using the symmetry-reduced action of a free massless scalar in a flat FLRW universe*

$$S[N, a, \phi] = \int_{\mathbb{R}} \mathrm{d}t \, \frac{a^3}{2N} \dot{\phi}^2 \,, \tag{2.19}$$

show that $\pi_\phi \equiv \frac{a^3}{N}\dot{\phi}$ is a constant of motion and that using ϕ as time implies the lapse $N = a^3/\pi_\phi$.

The total Hamiltonian for tensor perturbations is thus given by

$$N\delta\mathcal{C} = \frac{1}{2}\sum_{\vec{k}} \left[\frac{1}{\pi_\phi} P_{\vec{k}}^2 + \frac{a^4}{\pi_\phi} k^2 Q_{\vec{k}}^2 \right] . \tag{2.20}$$

a^4 and π_ϕ are background quantities, which correspond to operators in homogeneous LQC. One now again uses an approximation in which the total wavefunction for background and perturbations is of product form $\psi = \psi_0 \prod_{\vec{k}} \psi_{\vec{k}}$. The Wheeler–DeWitt equation is then first solved for the background wavefunction ψ_0; this equation takes the form

$$\mathrm{i}\frac{\partial}{\partial\phi}|\psi_0\rangle = \hat{\mathcal{H}}_0|\psi_0\rangle \tag{2.21}$$

which is the "square root" of the usual Wheeler–DeWitt equation. This form can be seen as a restriction of the original second order Wheeler–DeWitt equation to positive frequency modes with respect to ϕ. After solving (2.21), each perturbation mode wavefunction $\psi_{\vec{k}}$ needs to satisfy

$$|\psi_0\rangle \mathrm{i}\frac{\partial}{\partial\phi}|\psi_{\vec{k}}\rangle = \widehat{N\delta\mathcal{C}}\left(|\psi_0\rangle|\psi_{\vec{k}}\rangle\right) = \widehat{\frac{1}{2\pi_\phi}}|\psi_0\rangle \hat{P}_{\vec{k}}^2|\psi_{\vec{k}}\rangle + \widehat{\frac{a^4}{2\pi_\phi}}|\psi_0\rangle k^2 \hat{Q}_{\vec{k}}^2|\psi_{\vec{k}}\rangle \,. \tag{2.22}$$

Taking the scalar product of this equation with $\langle\psi_0|$ one finds the effective Wheeler–DeWitt equation for perturbations

$$\mathrm{i}\frac{\partial}{\partial\phi}|\psi_{\vec{k}}\rangle = \frac{1}{2}\langle\psi_0|\widehat{\pi_\phi^{-1}}|\psi_0\rangle \, \hat{P}_{\vec{k}}^2|\psi_{\vec{k}}\rangle + \frac{1}{2}\langle\psi_0|\widehat{a^4\pi_\phi^{-1}}|\psi_0\rangle k^2 \hat{Q}_{\vec{k}}^2|\psi_{\vec{k}}\rangle \,. \tag{2.23}$$

We see that the effect of a quantised background on perturbations can be captured in two expectation values of the background wavefunction ψ_0; these expectation values can be interpreted as defining a *dressed metric* on which LQC perturbations propagate. For suitably semiclassical states ψ_0, these expectation values again follow big bounce trajectories as we saw in the previous discussion. One can repeat the discussion for scalar perturbations and find a similar equation, the only difference being that now a third expectation value appears which includes the scalar potential $V(\phi)$ (often negligible in the background evolution within LQC).

A bouncing scenario provides a preferred point for setting initial conditions, namely at the bounce itself where $\rho = \rho_c$. In the setting we have been describing here, spacetime remains semiclassical and perturbations small throughout the bouncing phase, making it possible to set initial conditions there. Spacetime is not (even approximately) deSitter here; so instead of the Bunch–Davies vacuum one chooses a regular, maximally symmetric vacuum state (precisely, a suitable fourth order adiabatic vacuum) at the bounce point.

The LQC pre-inflationary dynamics can now influence the initial conditions for the usual slow-roll phase[13]. One way to see this is the following: consider again tensor perturbations for simplicity. Written in terms of a suitable variable χ these satisfy

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\eta^2} + \left(k^2 - \frac{a''}{a}\right)\chi = 0 \tag{2.24}$$

where η is conformal time. The two terms in brackets show a competition between the physical wavenumber k/a of a mode and the scale set by the curvature (the Ricci scalar is proportional to a''/a^3). Modes whose wavelengths are much shorter than the curvature radius propagate as in flat space whereas modes that can feel spacetime curvature can get excited. A key consequence of modifying the FLRW dynamics *à la* LQC is that some long-wavelength perturbation modes relevant for the CMB now feel spacetime curvature before inflation, unlike in standard general relativity. This leads to enhanced power for small wavenumbers, or on large scales, which might explain the excess on large scales seen in the observations of CMB power spectra compared to the standard predictions from inflation.

