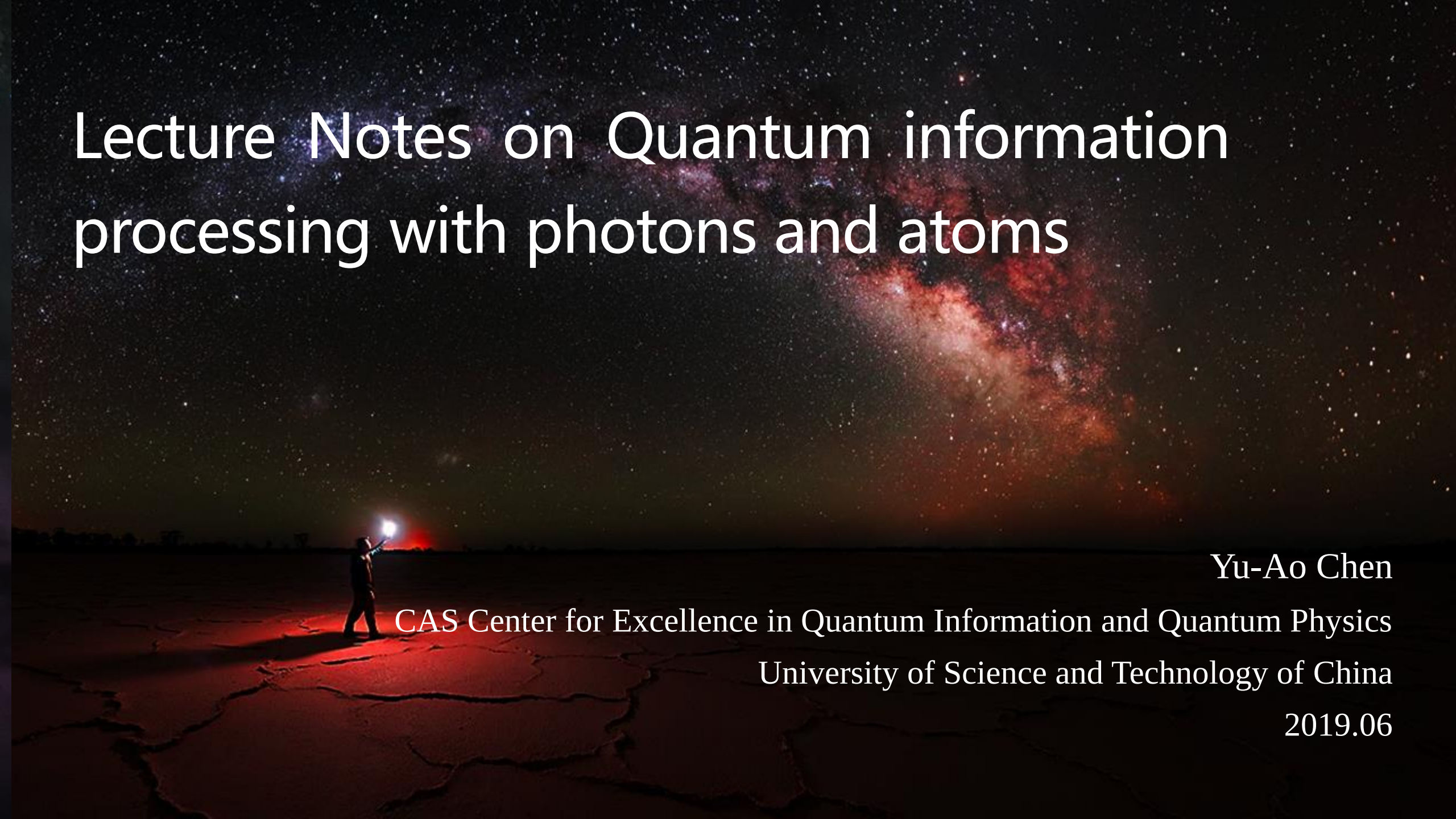
Lecture Notes on Quantum information processing with photons and atoms



Yu-Ao Chen

CAS Center for Excellence in Quantum Information and Quantum Physics

University of Science and Technology of China

2019.06

Introduction to our group

Excellence Center for Quantum Information and Quantum Physics

- Jointly supported by CAS and the Ministry of Education

Hosted by **USTC**

includes top institutes and universities on quantum physics

Shanghai Institute of Technical Physics

Institute of Semiconductor

Institute of Optics and Electronics

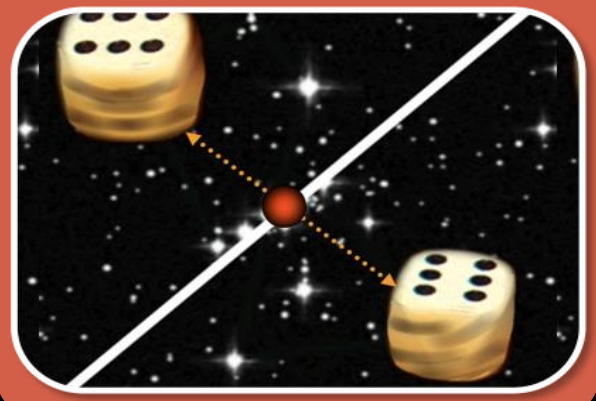
Nanjing University

and excellence groups among China's universities:

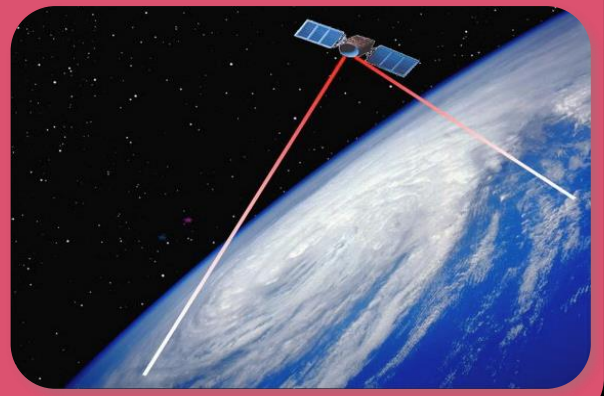
Tsinghua University, Peking University, Jiaotong University, etc.

Introduction to our group

Quantum Foundations



Quantum Communication



Quantum Physics & Quantum Information

Quantum Metrology

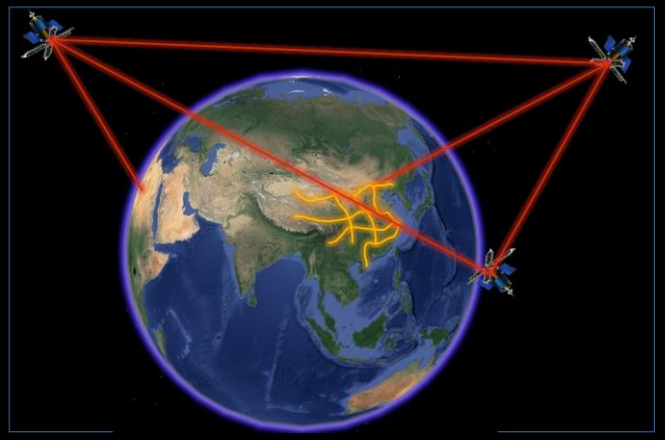
Quantum Computation & Simulation

Visit us at http://quantum.ustc.edu.cn

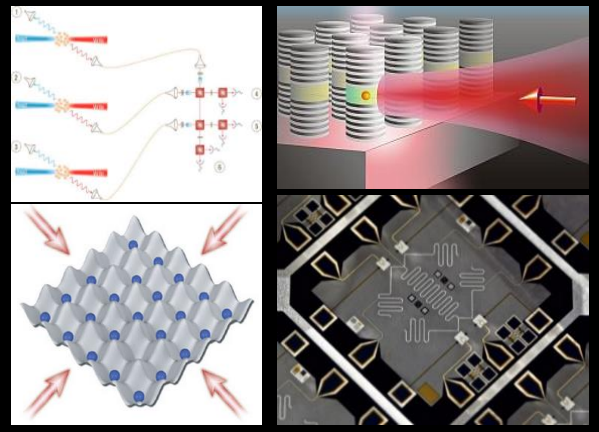
China’s Future National Projects

The Center is now playing a leading role in organizing

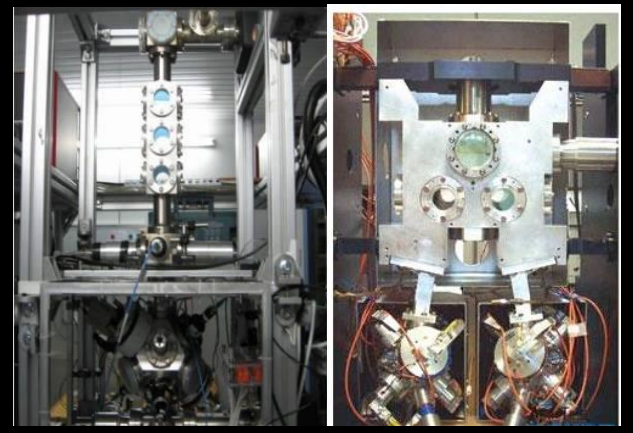
- National Science and Technology Project on Quantum Information in the next 15 years, similar to European Quantum Technologies Flagship
- National Laboratory for Quantum Information Sciences (NLQIS)



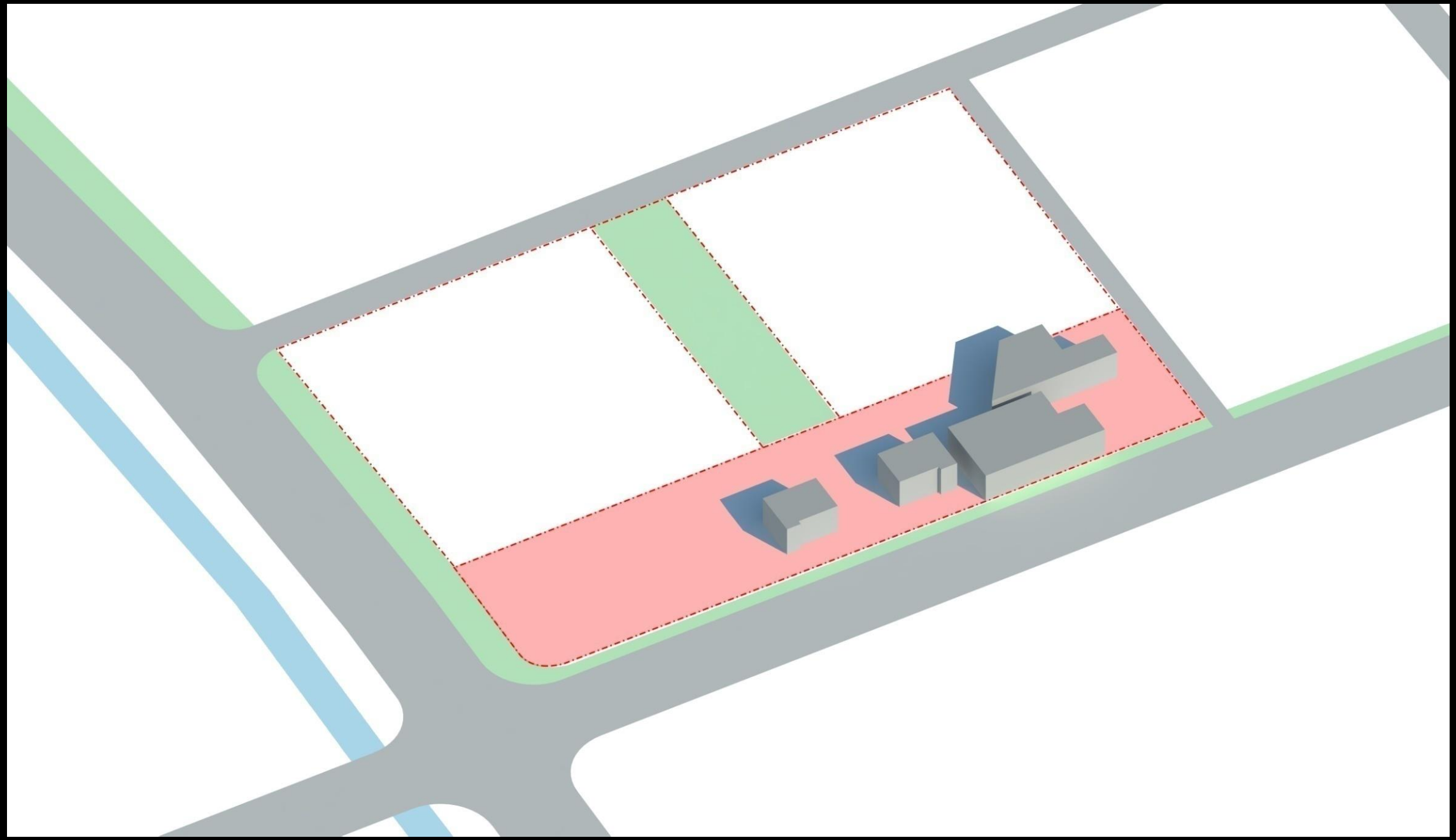
Global Quantum Communication Networks

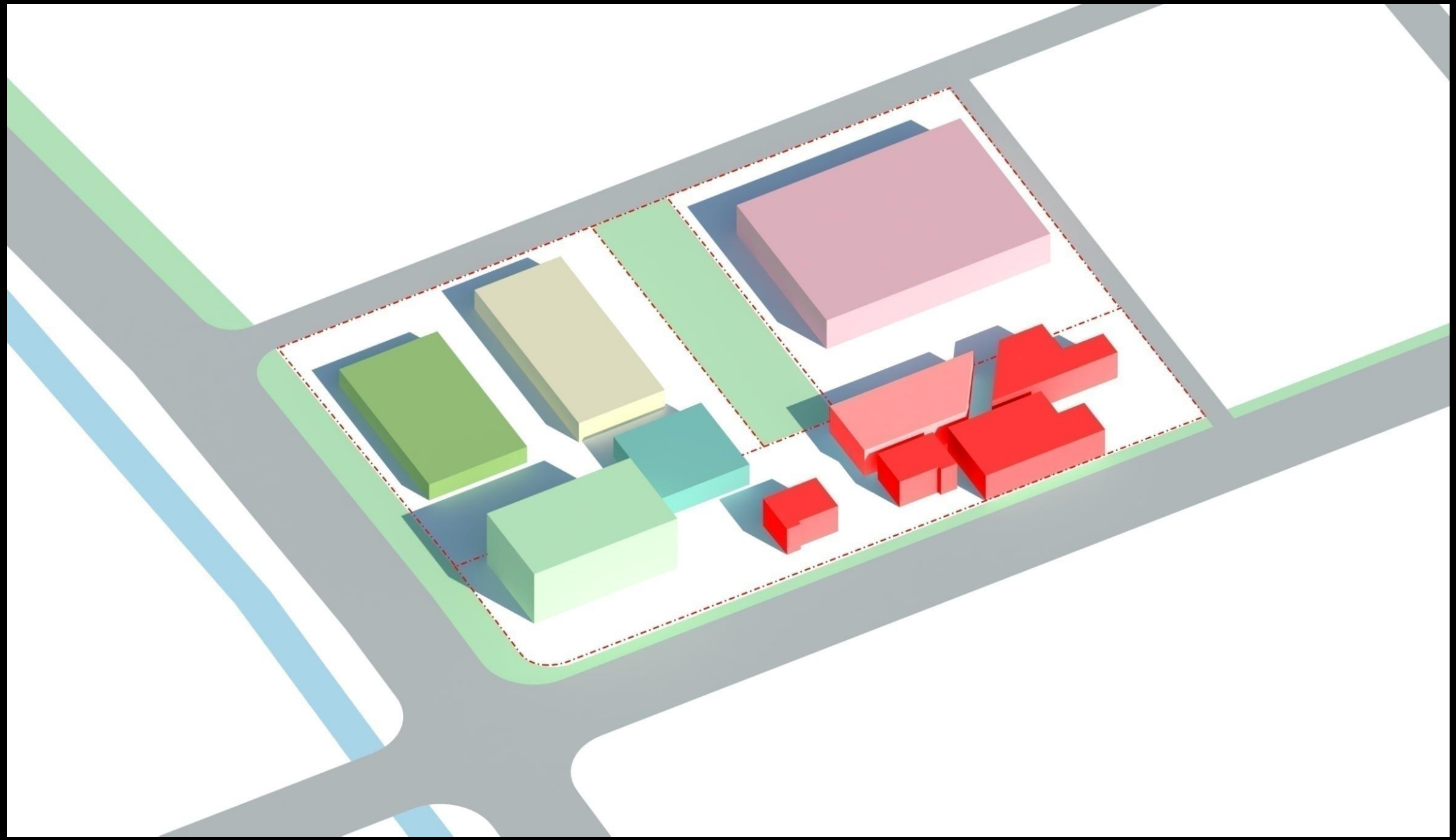


Scalable Quantum Computation and Quantum Simulation



Super-resolution Quantum Metrology






Ph. D & Postdoc position available

Lecture 1: Introduction to Quantum Physics and Quantum Information

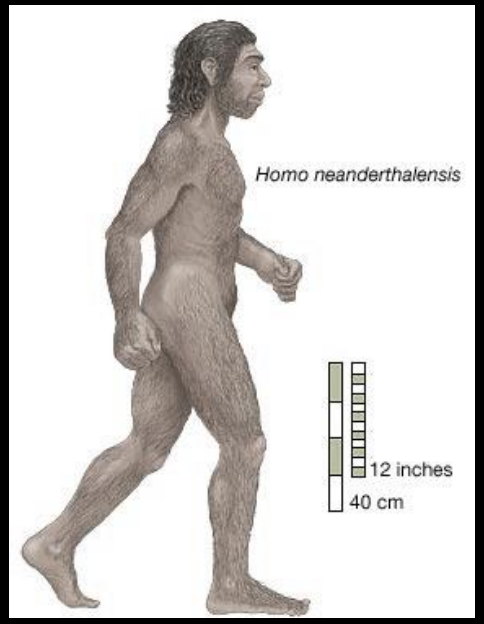
Part 1: Quantum Foundations

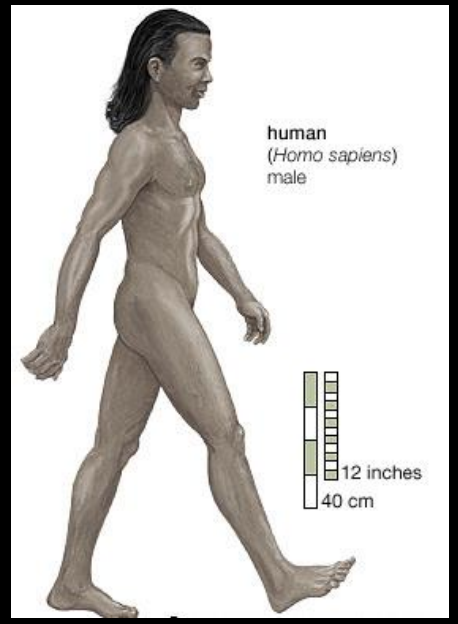
Information Exchange in Human Evolution

About 100,000 years ago, *Homo Neanderthalian* and *Homo Sapien* co-existed in Europe

Homo Neanderthalian ➡

- Stronger
- With even larger cranial capacity than modern human!





⬅ Homo Sapiens

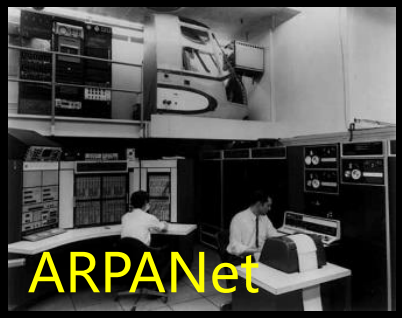
Individually weaker than Homo Neanderthalian, but developed proto- symbol and language

Why did Homo Sapien win the evolution battle and become our ancestors?

Information exchange ➡ ***Coordinating groups***

Social Advancement & Privacy Protection

Privacy ➡ Freedom of thoughts ➡ Innovation and social advancement!



1765 1876 1946 1969

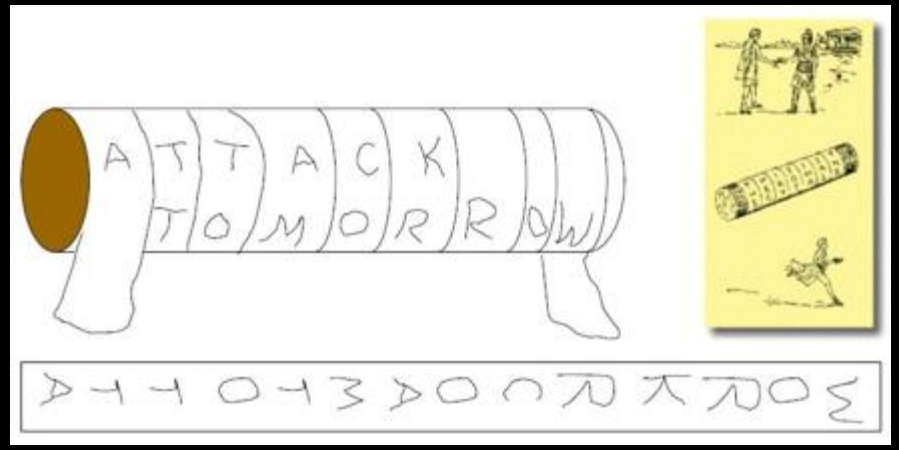
Social progress makes information exchange more efficient

The information exchange has been and will continue to be accompanied by the human evolution and social development

Ever lasting questions:

- How to make information exchange more efficient? ⇒ **Computing power**
- How to protect privacy? ⇒ **Information security**

Challenges in Information Security



Ancient Greek scytale, 400 BC

Caesar cipher, 50 BC

《送杜少府之任蜀川》

——王勃

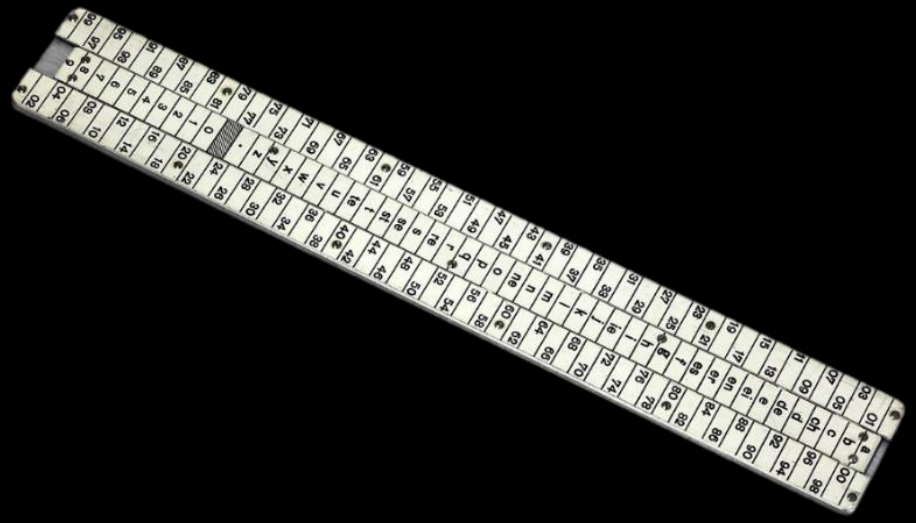
城阙辅三秦，风烟望五津。
与君离别意，同是宦游人。
海内存知己，天涯若比邻。
无为在歧路，儿女共沾巾。

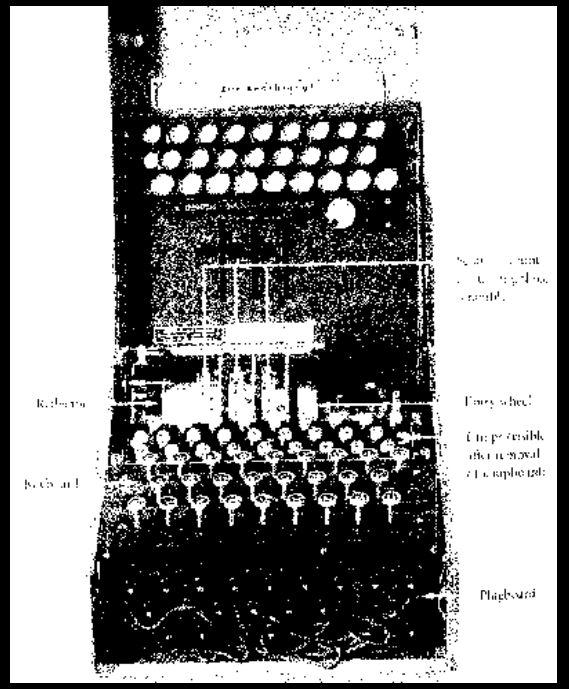
1请弓、2请箭、3请刀、4请甲、5请枪旗、6请锅幕、7请马、8 请衣赐、9请粮料、10请草料、…、39都将病、40战小胜

11th century, 北宋《武经总要》：**替换式密码**

16th century, France, Vigenère cipher

Challenges in Information Security



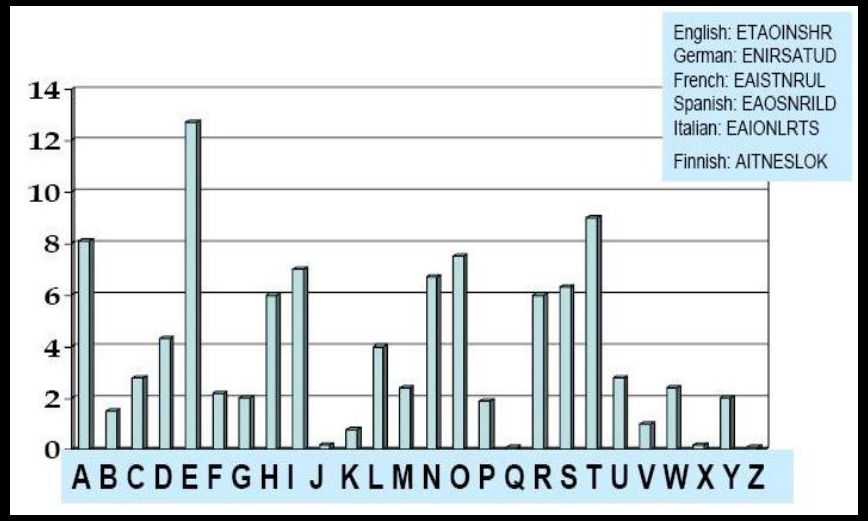


20th Century, Switzerland

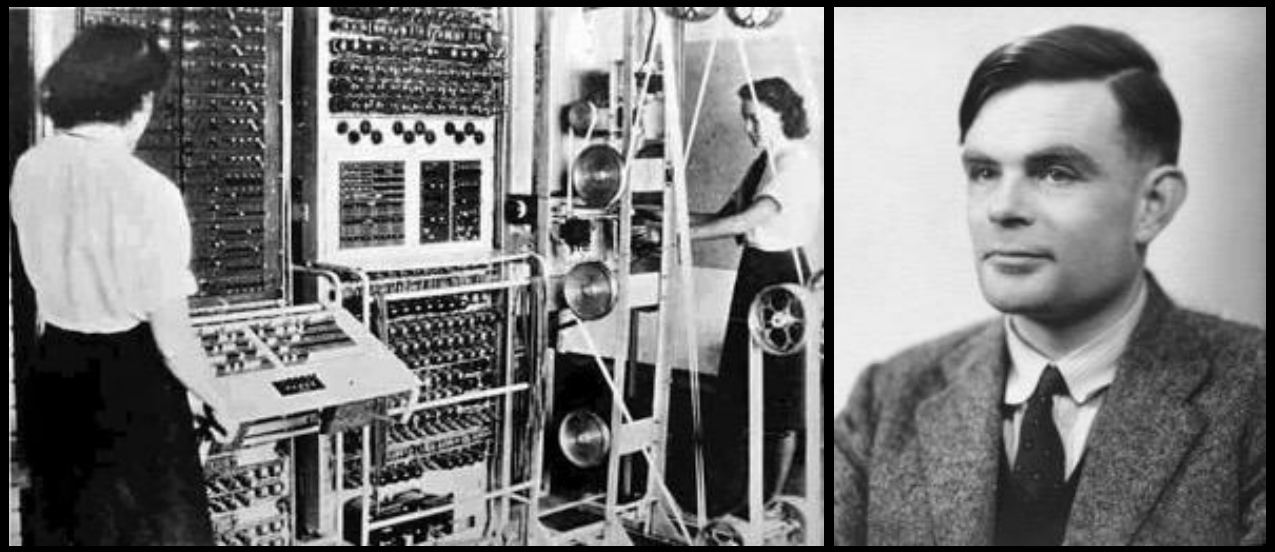
Enigma Machine

Classical encryption based on computational complexity

Challenges in Information Security



Crack via variations in the frequency of the occurrence of letters, by Al-Kindi (800-873)



Enigma machine broken by Alan Turing's Bombe machine

- ☒ RSA 512: cracked in 1999

  RSA 768: cracked in 2009

  RSA 1024: ? shall not be used after December 31, 2013 by NIST
- ☒ The next-generation code "pairing-based cryptography" Cracked in 2012……
- ☒ Feb. 2017, SHA-1 cracked by Google

All the classical encryption methods that depend on computational complexity, can be cracked in principle!

"……human ingenuity cannot concoct a cipher which human ingenuity cannot resolve"

—*A few words on secret writing*, Edgar Alan Poe (1841)

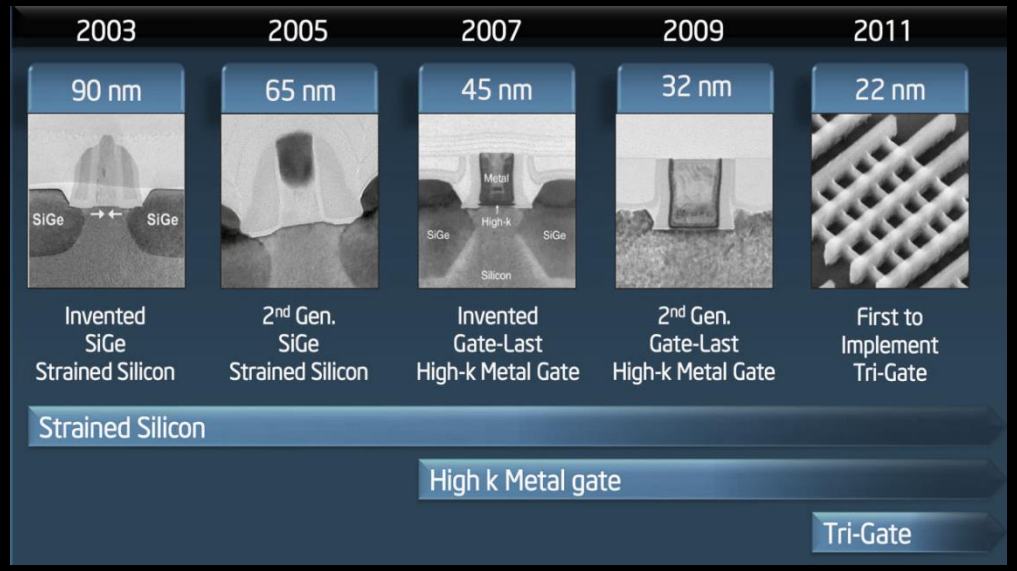
Challenges in the Computational Capacity

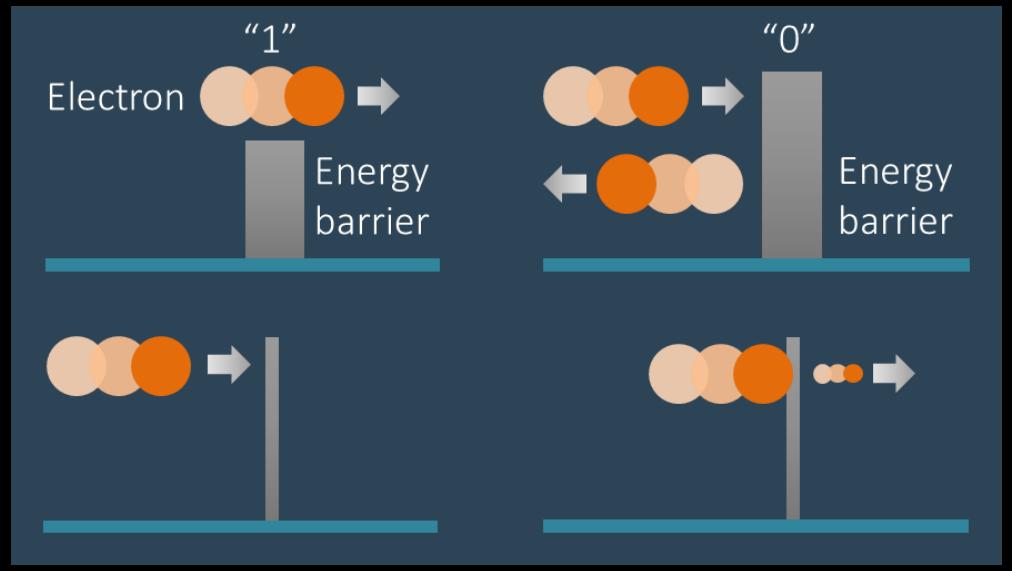
Classical computational bottleneck

The world's total computing power is insufficient to search a target in $2^{80\text{-}90}$ database within a year

A technological limit

The Moore's law that predicts the transistor density doubles every 18 months has come to an end





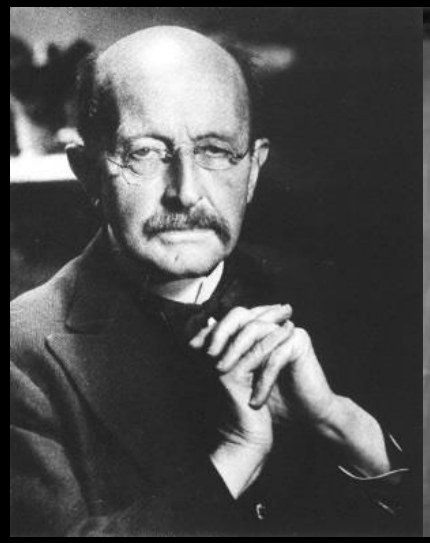
2017, 14 nm ➡ 2022, 4 nm ➡

0.2 nm (atomic scale) ➡ ???