We hence see how combining the new pre-inflationary physics suggested by LQC with the standard formalism of inflation can change the predictions of inflation in a subtle, but potentially physically very significant way.

2.5 Relation of LQC to LQG

We have seen that LQC uses key features of LQG, most notably the replacement of continuum connection fields by holonomies along nontrivial curves, but requires additional input in the construction of models that can then be used for cosmological phenomenology. Going back to the beginning, we saw that holonomy corrections lead to the replacement

$$c \to \frac{\sin(\mu c)}{\mu}\,. \tag{2.25}$$

where μ is usually taken to be proportional to $|p|^{-1/2}$. The physical reasoning behind this is that the "elementary loop" used to define the holonomy should be of Planckian physical length, so that its coordinate length must scale as $1/a$ with the expansion of the universe.

This is a physically reasonable assumption which leads to interesting modifications in cosmology. It also avoids self-consistency issues arising from other possible choices of μ. Nevertheless, it is an open problem how such cosmological dynamics could arise from full LQG.

To model a homogeneous, isotropic universe in full LQG one often uses a regular graph that appears homogeneous and isotropic on large scales; for instance, a cubic lattice in which all edges are required to have the same length and expand uniformly. One could then compute expectation

[13] I. Agullo, A. Ashtekar and W. Nelson, "The pre-inflationary dynamics of loop quantum cosmology: confronting quantum gravity with observations," Class. Quant. Grav. **30** (2013) 085014, `arXiv:1302.0254`

values for an LQG Hamiltonian constraint operator on such a graph, and this might reduce to an LQC-like Hamiltonian constraint at some initial time. However, to be consistent with the idea that edge lengths always remain Planckian, such dynamics would have to change the graph in a precise way, constantly generating edges and vertices while preserving overall homogeneity and isotropy. No proposal is known yet in full LQG that would achieve this.

A less fundamental issue is that in the construction detailed above the dynamics was reduced to FLRW universes at the classical level before LQG-like corrections were implemented. One could instead implement an LQG-like regularisation of the constraint before imposing FLRW symmetry. For instance, the Hamiltonian constraint in full general relativity in Ashtekar–Barbero variables consists of two terms,

$$\mathcal{C} = \frac{1}{16\pi G\sqrt{\det E}} E^i_a E^j_b \left(\epsilon^{ab}{}_c F^c_{ij}[A] - 2(1+\gamma^2) K^a_{[i} K^b_{j]} \right) \tag{2.26}$$

where F^a_{ij} is the field strength of the Ashtekar–Barbero connection A, γ is the Barbero–Immirzi parameter and K^a_i is the extrinsic curvature. For a flat FLRW universe, the curvature F is entirely given in terms of the extrinsic curvature since each spatial slice is (by assumption) flat; concretely one has

$$\epsilon^{ab}{}_c F^c_{ij}[A] = 2\gamma^2 K^a_{[i} K^b_{j]} \tag{2.27}$$

and the γ-dependent piece disappears. However, in the full theory the two different terms in $\mathcal{C}$ are typically regularised in a different way. Including these different regularisations into LQC before imposing (2.27) leads to an effective Hamiltonian constraint of the form

$$\gamma^2 \frac{\sin^2(\mu_0 b) v}{\mu_0^2} - \frac{1+\gamma^2}{4\mu_0^2} \sin^2(2\mu_0 b) v + \text{matter} \tag{2.28}$$

which clearly differs from the standard LQC expression

$$-\frac{\sin^2(\mu_0 b) v}{\mu_0^2} + \text{matter} \tag{2.29}$$

leading to different dynamics and phenomenology. In particular one now finds an emergent deSitter-like phase with Planck-size cosmological constant, which then transitions into the low-energy flat expanding universe[14]. (The matter part of the dynamics is unaffected by these ambiguities, as everywhere in LQC.) Other LQC-like models have been proposed recently, again using different combinations of ingredients of full LQG. We see that the passage from quantum gravity to cosmological models involves additional choices which cannot always be fundamentally justified. In models related to LQG these are typically related to discretisation or regularisation choices.

2.6 Summary

Loop quantum cosmology provides a setting which can in many ways be seen as an improvement of Wheeler–DeWitt quantum cosmology; where the latter must be defined intrinsically without guidance from what the full theory of quantum gravity might be, LQC models take various ingredients

[14] M. Assanioussi, A. Dapor, K. Liegener and T. Pawłowski, "Emergent de Sitter Epoch of the Quantum Cosmos from Loop Quantum Cosmology," Phys. Rev. Lett. **121** (2018) no.8, 081303, `arXiv:1801.00768`

that are crucial in the construction of LQG. The most important ingredient is the replacement of a continuum connection or curvature by regularised objects defined via finite holonomies and graphs. This introduces Planck-scale corrections into the dynamics already at the classical level and leads to a different Wheeler–DeWitt equation in the quantum theory.

The main achievement of this approach is a generic resolution of cosmological singularities, which arises from the replacement of unbounded by bounded functions of the curvature, implying an upper bound on the energy density of matter. Many features of the resulting cosmological dynamics can be studied in terms of effective classical Hamiltonians, but it is important to check which types of quantum states are well described by such effective equations.

Perturbations can be added in a way similar to standard quantum cosmology, and the modified background dynamics lead to corrections to predictions, e.g., of inflationary cosmology. The corrections are small but might become significant on largest scales. A remaining major challenge is to clarify further the relation of cosmological models to the full theory.

3 Cosmology from Group Field Theory

Towards the end of the last section we encountered some challenges in deriving LQC models more systematically from loop quantum gravity. In particular, one the main physical ideas in the derivation of LQC holonomy corrections was that the discreteness of spacetime – as given by the finite holonomies of a graph underlying the definition of LQG states – should correspond to a fixed (presumably Planckian) physical scale, meaning that the expansion of the universe corresponds to the generation of additional discrete degrees of freedom. In this part we will look at a related, but different approach to quantum gravity, *group field theory* (GFT) in which this key idea can be implemented more straightforwardly. The dynamics are now no longer defined in terms of a Hamiltonian constraint; the Friedmann equations of cosmology will instead appear as effective descriptions of GFT dynamics. Comparisons with other formalisms such as loop quantum cosmology then happen at the level of these effective semiclassical Friedmann equations.

3.1 Basic Ideas of Group Field Theory

One way of introducing group field theory is as a "second quantisation of spin networks". Recall from the introduction to loop quantum gravity that LQG quantum states are defined on graphs with a number of edges and vertices, wavefunctions are functions on the space of holonomies associated to graph edges and gauge transformations act on vertices implementing a notion of gauge invariance. In the reformulation of spin networks in GFT, one sees these as analogous to v-particle wavefunctions in normal quantum mechanics where v is the number of vertices in the graph.

Consider a restriction of all spin networks to spin networks built from 4-valent graphs. Such a restriction can be motivated in different ways; a perhaps intuitive one is that an n-valent vertex in LQG is interpret as representing a polyhedron with n faces and this would be the case in which the discrete geometry of space is only built from tetrahedra, which is the simplest possibility.

The elementary graph one can then consider has one vertex and four outgoing edges. LQG wavefunctions on such a graph are of the form

$$\psi(g_1, g_2, g_3, g_4)\,, \quad \psi(g_1, g_2, g_3, g_4) = \psi(g_1 h, g_2 h, g_3 h, g_4 h) \quad \forall\, h \in \mathrm{SU}(2) \tag{3.1}$$

where the equality arises from gauge transformations acting on the vertex. Following the usual recipe of "second quantisation", we now promote ψ to a quantum field and define a Fock space

$$\mathcal{H}_{\mathrm{Fock}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \tag{3.2}$$

where each $\mathcal{H}_n$ is a Hilbert space of spin networks built from 4-valent graphs with n vertices. As usual one can define this Fock space in terms of annihilation and creation operators where the annihilation operators map $\mathcal{H}_n$ to $\mathcal{H}_{n-1}$ (and the Fock vacuum $\mathcal{H}_0$ is mapped to zero), and the creation operators map $\mathcal{H}_n$ to $\mathcal{H}_{n+1}$. We will do this explicitly below.

In cosmological applications we would like to add matter degrees of freedom as well (recall that the group elements g_i above represent holonomies of the Ashtekar–Barbero connection, i.e., gravitational degrees of freedom). Scalars are naturally associated to 0-dimensional submanifolds, i.e., the vertices of a spin network. In our case we would then extend the elementary vertex wavefunction with an additional scalar-valued argument to obtain a wavefunction

$$\psi(g_1, g_2, g_3, g_4, \phi) \tag{3.3}$$

on $\mathrm{SU}(2)^4 \times \mathbb{R}$, or after a second quantisation a quantum field with domain space $\mathrm{SU}(2)^4 \times \mathbb{R}$. Note that this domain space does not have a spacetime interpretation; we are defining a quantum field theory *of*, not *on* spacetime. Indeed spacetime geometry only arises from the quantum excitations of the GFT field, which come with holonomy and matter degrees of freedom.

A main advantage of this "second quantised" reformulation of LQG is that it allows more easily dealing with changing particle number, just as it does in standard formulations of particle and condensed matter physics. This advantage will be key in developing interesting cosmological models which require a changing number of spacetime quanta.