Tunneling induced leakage ➡

The "0/1" logic in the transistors will fail

Quantum physics, after one century's development, comes to the rescue for the problems confronted in the classical information technologies


Max Planck


Albert Einstein


Niels Bohr


Erwin Schrödinger


Werner Heisenberg

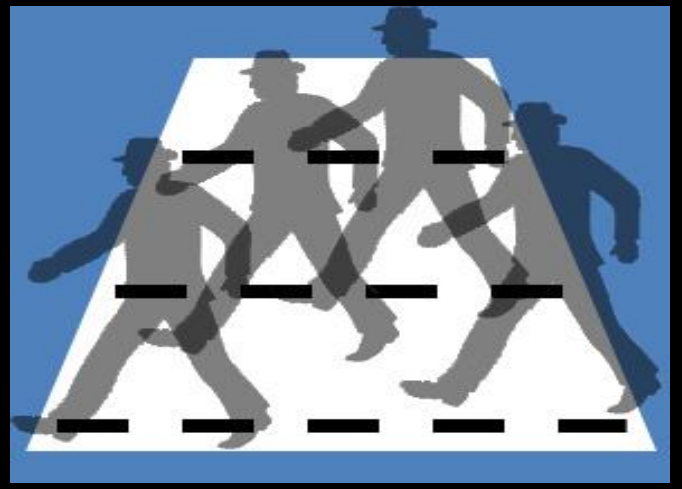

Paul Dirac


Quantum Physics
for Babies
by Chris Ferrie

Quantum Superposition

Classical World

$|here\rangle$ or $|there\rangle$

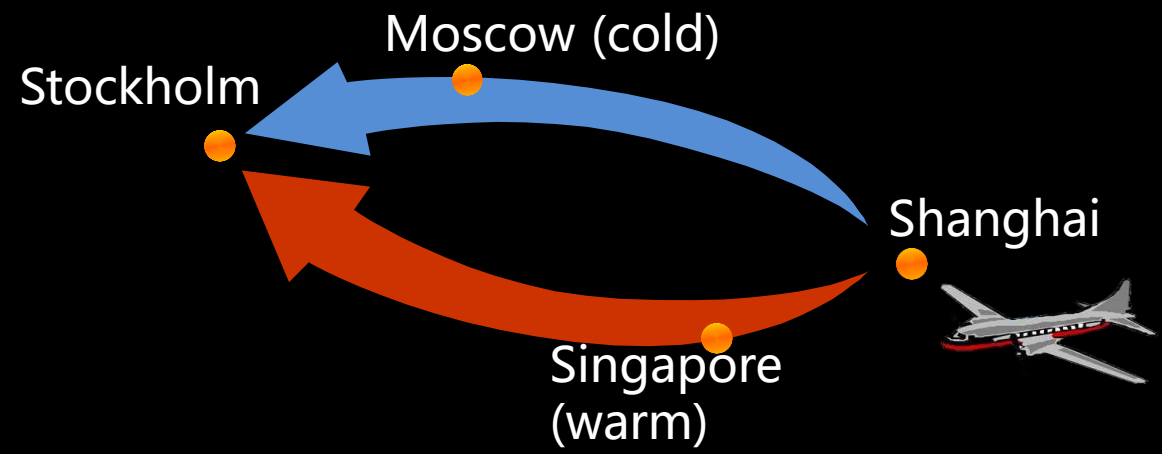


Quantum World

$|here\rangle + |there\rangle$

Quantum Superposition

A “quantum flight”: from Shanghai to Stockholm, two possible routes:



When arrived

- If I fell asleep during flight (do not know which route I take), I will feel : “both cold and warm”

I took both routes in one flight!?

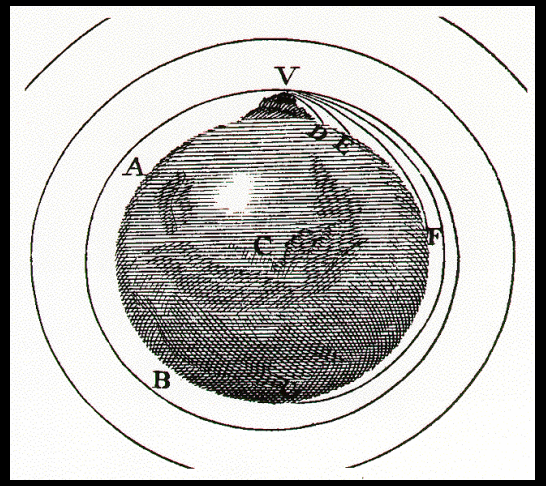
- If I was awake during flight and checked which route I take, I will Feel either cold (Moscow) or warm (Singapore)

It confirms I can only take one of the routes!

In quantum world, the state of a quantum object can be affected by measurement!

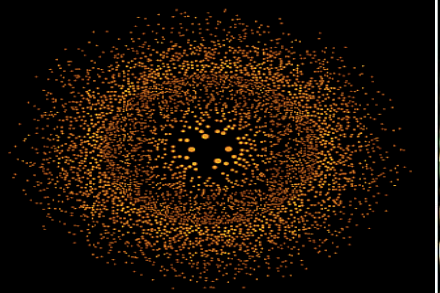
When Classical Physics Meets Life Philosophy

Newton's law precisely predicts every single movement for all objects in our daily life

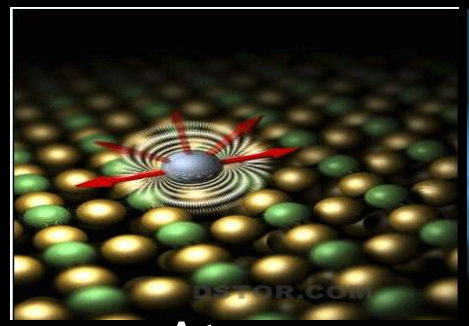


- *A manifest of the beauty and power of physics!*
- *However, does it imply determinism?*
- *Does it mean everything (e.g. lectures today) is already determined from Big-bang?*
- *Efforts meaningless?*
- ***Fortunately, quantum mechanics tells that your act (measurement) can affect the world!***

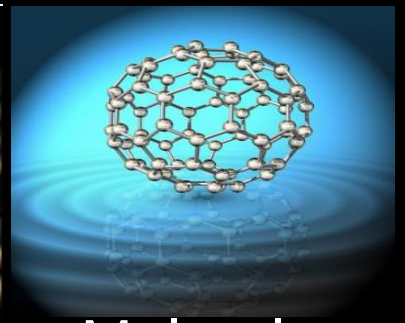
Qubit & Quantum Superposition



Photon

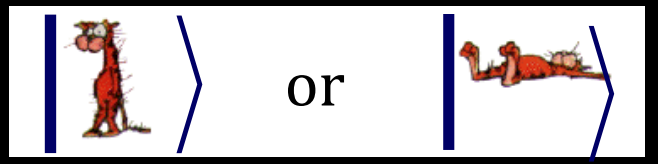


Atom

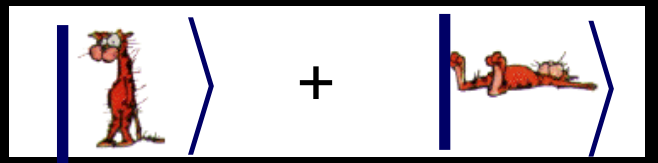


Molecular

Classical Physics: “bit”

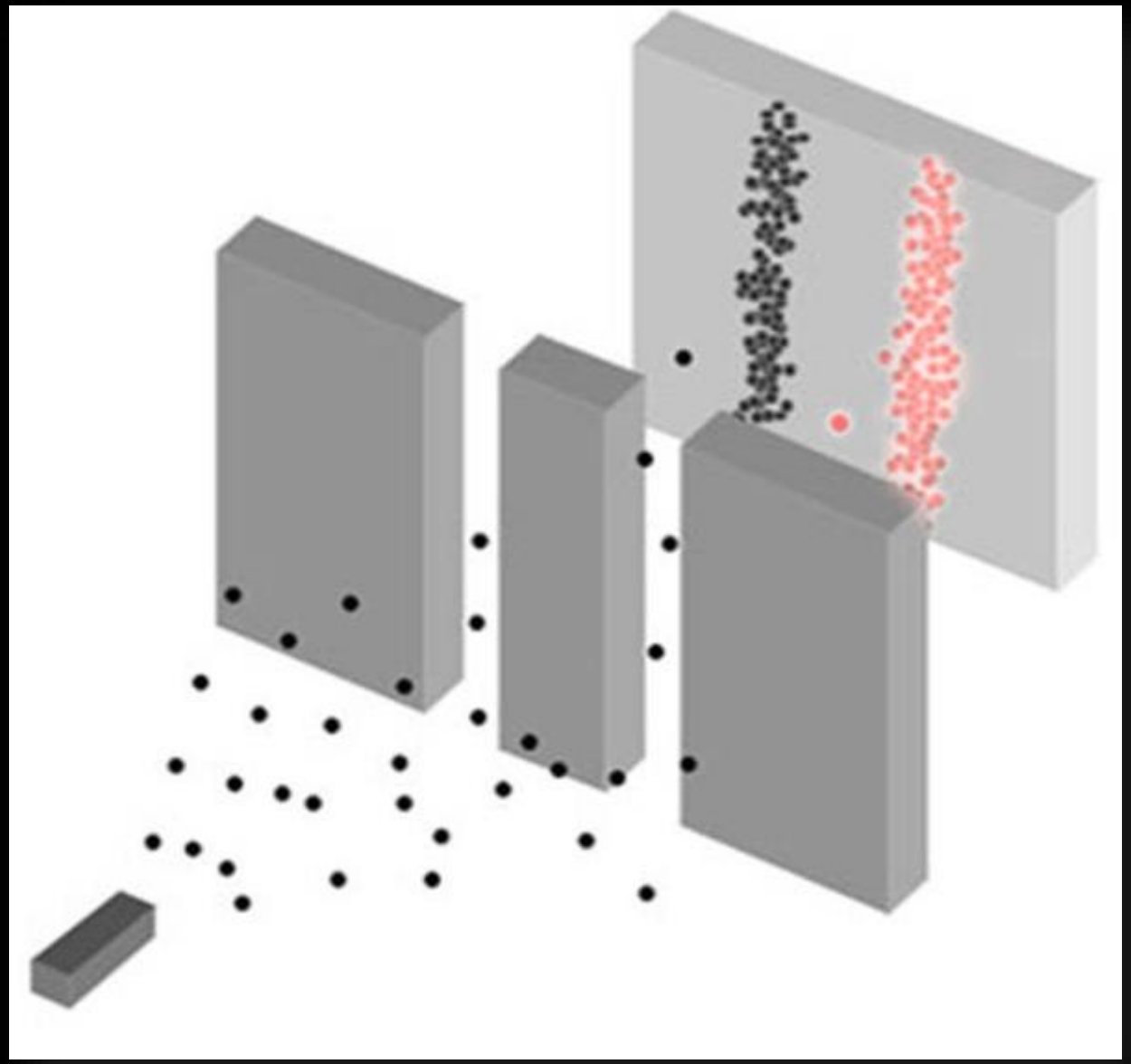


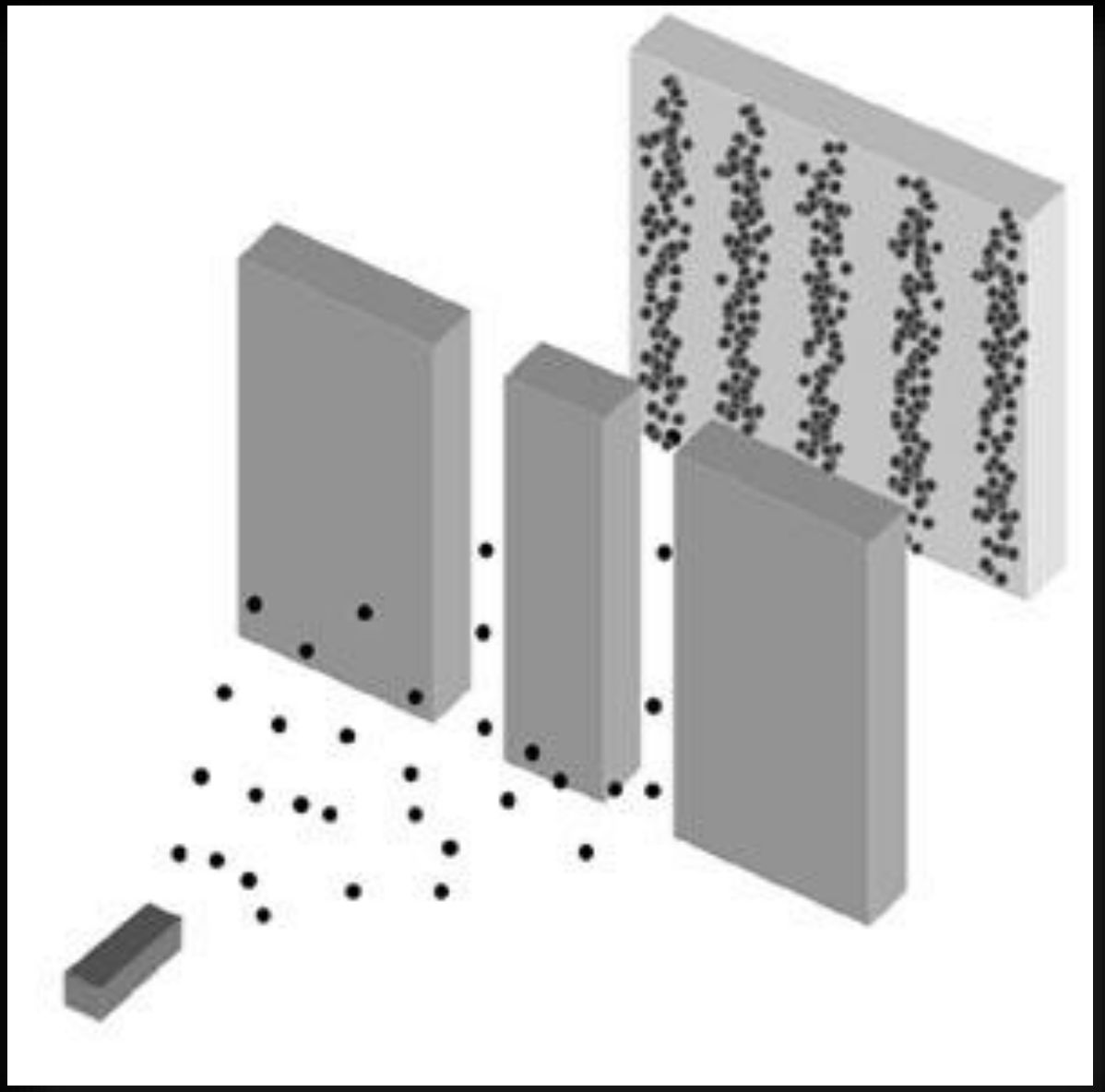
Quantum Physics: “qubit”