3.2 Hamiltonian Formalism and Toy Model

GFT define a relatively general framework for quantum gravity in which one can now consider different models based on different actions. One may see the choice of action as a proposal for the dynamics of LQG, more concretely for the dynamics of 4-valent spin networks. Different models mainly differ in their choice of interactions between the building blocks of space (which one may picture as geometric tetrahedra) to form a macroscopic spacetime. In the cosmological application of GFT one assumes that these interactions are subdominant with respect to the kinetic term; this encodes the distinguishing feature of homogeneous cosmological models that time evolution dominates over the interactions (gradients) between different points of space. As a result the details of the GFT interaction terms will not be too important in much of what follows.

We will now present a Hamiltonian formalism for GFT, valid for a wide class of actions. As in QFT on Minkowski spacetime, this formalism makes use of a mode decomposition of the GFT quantum field φ which is defined on $\mathrm{SU}(2)^4 \times \mathbb{R}$. This *Peter–Weyl* decomposition is of the form

$$\varphi(g_1, \ldots, g_4, \phi) = \sum_{j_i, m_i, n_i, \iota} \varphi^{\vec{\jmath},\iota}_{\vec{m}}(\phi)\, \mathcal{I}^{\vec{\jmath},\iota}_{\vec{n}} \prod_{a=1}^{4} \sqrt{2j_a + 1}\, D^{j_a}_{m_a n_a}(g_a) \tag{3.4}$$

where the sum is over irreducible representations j_i (or spins) of SU(2); $D^j_{mn}(g)$ is the matrix representation of $g \in \mathrm{SU}(2)$ in the representation j; and $\mathcal{I}^{\vec{\jmath},\iota}_{\vec{n}}$ denotes an intertwiner for the representations j_i, i.e., an invariant map from the tensor product $j_1 \otimes j_2 \otimes j_3 \otimes j_4$ to the singlet $j = 0$. There may be multiple (or no) such intertwiners depending on the values of j_i, and these are labelled by ι. These details of SU(2) representation theory will not be important in the construction of cosmological models so readers that are not interested in them may safely ignore them.

We now consider actions of the form[15]

$$S[\varphi] = \int \mathrm{d}\phi \sum_{j_i, m_i, \iota} \overline{\varphi^{\vec{\jmath},\iota}_{\vec{m}}(\phi)} \mathcal{K}_{\vec{\jmath},\vec{m},\iota} \varphi^{\vec{\jmath},\iota}_{\vec{m}}(\phi) + \mathcal{V}[\varphi] \tag{3.5}$$

where we will assume that the field φ is real which implies the following relations for the Peter–Weyl components,

$$\overline{\varphi^{\vec{\jmath},\iota}_{\vec{m}}(\phi)} = (-1)^{\sum_i (j_i - m_i)} \varphi^{\vec{\jmath},\iota}_{-\vec{m}}(\phi)\,. \tag{3.6}$$

The coefficients $\mathcal{K}_{\vec{\jmath},\vec{m},\iota}$ are then also real. All terms higher than second order in the fields are part of $\mathcal{V}[\varphi]$ but as we commented above, the contribution from these terms will be neglected initially. As an example, a typical form for a quadratic GFT action would be

$$S[\varphi] = \int \mathrm{d}\phi\, \mathrm{d}^4 g\, \varphi(g_I, \phi) \left(\mu + \alpha \sum_i \Delta_{g_i} + \beta \partial^2_\phi \right) \varphi(g_I, \phi) \tag{3.7}$$

whree Δ_{g_i} is the SU(2) Laplacian acting on the i-th argument of φ. When written in the Peter–Weyl form (3.5) this would correspond to $\mathcal{K}_{\vec{\jmath},\vec{m},\iota} = \mu - \alpha \sum_i j_i(j_i + 1) + \beta \partial^2_\phi$.

The key idea in setting up the Hamiltonian formalism for GFT is to again view the scalar field ϕ as a matter clock, and hence to define derivatives with respect to ϕ as velocities of the field φ[16]. This is a *deparametrised* formalism in which one of the dynamical degrees of freedom is singled out as a time coordinate, rather similar to what we saw for LQC. One can show that, in order for ϕ to correspond to a free massless scalar field, the $\mathcal{K}_{\vec{\jmath},\vec{m},\iota}$ are of the form

$$\mathcal{K}_{\vec{\jmath},\vec{m},\iota} = \mathcal{K}^{(0)}_{\vec{\jmath},\vec{m},\iota} + \mathcal{K}^{(2)}_{\vec{\jmath},\vec{m},\iota} \partial^2_\phi \tag{3.8}$$

[15] For details of the derivation see S. Gielen, A. Polaczek and E. Wilson-Ewing, "Addendum to "Relational Hamiltonian for group field theory"," Phys. Rev. D **100** (2019) 106002, `arXiv:1908.09850`

[16] E. Wilson-Ewing, "Relational Hamiltonian for group field theory," Phys. Rev. D **99** (2019) no.8, 086017, `arXiv:1810.01259`.

with no further ϕ dependence on the right-hand side. In other words, each GFT field mode $\varphi^{\vec{\jmath},\iota}_{\vec{m}}(\phi)$ comes with a quadratic potential and a free coefficient in front of its kinetic term. Both free coefficients in (3.8) can take either sign, so that the dynamics of each mode is that of either a harmonic oscillator or that of an upside-down harmonic oscillator with *negative* quadratic potential. The latter case is the one of interest, since it allows a field mode to run away to infinity, corresponding to an exponentially expanding universe.