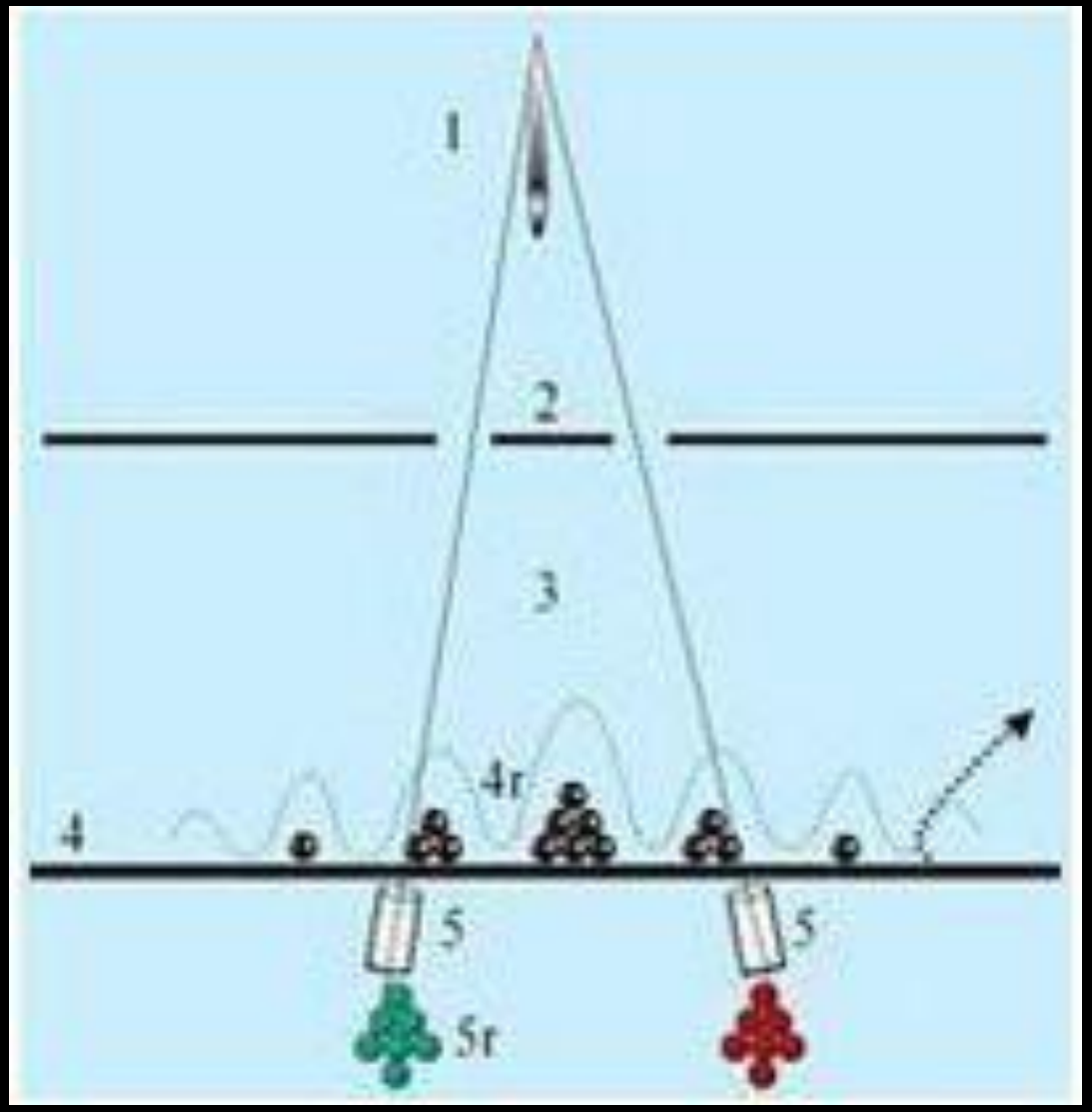
Quantum Superposition

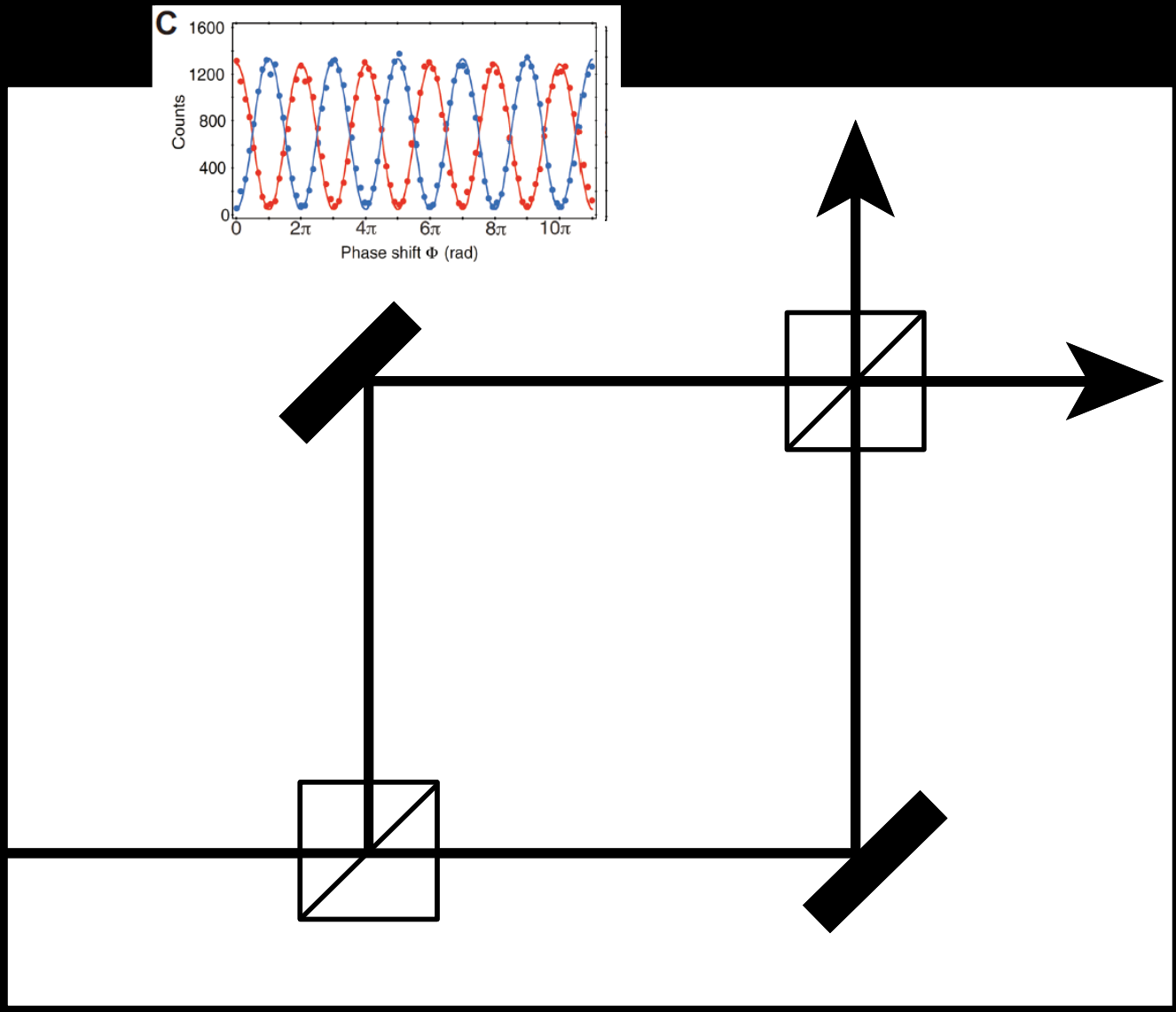
Quantum Superposition



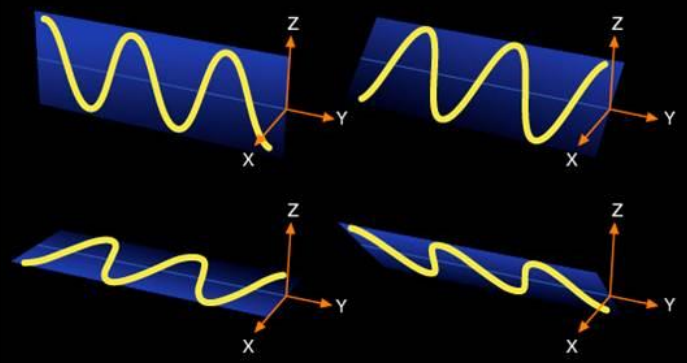


Quantum Superposition





Qubits: Polarization of Single Photon



- One bit of information per photon (encoded in polarization)

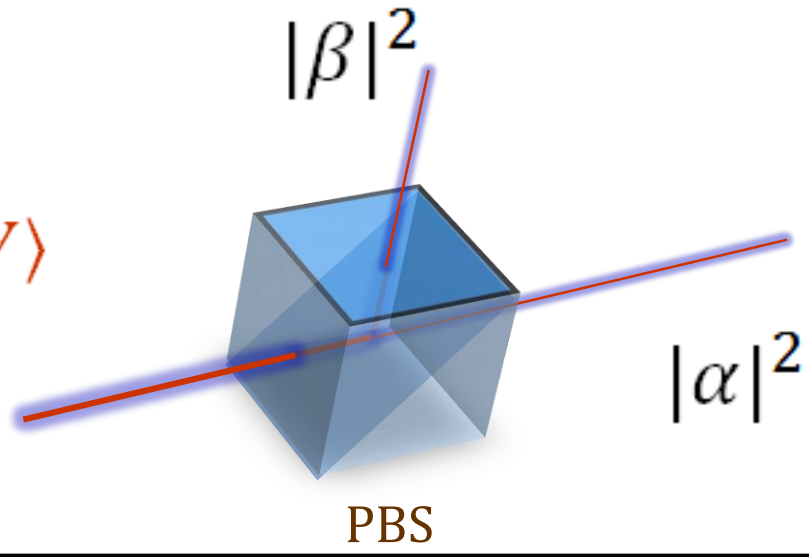
$|H\rangle$ >=|"0">

$|V\rangle$ >=|"1">

- Qubit:

$$|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$



- Non-cloning theorem:

  An unknown quantum state can not be copied precisely!

$00 \to 00,$

$10 \to 11,$

$(0+1)0 \to 00+11$

$\neq (0+1)(0+1)$

Single-Qubit Operations

Column vector represent of two-dimensional quantum states

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Pauli matrix $\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Two eigenstates:

$|+\rangle = 1/\sqrt{2}\,(|H\rangle + |V\rangle)$

$|-\rangle = 1/\sqrt{2}\,(|H\rangle - |V\rangle)$

Unitary rotation:

$\sigma_X|H\rangle = |V\rangle$

$\sigma_X|V\rangle = |H\rangle$

Pauli matrix $\sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Two eigenstates:

$|L\rangle = 1/\sqrt{2}\,(|H\rangle - i|V\rangle)$

$|R\rangle = 1/\sqrt{2}\,(|H\rangle + i|V\rangle)$

Unitary rotation:

$\sigma_Y|H\rangle = -i|V\rangle$

$\sigma_Y|V\rangle = i|H\rangle$

Pauli matrix $\sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Two eigenstates:

$|H\rangle$

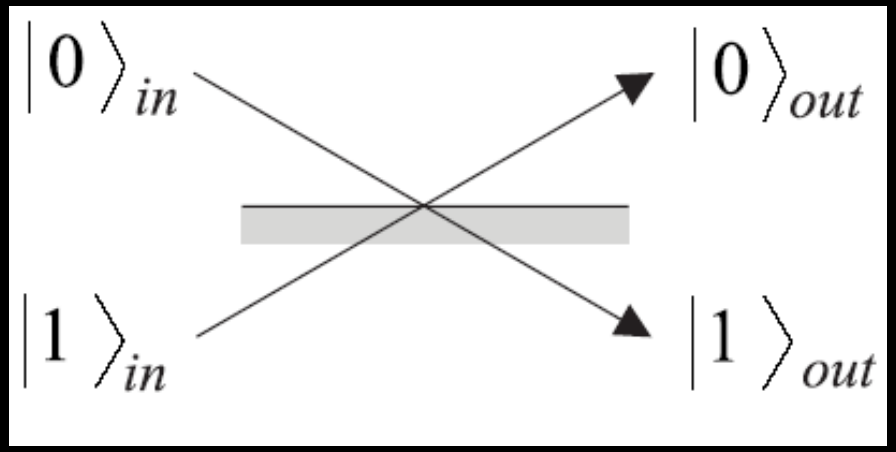
$|V\rangle$

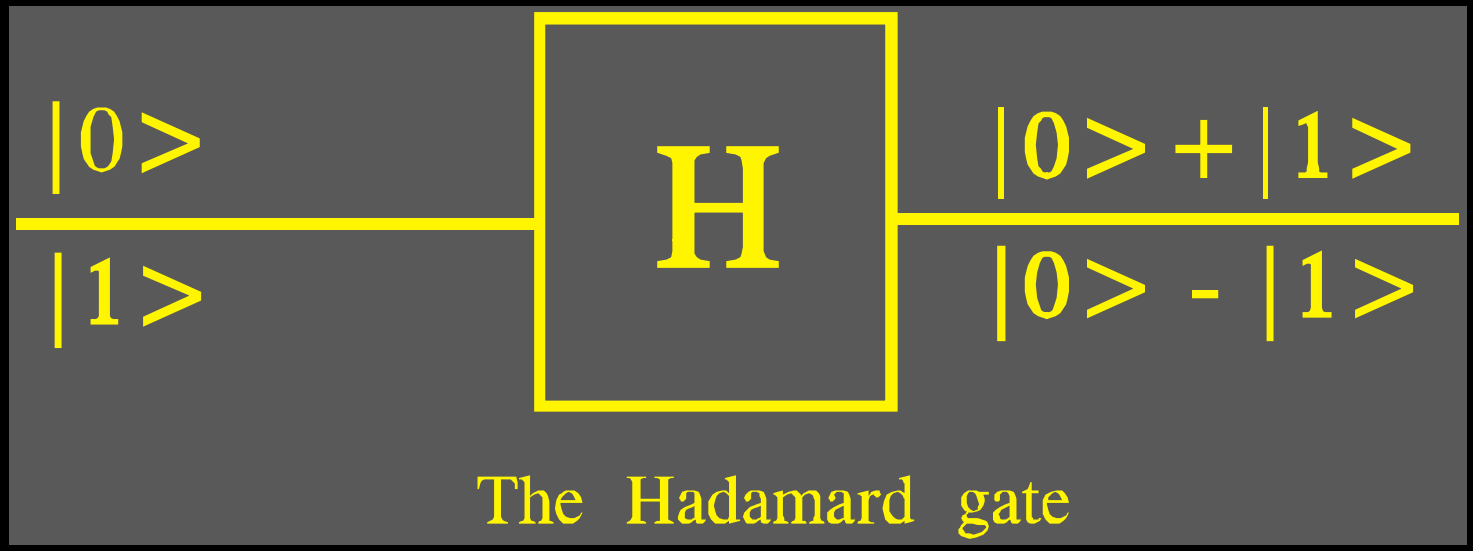
Unitary rotation:

$\sigma_Z|H\rangle = |H\rangle$

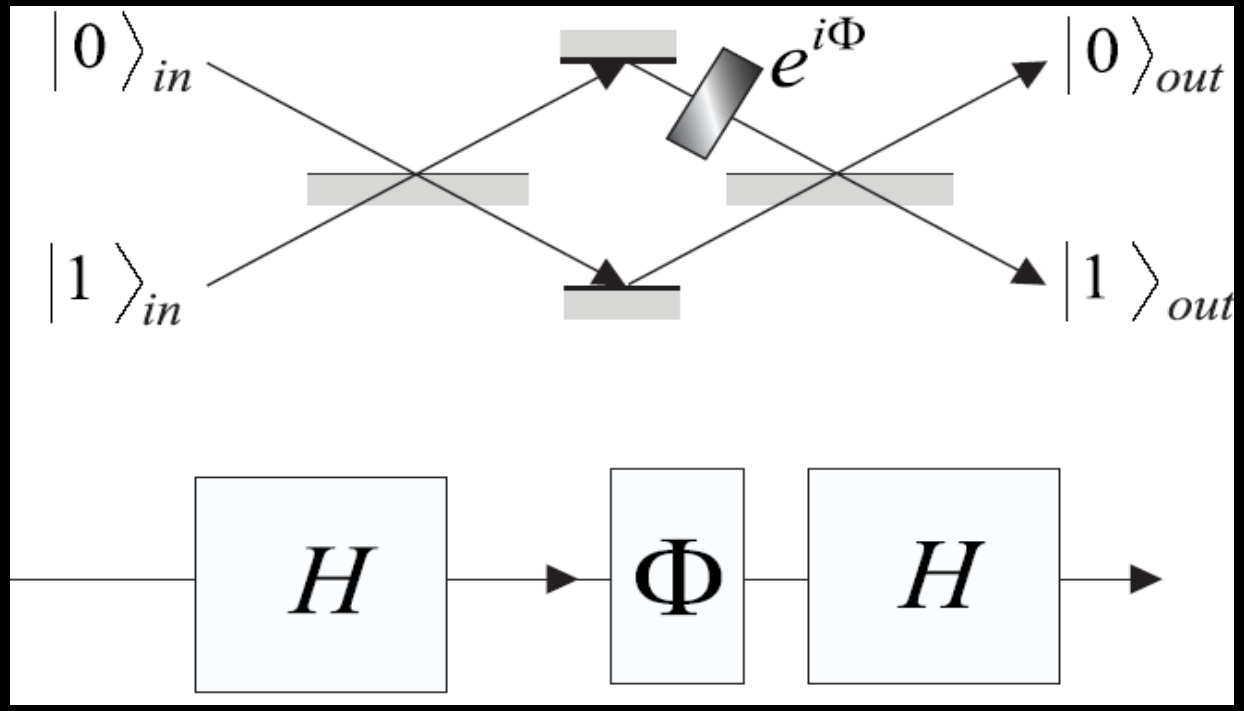
$\sigma_Z|V\rangle = -|V\rangle$

Single-Qubit Operations



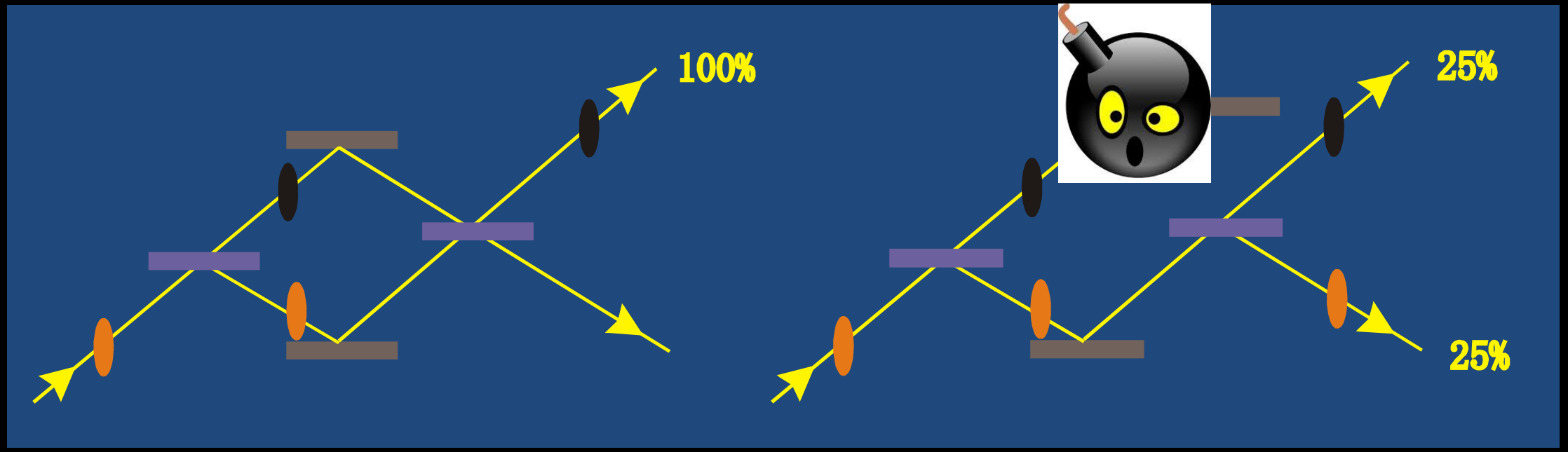


Mach-Zehnder Interferometer



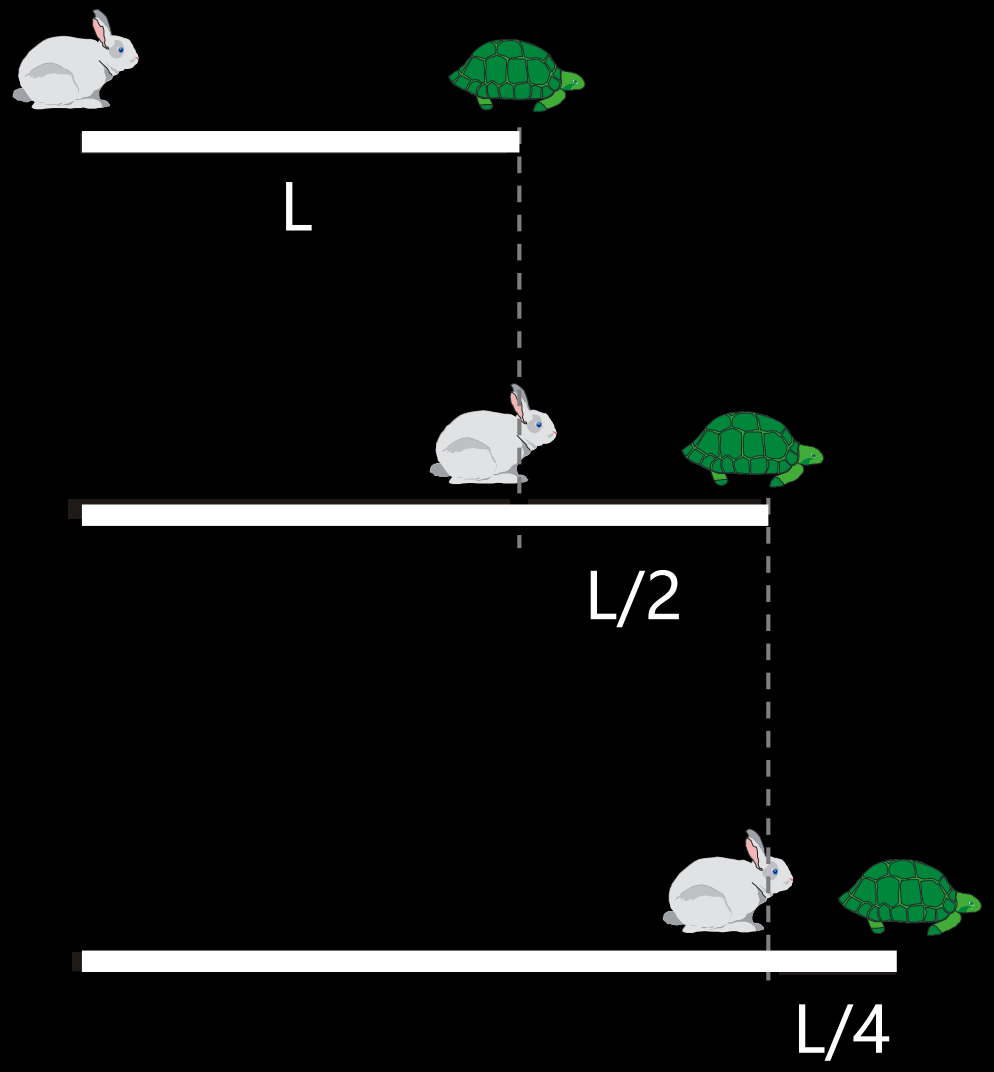
$$H\Phi H|0\rangle = \frac{1}{2}((e^{i\varphi}+1)|0\rangle + (e^{i\varphi}-1)|1\rangle)$$

Mach-Zehnder Interferometer



Interaction-free measurement !

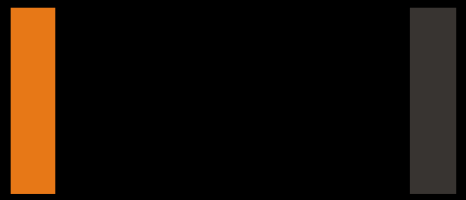
Zeno Paradox



Origin of Zeno effect

Can the rabbit overtake the turtle?

Quantum Zeno Effect



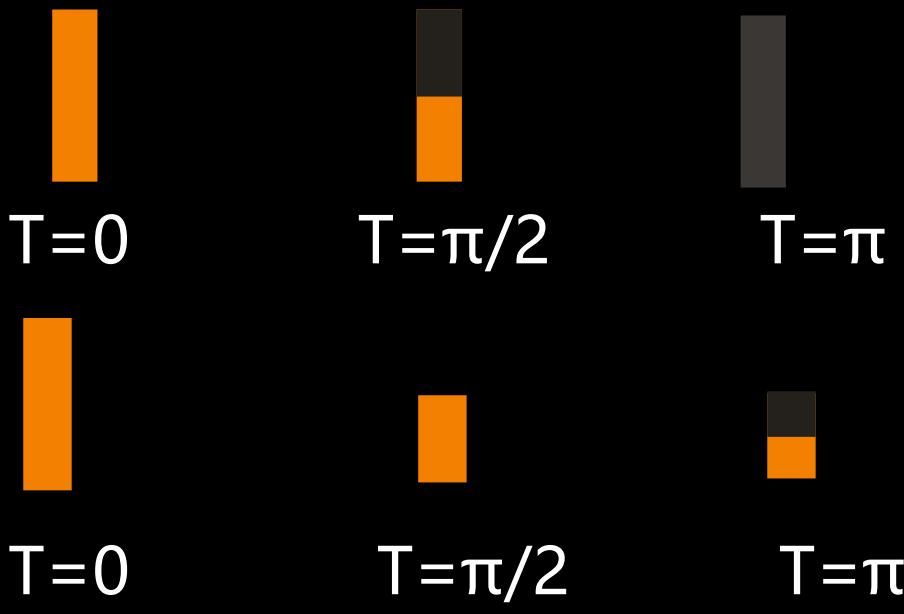
Considering neutron spin evolving in magnetic field, the probability to find it still in spin up state after time T is

$$P = \cos^2\left(\frac{\omega T}{2}\right)$$

where ω is the Larmor frequency

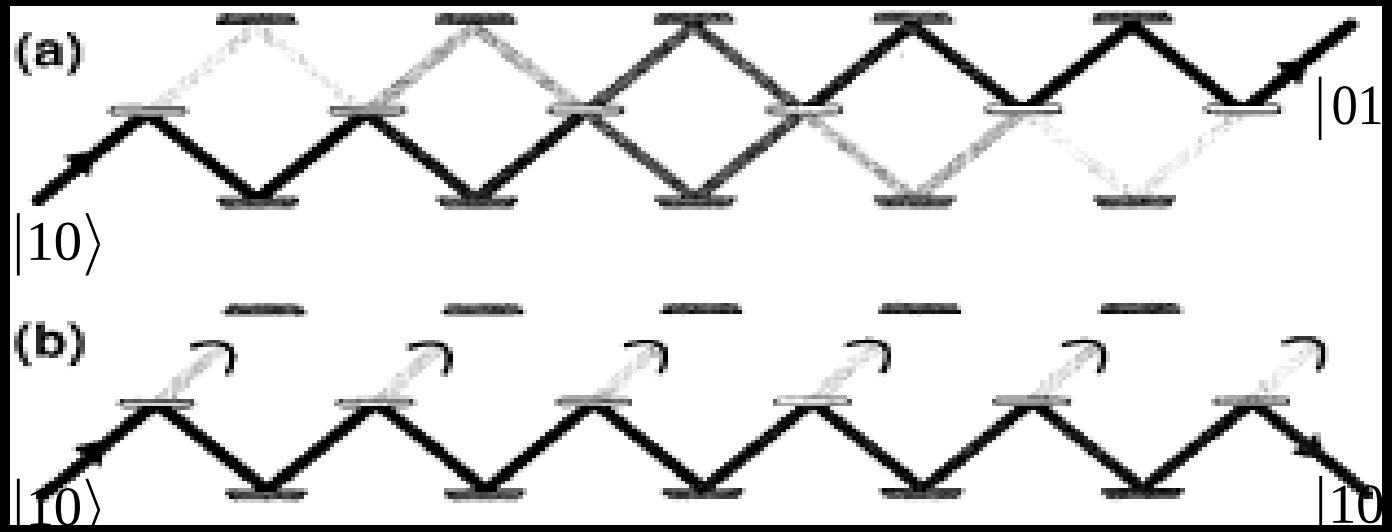
Quantum Zeno Effect

$$G\ (\text{cake is good}) = G_0 \times \frac{1+\cos T}{2}, (\omega = 1)$$



If we cut the bad part of the cake at time T=π/2 ,
then at T=π we have $G=1/4\times G_0$

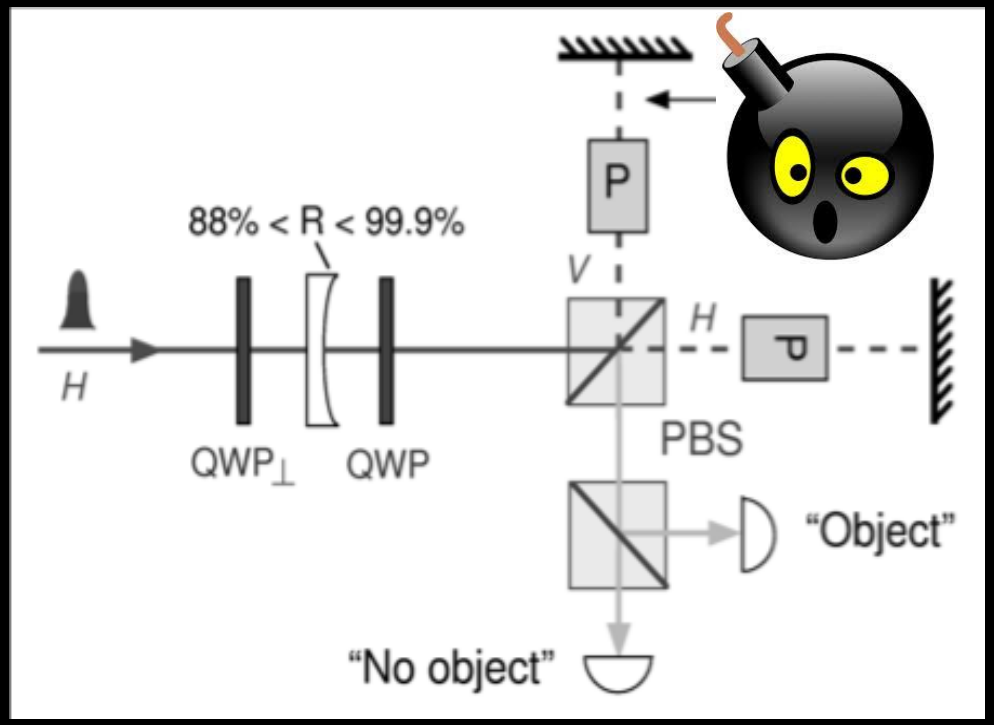
Experiment



$$P = \left[\cos^2\left(\frac{2\pi}{N}\right)\right]^N$$

In the limit of large N: $P = 1 - \frac{\pi^2}{4N} + O(N^{-2})$

Kwiat *et al.*, PRL 74, 4763 (1995)