Exercise 3.1 *Consider an upside-down harmonic oscillator Hamiltonian*

$$\mathcal{H} = \frac{1}{2m}\left(p^2 - m^2\omega^2 x^2\right) . \tag{3.9}$$

Show that the classical solutions generated by this Hamiltonian are of the form

$$x(t) = Ae^{-\omega t} + Be^{\omega t} = A_1\cosh(\omega t) + A_2\sinh(\omega t) . \tag{3.10}$$

To complete the Hamiltonian analysis one defines the usual harmonic oscillator annihilation and creation operators $\hat{a}_{\vec{\jmath},\vec{m},\iota}$ and $\hat{a}^\dagger_{\vec{\jmath},\vec{m},\iota}$. The total GFT Hamiltonian is then

$$\hat{\mathcal{H}} = \sum_{\vec{\jmath},\vec{m},\iota} \hat{\mathcal{H}}_{\vec{\jmath},\vec{m},\iota} \tag{3.11}$$

with each $\hat{\mathcal{H}}_{\vec{\jmath},\vec{m},\iota}$ either of the form

$$\hat{\mathcal{H}}_{\vec{\jmath},\vec{m},\iota} = \omega_{\vec{\jmath},\vec{m},\iota}\left(\hat{a}^\dagger_{\vec{\jmath},\vec{m},\iota}\hat{a}_{\vec{\jmath},\vec{m},\iota} + \frac{1}{2}\right) \tag{3.12}$$

for an effective positive mass term or

$$\hat{\mathcal{H}}_{\vec{\jmath},\vec{m},\iota} = -\frac{1}{2}\omega_{\vec{\jmath},\vec{m},\iota}\left(\hat{a}^\dagger_{\vec{\jmath},\vec{m},\iota}\hat{a}^\dagger_{\vec{\jmath},-\vec{m},\iota} + \hat{a}_{\vec{\jmath},\vec{m},\iota}\hat{a}_{\vec{\jmath},-\vec{m},\iota}\right) \tag{3.13}$$

for an effective negative mass term. These are the standard expressions for a harmonic oscillator and an upside-down harmonic oscillator. Notice that the Fock vacuum annihilated by the operators $\hat{a}_{\vec{\jmath},\vec{m},\iota}$ is a ground state only for the first form, but not for the second which has no ground state, corresponding to its dynamical instability.

We will focus on the second type of Hamiltonian which is unbounded from below; this is a *squeezing* operator since states of the form

$$\exp(\mathrm{i}\hat{\mathcal{H}}\phi)|\psi_0\rangle = \exp\left[-\frac{\mathrm{i}}{2}\omega\phi\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right)\right]|\psi_0\rangle , \tag{3.14}$$

for suitable initial states $|\psi_0\rangle$, e.g., the Fock vacuum, are known as *squeezed states* in quantum optics and quantum field theory. Such states have highly semiclassical properties making them suitable candidates for an effective semiclassical universe coming out of GFT.

Including all GFT field modes into an analysis of the dynamics would be rather complicated. We will focus on a toy model in which only a single field mode is excited; we assume that this is one of the modes for which the dynamics are given by an upside-down harmonic oscillator and will hence consider a one-mode squeezing Hamiltonian

$$\hat{\mathcal{H}} = -\frac{\omega}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right) . \tag{3.15}$$

The corresponding Fock space is that of a single harmonic oscillator. The two main quantities of interest for cosmology are the energy $\langle\hat{\mathcal{H}}\rangle$ (corresponding to the canonical momentum π_ϕ of the massless scalar) and the volume given by the volume operator

$$\hat{V} = v_0\,\hat{a}^\dagger\hat{a}\,. \tag{3.16}$$

This form of the volume operator is taken from the interpretation of GFT states in LQG: the LQG volume operator is diagonal in the Peter–Weyl basis given by SU(2) representations. In other words, a 4-valent spin network vertex in LQG defines a *quantum of space* with volume given as a function of its representation labels j_i. Here we fix a single field mode, or the values for the j_i, and v_0 is then simply the volume of a quantum of the GFT field.

The relation (3.16) follows exactly the physical insights of the improved dynamics prescription of LQC: in this simple trunaction of GFT each quantum of space comes with a fixed volume and the total volume is then proportional to the number of quanta. Expansion of the universe can only proceed by generating new quanta. This is key to the emergence of LQC-type dynamics when we now derive effective cosmological Friedmann equations from this model.

3.3 Effective Friedmann Equations

Solving the Heisenberg equations for the volume operator $\hat{V}$ one finds

$$\hat{V}(\phi) = -\frac{v_0}{2} + \left(\hat{V}(0) + \frac{v_0}{2}\right)\cosh(2\omega\phi) + \frac{\mathrm{i}}{2}v_0\left(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}\right)|_{\phi=0}\sinh(2\omega\phi)\,. \tag{3.17}$$

The dynamics of this system hence does not only involve the Hamiltonian $\hat{\mathcal{H}}$ and volume $\hat{V}$ but a third operator

$$\mathrm{i}\left(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}\right) \tag{3.18}$$

with no direct geometric or cosmological interpretation. Indeed, these three operators together form a closed Lie algebra $\mathfrak{su}(1,1)$[17]. The expression for $\hat{V}(\phi)$ corresponds to a big bounce: for generic states, $\langle\hat{V}(\phi)\rangle > 0$ for all ϕ whereas for very early or late times $|\omega\phi| \gg 1$ the expectation value follows the exponentially expanding or contracting solutions for general relativity with a massless scalar field. A nonvanishing expectation value $\langle\mathrm{i}\left(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}\right)|_{\phi=0}\rangle$ introduces a time asymmetry in the effective cosmology, i.e., different pre- and post-bounce phases.

Taking an expectation value and after some simple algebraic manipulations, one now finds an *effective GFT Friedmann equation*

$$\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = 4\omega^2\left(1 - \frac{\rho_{\text{eff}}(\phi)}{\rho_c} + \frac{v_0}{V(\phi)}\right) \tag{3.19}$$

[17] S. Gielen and A. Polaczek, "Generalised effective cosmology from group field theory," `arXiv:1912.06143`.

where $V(\phi) := \langle \hat{V}(\phi) \rangle$, the *effective energy density* is defined as

$$\rho_{\rm eff}(\phi) = \frac{\langle \hat{\mathcal{H}} \rangle^2}{2V(\phi)^2} + \frac{\omega^2 V(0)\,(V(0)+v_0)}{2v_0^2 V(\phi)^2} - \frac{\omega^2 \langle (\hat{a}^\dagger)^2_{\phi=0} \rangle \langle \hat{a}^2_{\phi=0} \rangle}{2V(\phi)^2}\,, \tag{3.20}$$

and $\rho_c := \omega^2/(2v_0^2)$. Let us make a number of observations regarding the effective Friedmann equation (3.19).

1. At large enough volume $V(\phi)$, only the leading term $4\omega^2$ survives on the right-hand side. Consistency with the low-energy (or low-curvature) limit given by general relativity then fixes the GFT coupling constant to be $\omega = \sqrt{3\pi G}$. In this sense, in GFT Newton's constant is an emergent quantity arising from fundamental quantum gravity couplings. (Recall or rederive, comparing with our discussion in LQC earlier, that classically a flat FLRW universe filled with a massless scalar has $V(\phi) = V_0 \exp(\sqrt{12\pi G}\phi)$.)

2. The effective energy density consists of the first term which corresponds to the classical energy density $\rho = \pi_\phi^2/(2V(\phi)^2)$ – since ϕ is used as time, the Hamiltonian generating evolution in ϕ should be interpreted as the conjugate momentum π_ϕ of ϕ – and two other terms that can be seen as quantum corrections. The magnitude of these corrections depends on the choice of initial state. Note however that these terms also scale as $1/V(\phi)^2$ and so their net effect is to shift the scalar field momentum π_ϕ from its classical value $\langle \hat{\mathcal{H}} \rangle$.

3. As in standard LQC we find a critical density ρ_c appearing in the Friedmann equation. Using the value for ω in terms of Newton's constant we find that this is of the form

$$\rho_c = \frac{\omega^2}{2v_0^2} = \frac{3\pi G}{2v_0^2} = \frac{3\pi}{2}\rho_{\rm Pl}\left(\frac{v_{\rm Pl}}{v_0}\right)^2 \tag{3.21}$$

   which is generally of order of the Planck density. (Assuming that v_0 is also Planckian. If the quanta of geometry are chosen to be large compared to the Planck scale $v_0 \gg v_{\rm Pl}$, this would make ρ_c substantially lower than Planck density.)