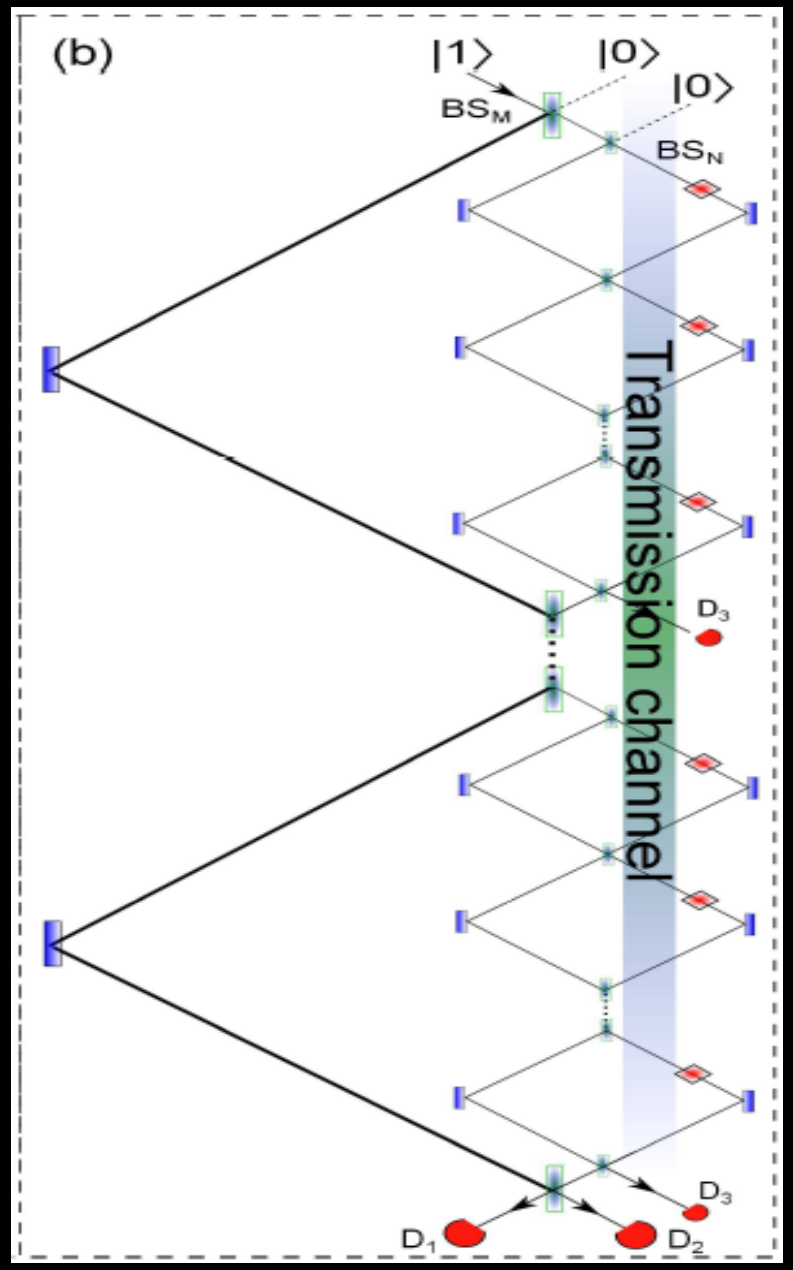


$|10\rangle \rightarrow |10\rangle$ with

$$P = \left[\cos^2\left(\frac{\pi}{2N}\right)\right]^N$$

Kwiat, *et al.*, PRL 83 4725 (1999)

Quantum Zeno Effect



More complex strcture: A nested and chained version of MZI

All-Pass:

$$|100\rangle \rightarrow \cos^{m-1} q_M \left(\cos q_M |100\rangle + \sin q_M |010\rangle\right)$$

$$\xrightarrow{m=M} |100\rangle$$

All-Block:

$$|100\rangle \rightarrow \cos m q_M |100\rangle + \sin m q_M |010\rangle$$

$$\xrightarrow{m=M} |010\rangle$$

Salih, *et al.*, PRL 110, 170502 (2013)

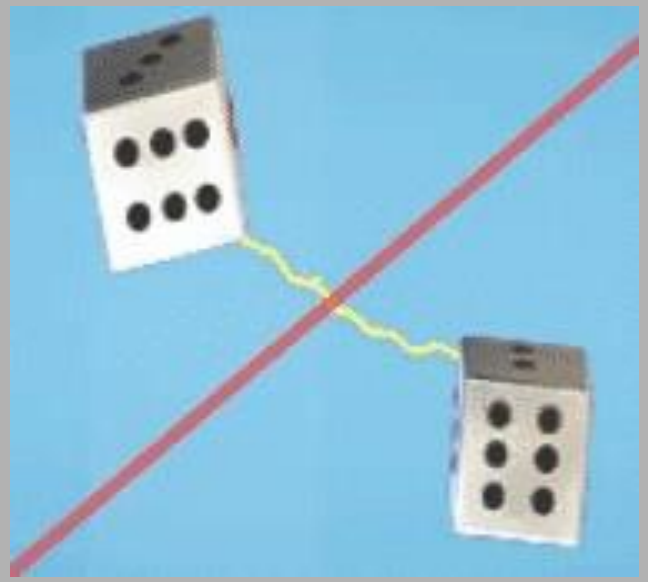
Quantum entanglement

- Quantum entanglement

- Bell states –
  maximally entangled states:

$$|\mathrm{F}^{\pm}\rangle_{12} = \frac{1}{\sqrt{2}}\left(|H\rangle_1|H\rangle_2 \pm |V\rangle_1|V\rangle_2\right)$$

$$|\mathrm{Y}^{\pm}\rangle_{12} = \frac{1}{\sqrt{2}}\left(|H\rangle_1|V\rangle_2 \pm |V\rangle_1|H\rangle_2\right)$$



Two dices are entangled

Spooky action at a distance

--Albert Einstein

Quantum entanglement

GHZ states: three-photon maximally entangled states

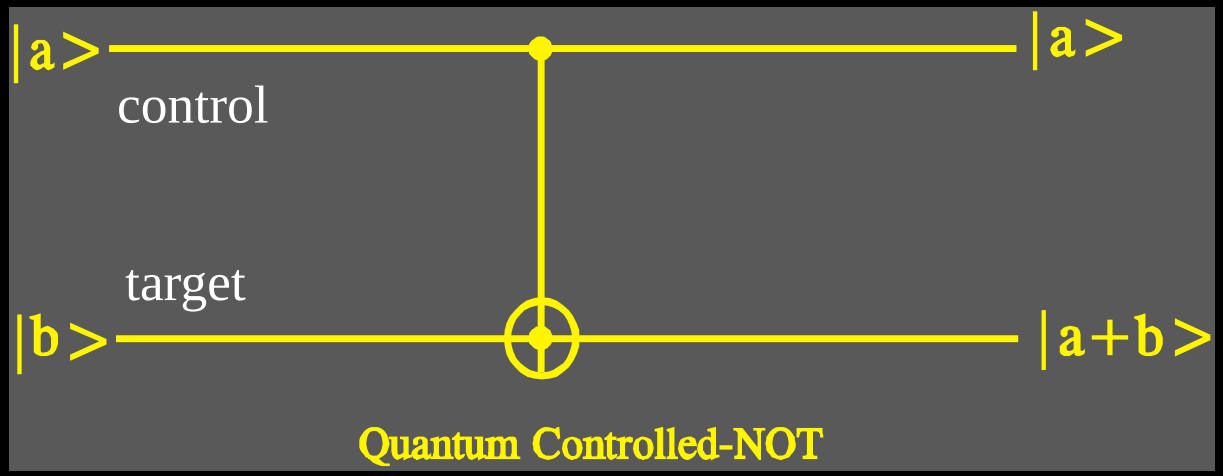
$$|\Phi^{\pm}\rangle_{123} = \frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_2|H\rangle_3 \pm |V\rangle_1|V\rangle_2|V\rangle_3)$$

$$|\Psi^{\pm}\rangle_{123} = \frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_2|V\rangle_3 \pm |V\rangle_1|V\rangle_2|H\rangle_3)$$

$$|\Xi^{\pm}\rangle_{123} = \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2|H\rangle_3 \pm |V\rangle_1|H\rangle_2|V\rangle_3)$$

$$|\Theta^{\pm}\rangle_{123} = \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2|V\rangle_3 \pm |V\rangle_1|H\rangle_2|H\rangle_3)$$

Manipulation of Entanglement



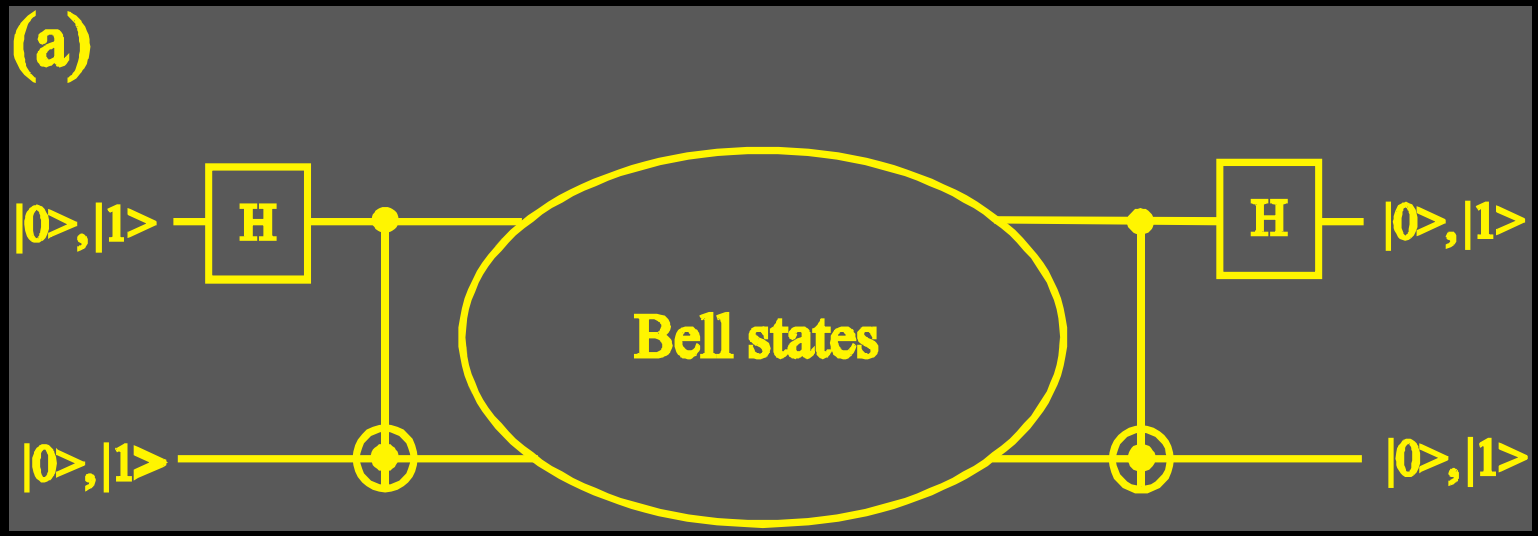
Flip the target when control=1:

$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |11\rangle$

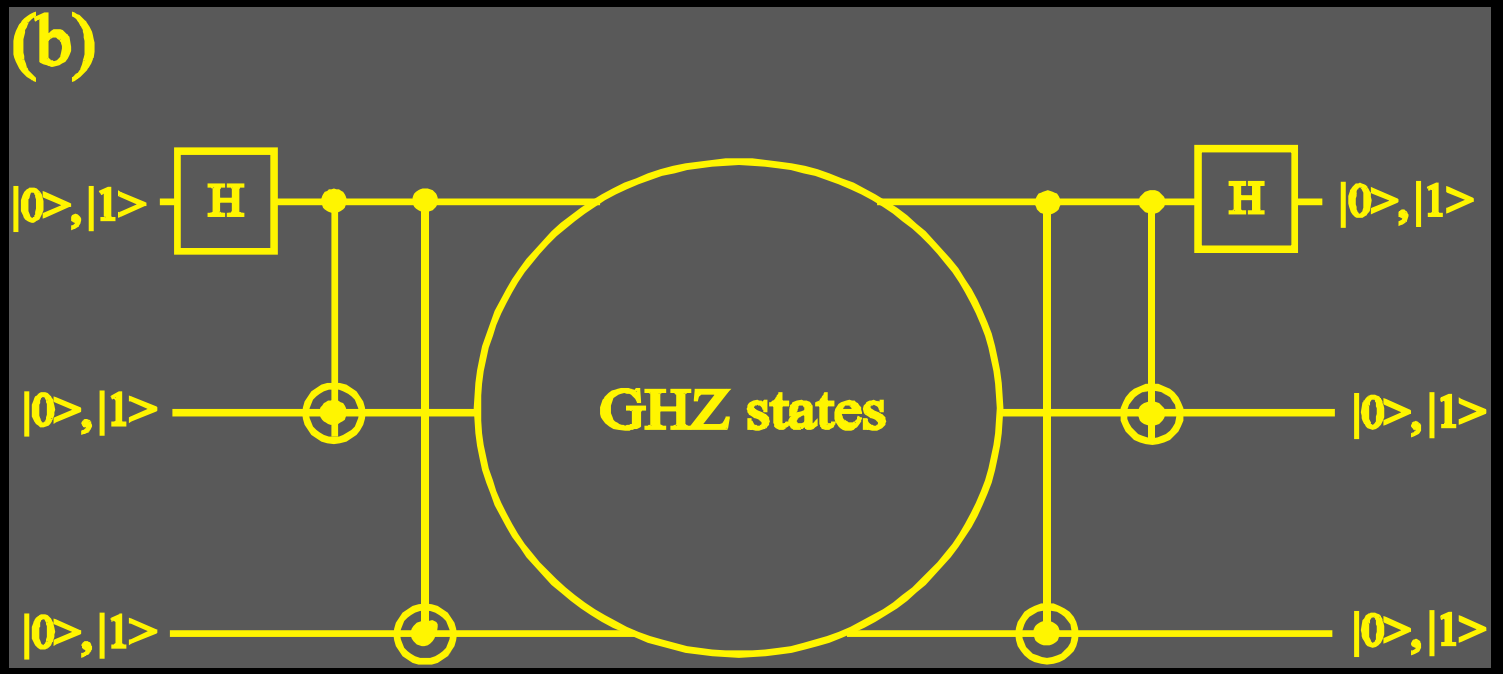
$|11\rangle \rightarrow |10\rangle$



$$00 \rightarrow (0+1)0 = 00 + 10 \rightarrow \mathbf{(00 + 11)}$$

$$\rightarrow (00 + 10) = (0 + 1)0 \rightarrow 00$$

Manipulation of Entanglement



$$000 \rightarrow (0+1)00 = (00+10)0 \rightarrow (00+11)0 = 000 + 110$$
$$\rightarrow 000 + 111 \rightarrow 000 + 110 = (00+11)0 \rightarrow (00+10)0$$
$$= (0+1)00 \rightarrow 000$$

Spooky Action at a Distance?

Entangled pair

Measurement time: Δt
Space-like separation: $L > \Delta t$

→ **Local Realism**

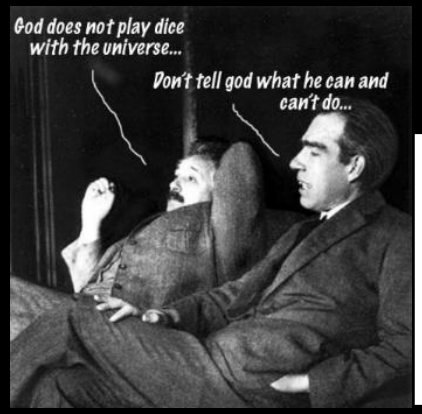
Measurement on particle A will not affect particle B

Quantum entanglement:

→ **Quantum Non-locality**

Measurement on particle A will cause instant collapse on particle B

Spooky Action at a Distance?



A philosophical argument

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

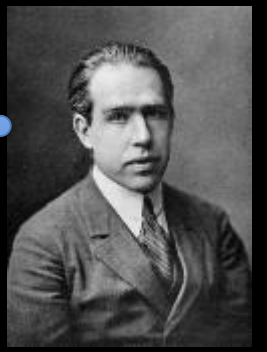
Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. EINSTEIN, B. PODOLSKY AND N. ROSEN, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

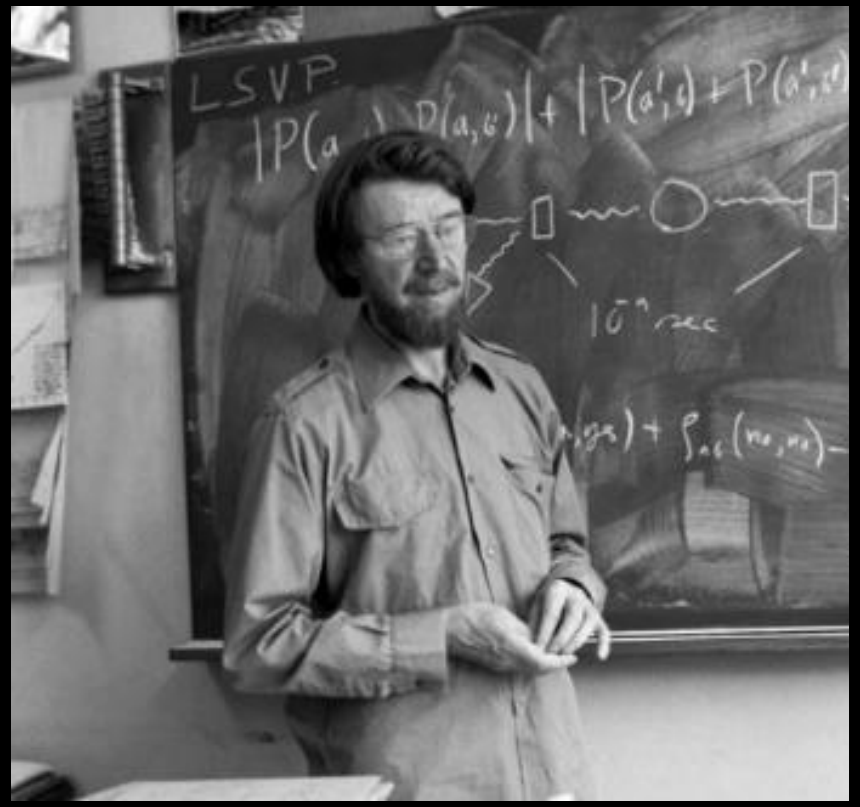
Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the 'old one'. I, at any rate, am convinced that He does not throw dice.