Exercise 3.2 *Derive the effective GFT Friedmann equation (3.19) from the explicit solution (3.17).*

Notice that the effective Friedmann equation derived here is completely general; so far we have made no assumptions about the initial state. The price for this generality is that the quantum corrections appearing in the energy density, which depend on the initial state, are not determined at this point. One can now restrict these equations further by making a suitable choice of initial state. As in the other approaches to quantum cosmology we have been discussing, the quantum state should be semiclassical in a certain sense. In this GFT toy model, one criterion for semiclassicality is that the relative uncertainties in volume and energy become very small at late times,

$$\frac{(\Delta \mathcal{H})}{\langle \hat{\mathcal{H}} \rangle} \ll 1\,, \quad \frac{(\Delta V)}{\langle \hat{V} \rangle} \ll 1 \quad \text{as } \phi \to \pm\infty. \tag{3.22}$$

This is a rather minimal criterion for semiclassicality, only demanding that when the universe is large and far from the Planck regime, quantum fluctuations around the expectation values for the volume and energy become small. This criterion can be satisfied for example by taking the initial state to be a *Fock coherent state*

$$|\psi_0\rangle \propto \exp\left(\sigma\hat{a}^\dagger - \bar{\sigma}\hat{a}\right)|0\rangle \tag{3.23}$$

with $|\sigma| \gg 1$ and avoiding certain specific values such as $\sigma = |\sigma|\exp(\mathrm{i}\pi/4)$ for which $\langle\hat{\mathcal{H}}\rangle = 0$. One can show that for such a Fock coherent state

$$\rho_{\rm eff}(\phi) = \frac{\langle\hat{\mathcal{H}}\rangle^2}{2V(\phi)^2} + \frac{\omega^2 V(0)}{2v_0 V(\phi)^2}\,, \tag{3.24}$$

so that, while there is still a dependence on the initial condition $V(0)$, this initial condition has a geometric interpretation as the volume in the initial state, unlike the general form (3.20). Generic states of this form contribute to the time asymmetry term in (3.17) so that a generic cosmology in this model has different pre- and post-bounce phases. This would seem to raise the question of how to set initial conditions, even within the class of Fock coherent states, in order for this asymmetry to be quantified. This question has not been studied much so far, it has only been shown that the asymmetry is large in the general case, i.e., symmetric bounces would be seen as fine tuned.

3.4 Extensions

The simple toy model we studied, in which the GFT action is truncated to quadratic order and only a single field mode is considered, already shows how LQC-type bouncing dynamics can be derived from a full theory of quantum gravity. These results provide tentative evidence that a GFT definition of LQG dynamics can provide some fundamental basis to the physical assumptions made behind the derivation of LQC models. Nevertheless, as in all models we have discussed in these lectures, assumptions had to be made which would need independent justification.

An obvious extension of the toy model we have discussed would be to add interactions, i.e., terms of higher than quadratic order to the Hamiltonian. One form which has been studied is

$$\hat{\mathcal{H}} = -\frac{\omega}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right) + \frac{\lambda\omega}{4}(\hat{a}+\hat{a}^\dagger)^4\,. \tag{3.25}$$

Using the definition of the creation and annhilation operators in terms of the original GFT field the newly added term would correspond to an interaction of φ^4 type in a GFT action. The simplifying assumption which is made is to still restrict to a single GFT field mode, whereas the GFT interactions generally couple different field modes in the full theory.

There is a choice of sign for the coupling λ. For positive λ, the Hamiltonian would now be bounded from below, leading to a finite expansion of the universe followed by a recollapse when interpreted in cosmological terms. In contrast, negative λ means a φ^4 interaction without lower bound which accelerates the expansion of the universe to infinite size. The second case is more likely to correspond to a standard cosmology and we adopt the choice $\lambda < 0$.

For this interacting Hamiltonian the quantum dynamics can no longer be solved exactly. Instead, various approximations can be considered. For instance, one can use a semiclassical approximation in which the operators commute, and expectation values dominate over fluctuations (i.e., $\langle X^2\rangle \approx \langle X\rangle^2$) for the quantities of cosmological interest, energy and volume. One can then derive an effective Friedmann equation for (3.25) which takes the form

$$\begin{aligned}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 &= -\frac{2\omega^2 v_0^2}{\lambda^2 V(\phi)^2}\left(1+4\lambda\left(E-\frac{V(\phi)}{v_0}\right)\right)\left(1-4\lambda\frac{V(\phi)}{v_0}\right.\\ &\quad \left.-\left(1-2\lambda\frac{V(\phi)}{v_0}\right)\sqrt{1+4\lambda\left(E-\frac{V(\phi)}{v_0}\right)}+2\lambda E-\frac{3}{2}\lambda^2\right). \qquad (3.26)\end{aligned}$$

Here E is a constant of motion corresponding to the scalar field momentum π_ϕ divided by ω.

Exercise 3.3 *Show that for small λ the effective Friedmann equation (3.26) approximates to*

$$\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = 4\omega^2\left(1-\frac{E^2-\frac{3}{4}}{V(\phi)^2}v_0^2\right)+O(\lambda) \qquad (3.27)$$

and that $-E^2 v_0^2/V(\phi)^2 = -\rho_\phi/\rho_c$ with ρ_c as defined before.