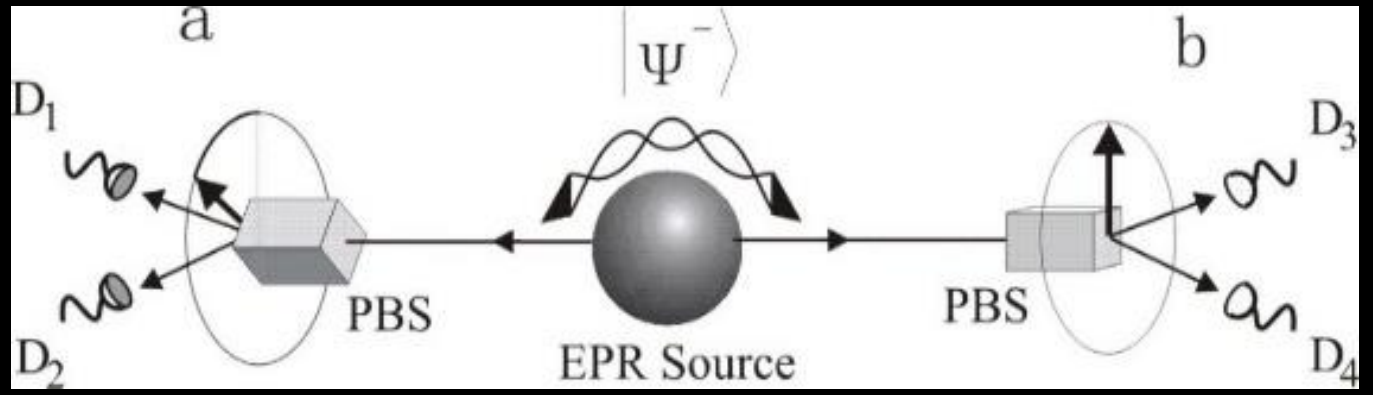
Einstein, stop telling God what to do!



Bell's Inequality: Testing This Battle

Experimental testable inequality:
Bell, Physics 1, 195 (1964)
Clauser *et al.*, PRL 23, 880 (1969)



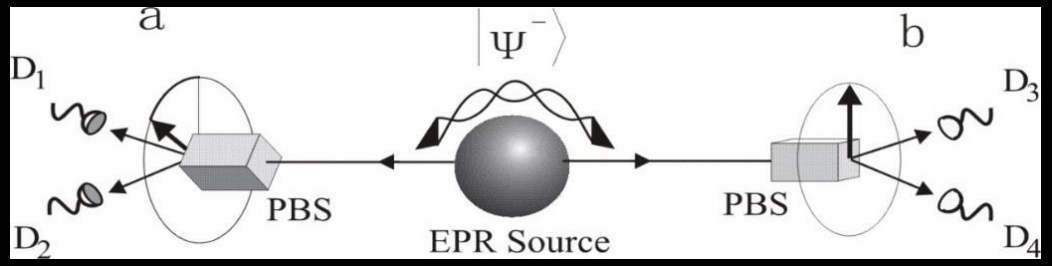


$$S = |E(\phi_A\phi_A) - E(\phi_A\phi'_B) + E(\phi'_A\phi_B) + E(\phi'_A\phi'_B)|$$

- Einstein's local realism: $S_{\max} \leq 2$
- Quantum mechanics: $S_{\max} = 2\sqrt{2}$

Bell's Inequality: Testing This Battle

A simplified case: Sakurai's Bell Inequality



	Alice			Bob			
measurement directions	A	b	c	a	b	c	probability
pre-determined outcomes	0	0	0	1	1	1	P_1
	0	0	1	1	1	0	P_2
	0	1	0	1	0	1	P_3
	0	1	1	1	0	0	P_4
	1	0	0	0	1	1	P_5
	1	0	1	0	1	0	P_6
	1	1	0	0	0	1	P_7
	1	1	1	0	0	0	P_8

Singlet state: anti-correlation of measurement results of two sides

$$|\Psi^-\rangle_{12} = \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2 - |V\rangle_1|H\rangle_2)$$

Pick three arbitrary directions **a**, **b**, and **c**:

$$P(\mathrm{a0}, \mathrm{b0}) = P_3 + P_4$$

$$P(\mathrm{a0}, \mathrm{c0}) = P_2 + P_4$$

$$P(\mathrm{c0}, \mathrm{b0}) = P_3 + P_7$$

$$P_3 + P_4 \leq P_3 + P_4 + P_2 + P_7$$

Bell's Inequality: Testing This Battle

Local realism requires:

$$P(a0,b0) \leq P(a0,c0) + P(c0,b0)$$

Quantum-mechanical prediction:

$$P(a0,b0) = \frac{1}{2}\sin\left(\frac{a-b}{2}\right)^2$$

For example $a = 90^\circ, b = 45^\circ, c = 0^\circ$, the inequality would require

$$\frac{1}{2}\sin^2 45^\circ \leq \frac{1}{2}\sin^2 22.5^\circ + \frac{1}{2}\sin^2 22.5^\circ \quad \Rightarrow \quad 0.2500 \leq 0.1464!$$

An unsatisfactory feature

In the derivation of BI such a local realistic and thus classical picture can explain perfect correlations and is only in conflict with statistical prediction of quantum mechanics

Conflict with Local Realism

Consider a three-photon GHZ state written in σ_Z basis

$$|\Psi_{123}\rangle = \frac{1}{\sqrt{2}}(|H_1\rangle|H_2\rangle|H_3\rangle + |V_1\rangle|V_2\rangle|V_3\rangle)$$

Linear polarization basis

$$\sigma_x : |H'\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle + |V\rangle\right),$$
$$|V'\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle - |V\rangle\right).$$

Circular polarization basis

$$\sigma_y : |R\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle + i|V\rangle\right),$$
$$|L\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle - i|V\rangle\right).$$

Conflict with Local Realism

$$\sigma_{1y}\sigma_{2y}\sigma_{3x}:\ |\psi_{123}\rangle=\frac{1}{2}\left(R_1L_2H_3^{'}+L_1R_2H_3^{'}+R_1R_2V_3^{'}+L_1L_2V_3^{'}\right)$$

$$\sigma_{1y}\sigma_{2x}\sigma_{3y}:\ |\psi_{123}\rangle=\frac{1}{2}\left(R_1H_2^{'}L_3+L_1H_2^{'}R_3+R_1V_2^{'}R_3+L_1V_2^{'}L_3\right)$$

$$\sigma_{1x}\sigma_{2y}\sigma_{3y}:\ |\psi_{123}\rangle=\frac{1}{2}\left(H_1^{'}R_2L_3+H_1^{'}L_2R_3+V_1^{'}R_2R_3+V_1^{'}L_2L_3\right)$$

Therefor state $|\Psi_{123}\rangle$ is the eigenstate of operators

$\sigma_{1y}\sigma_{2y}\sigma_{3x}, \sigma_{1y}\sigma_{2x}\sigma_{3y}, \sigma_{1x}\sigma_{2y}\sigma_{3y}$ with value -1

Conflict with Local Realism

- EPR reality criterion: operator is predeterminethe individual value of any local d
- There exists an element of local reality S_{ix} corresponding to operator

$$\sigma_{ix}\,(i=1,2,3).$$

All six of the elements of reality S_{ix} and S_{iy} have to be there, each with the values +1 and –1!

$$S_{1y}S_{2y}S_{3x}=-1,$$
$$S_{1y}S_{2x}S_{3y}=-1,$$
$$S_{1x}S_{2y}S_{3y}=-1.$$

What Outcomes Are Possible?

Consider measurement of 45° linear polarization basis

Local realism:

$$\begin{aligned} S_{1x}S_{2x}S_{3x} &= S_{1x}(S_{1y})^2 S_{2x}(S_{2y})^2 S_{3x}(S_{3y})^2 \\ &= (S_{1x}S_{2y}S_{3y})(S_{1y}S_{2x}S_{3y})(S_{1y}S_{2y}S_{3y}) \\ &= -1 \end{aligned}$$

Possible outcomes:

$$V_1'V_2'V_3',\ H_1'H_2'V_3',\ H_1'V_2'H_3',\ V_1'H_2'H_3'$$

What Outcomes Are Possible?

Quantum physics

$$|\psi_{123}\rangle = \frac{1}{2}\left(H_1'H_2'H_3' + H_1'V_2'V_3' + V_1'H_2'V_3' + V_1'V_2'H_3'\right)$$

$$\Rightarrow S_{1x}S_{2x}S_{3x} = 1!$$

Possible outcomes:

$$H_1'H_2'H_3',\ H_1'V_2'V_3',\ V_1'H_2'V_3',\ V_1'V_2'H_3'$$

Whenever local realism predicts a specific result definitely to occur for a measurement for one of the photons based on the results for the other two, quantum physics definitely predicts the opposite result

Bell's Inequality: Testing the Battle

Chien-Shiung Wu

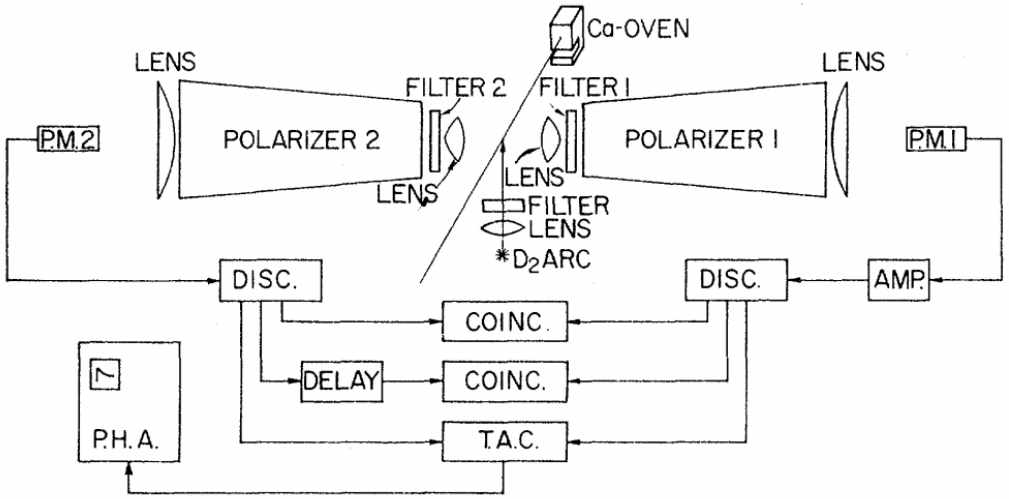
First observation of quantum entanglement

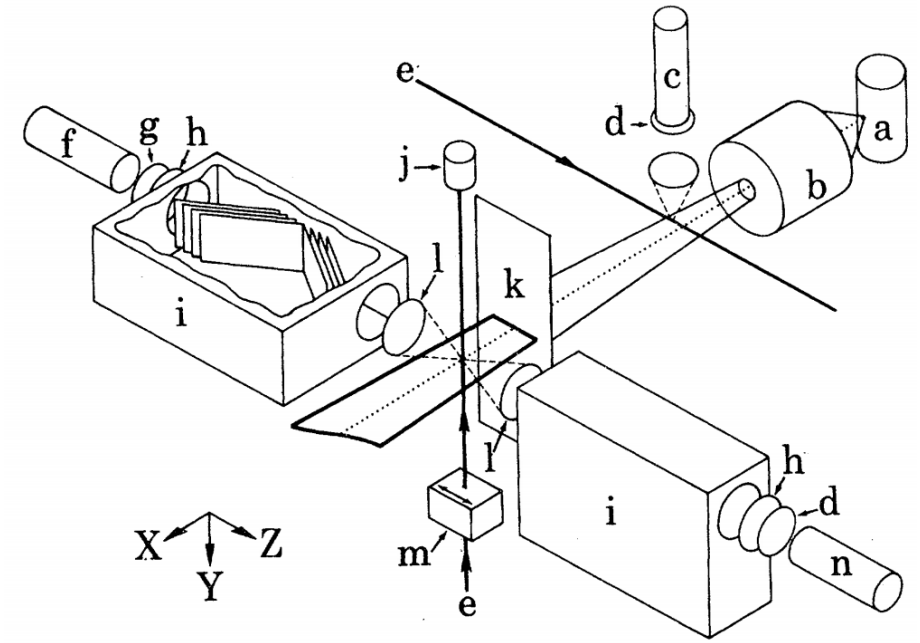
The Angular Correlation of Scattered Annihilation Radiation*

C. S. Wu and I. Shaknov
Pupin Physics Laboratories, Columbia University, New York, New York
November 21, 1949

Phys. Rev. 27, 136 (1950)

Experimental Test of Bell Inequality



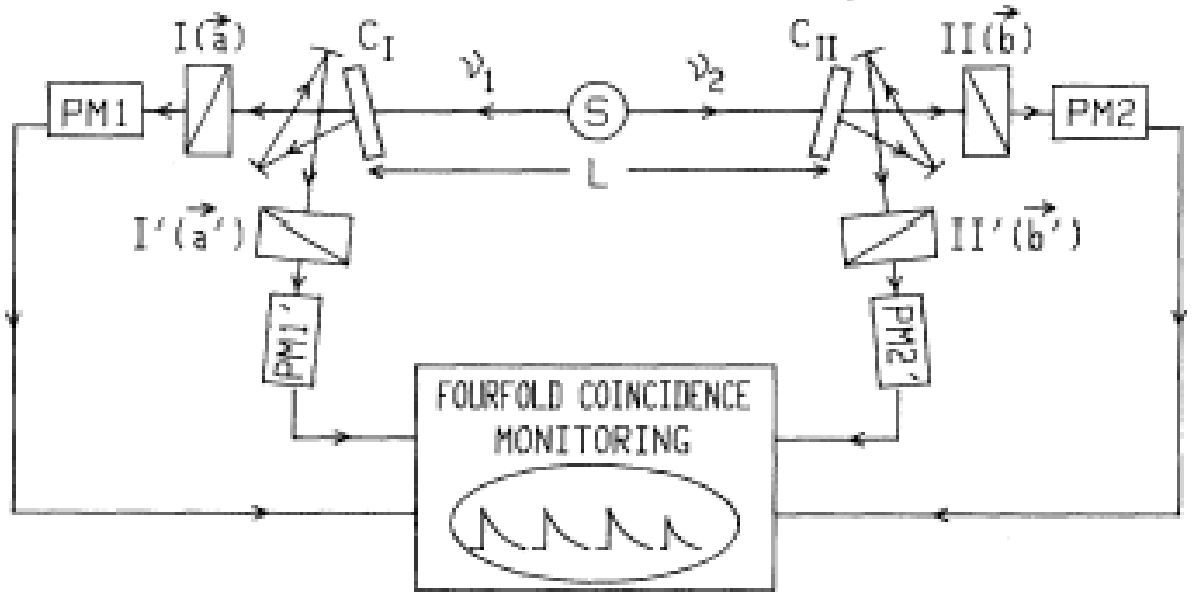


- Freedman & Clauser, PRL 28, 938 (1972)
- Fry & Thompson, PRL 37, 465 (1976)

Two measurement sites are not space-like separated

Experimental Test of Bell Inequality

Aspect *et al.*, PRL 49, 1804 (1982)



$S_{\text{exp}} = 0.101 \pm 0.020$ violates a generalized inequality $S \leq 0$ by 5 standard deviations