As expected, the non-interacting case $\lambda = 0$ reduces essentially to what we had previously, an LQC-like effective Friedmann equation; the low-energy limit would still demand that we set $\omega = \sqrt{3\pi G}$.

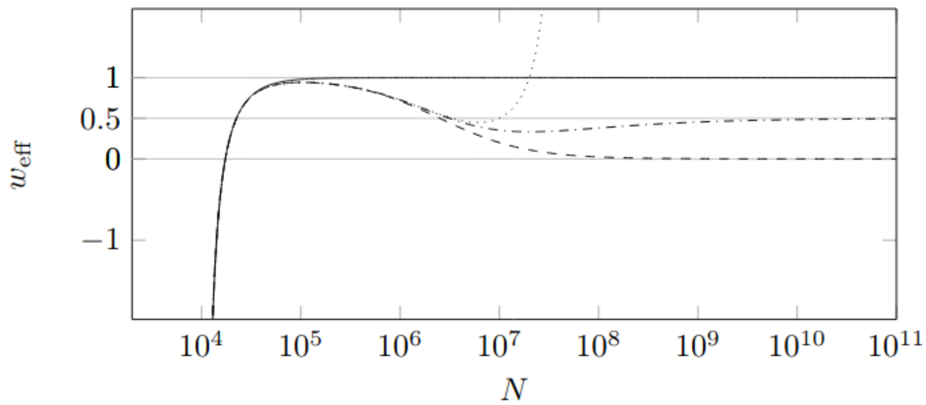
In this model the overall strength of the interactions depends on the number of quanta, since each quantum interacts with every other quantum through the Hamiltonian (3.25). Hence the free approximation where λ is small is valid for a small volume, or at high curvatures near the bounce. As the universe expands, the number of quanta grows exponentially and interactions become stronger until they dominate completely. In the cosmological interpretation of this model, this implies that an approximate form of the Friedmann equation at large volumes is no longer given by the classical dynamics for a free massless scalar field (or matter with equation of state parameter $w = 1$) but by an effective matter component that dilutes more slowly with the expansion of the universe. Indeed this is what one finds:

Exercise 3.4 *Show that for large volume the effective Friedmann equation (3.26) approximates to*

$$\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = 96\pi G\sqrt{-\frac{\lambda V(\phi)}{v_0}}+O(V^0). \qquad (3.28)$$

The right-hand side takes the form of matter with an effective equation of state parameter $w = \frac{1}{2}$. This is a fully non-perturbative regime dominated by the interaction term in the Hamiltonian.

These classical calculations can be extended to the quantum regime by numerically solving the Schrödinger equation for a choice of initial state such as a Fock coherent state. The results are largely in agreement with these classical calculations in that one seems to find a regime effective described by matter with $w = \frac{1}{2}$, but numerics typically break down quickly when the interactions become too strong.



(Image taken from S. Gielen and A. Polaczek, "Generalised effective cosmology from group field theory," `arXiv:1912.06143`.) The plots show an effective equation of state parameter as a function of N (the number of quanta proportional to the volume). The solid lines correspond to truncation at zeroth order in λ; dashed lines correspond to a truncation at first order and dotted lines correspond to a truncation at second order. The dash-dotted lines correspond to the full nonperturbative case.

This simple example thus already shows how the cosmology of GFT can develop rather nontrivial features as soon as one goes beyond the free case of a quadratic Hamiltonian. Notice however that the quadratic Hamiltonian still accurately captures the regime of the bounce itself, where the dynamics is of the form of the LQC improved dynamics.

3.5 Summary

In the discussion of loop quantum cosmology (LQC) models we encountered difficulties in deriving the physically motivated form of quantum gravity (holonomy) corrections from the dynamics of loop quantum gravity. Intuitively, we would like the expansion of the universe to correspond to the generation of new quanta of geometry, where these quanta themselves are of fixed Planckian size. It is not clear how to achieve this in the Hamiltonian setting of LQG.

We saw how the group field theory (GFT) framework provides a mechanism in which the physical picture of LQC can be realised. GFT defines a type of second quantisation of LQG degrees of freedom, in which there are annihilation and creation operators acting on a Fock space. In the truncation to a quadratic GFT action, the dynamics of each field mode are given either by a harmonic oscillator or an upside-down harmonic oscillator. The latter case is of interest as it leads to quantum squeezing, or exponentially growing solutions.

The dynamics of a quadratic GFT Hamiltonian could be studied explicitly, leading to LQC-type effective Friedmann equations which again predict a bounce. There are in general additional terms in these equations corresponding to a freedom of initial conditions, which can be specified further by choosing, e.g., a class of semiclassical states. We discussed the extension of the dynamics to interacting GFT dynamics, in which the resulting dynamics becomes more complicated and can be interpreted as the appearance of new effective matter components at late times.