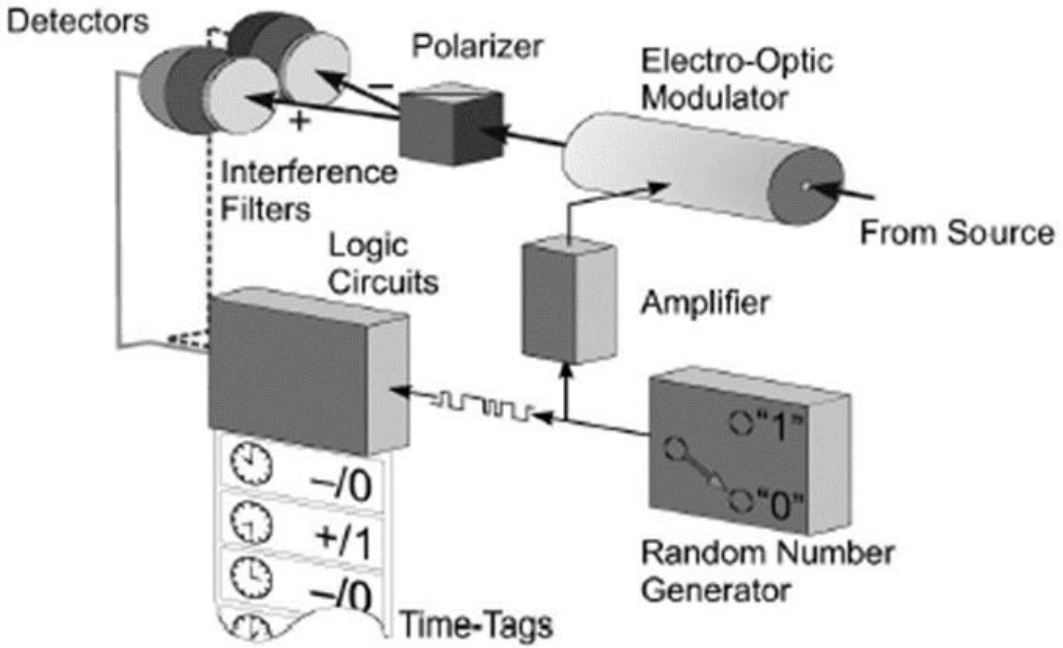
Drawbacks: 1. locality loophole
2. detection loophole

Locality Loophole

- ☒ Measurement devices may "tell" the EPR source their basis choices ➡ the source may "select" according events to violate Bell inequality
- ☑ Solution: basis choice and emission of EPR source must be also space-like separated (i. e., fast and random switch of measurement basis)

Experimental Test of Bell Inequality

Weihs *et al.*, PRL 81, 5039 (1998)



$S_{\text{exp}} = 2.73 \pm 0.02$ violates CHSH inequality $S \leq 2$ by 30 standard deviations

Drawback: detection loophole

Detection Loophole

☒ Detection efficiency of single photon detectors is not unity ➡
some events cannot contribute to $S > 2$ were not detected?

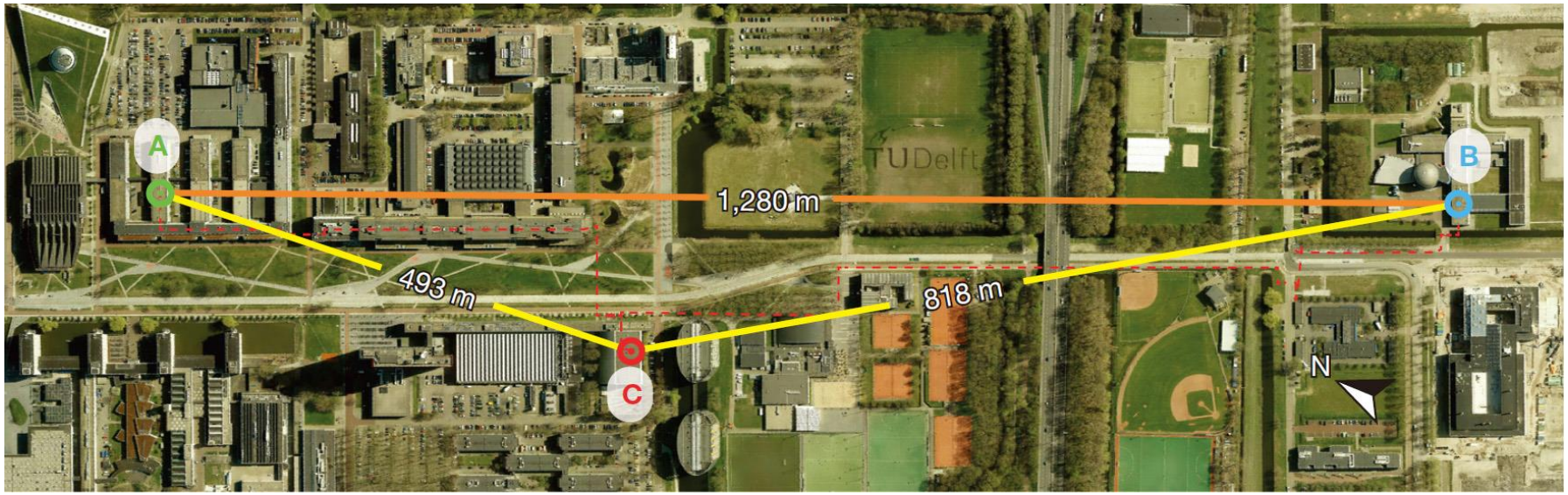
☑ Solution: high detection efficiency (>83%)

Pearle, PRD 2, 1418 (1970)

Garg & Mermin, PRD 35, 3831 (1987)

Experimental Test of Bell Inequality

Hensen *et al.*, Nature 526, 682 (2015)



Close both detection loophole and locality loophole

- Detection efficiency>95%
- Switch time: 480ns<493m/c

But still with loopholes…

Freedom of Choice and Collapse Locality Loophole

☒ Freedom of choice loophole: random number generators (RNGs) could be prior correlated ➡ the choice of measurement bases are not truly random

Brunner *et al.*, RMP 86, 419 (2014)

Schrödinger's cat

☒ Collapse locality loophole: measurement outcome is not defined until it is registered by a human consciousness ➡

Realized "events" have never been space-like separated

Kent, PRA 72, 012107 (2005)

Leggett, Compendium of Quantum Physics (Springer, 2009)

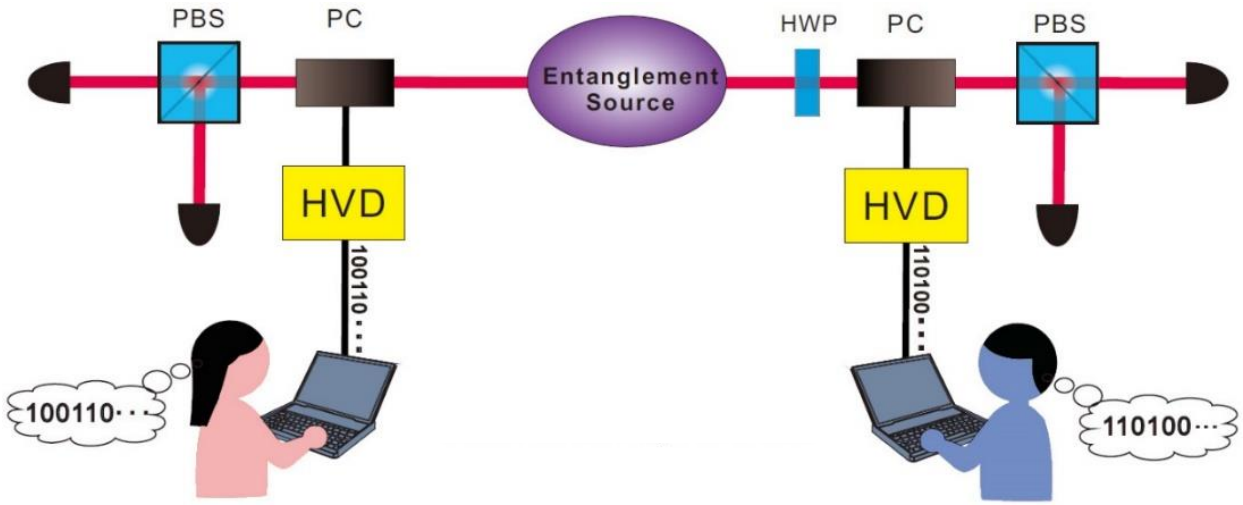
Bell-test experiment with human-observer

Solution for both loopholes: Bell-test experiment with human-observer!

☑ Basis choice by free will

☑ Measurement outcomes defined by consciousness

Leggett, Compendium of Quantum Physics (Springer, 2009)



Requirement:

Quantum signal transit time exceeds human reaction 100ms ➡ entanglement distribution at a distance on the order of one light-second

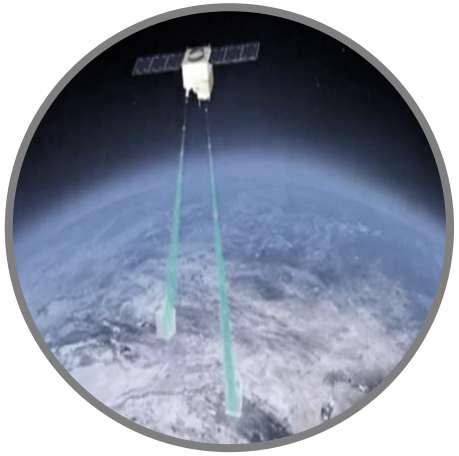
Quantum Information Processing (QIP)

Test of quantum nonlocality

Coherent manipulation of quantum systems

Enabling encode and process information in quantum states, outperform classical information systems in terms of

Unconditional security

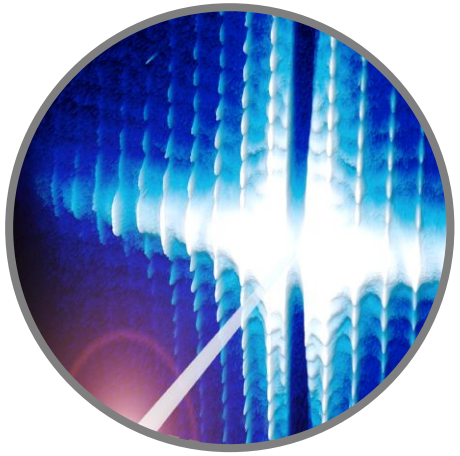


Quantum communication

Computational capacities

Quantum computation and simulation

Super-resolution

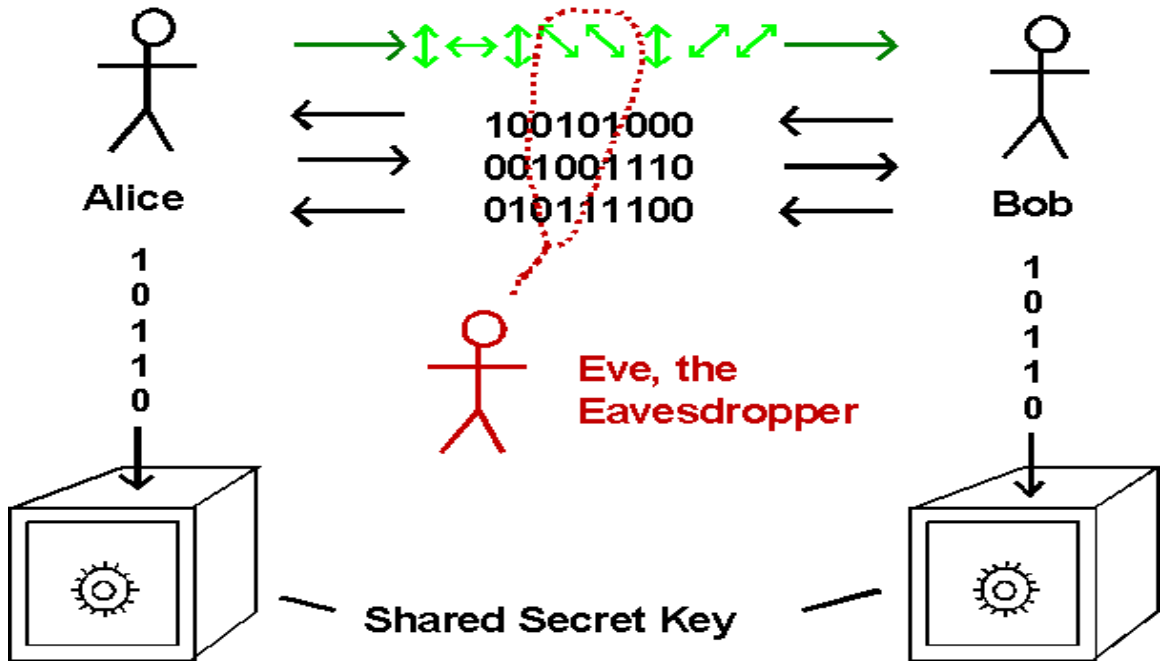


Quantum metrology

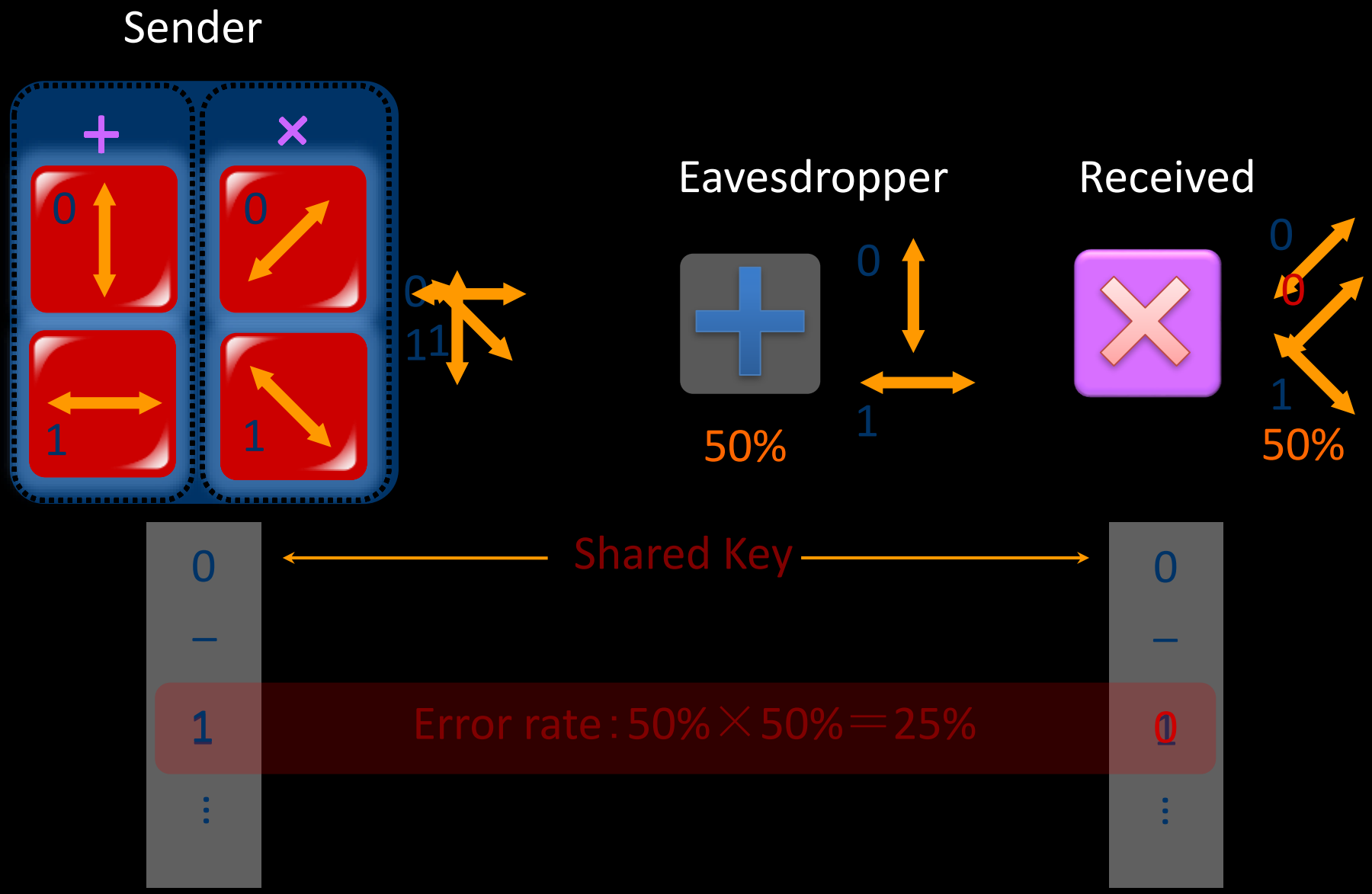
Part 2: Quantum Communication

Quantum Key Distribution (QKD)

- Single-photon-based key distribution: [Bennett & Brassard 1984 protocol]

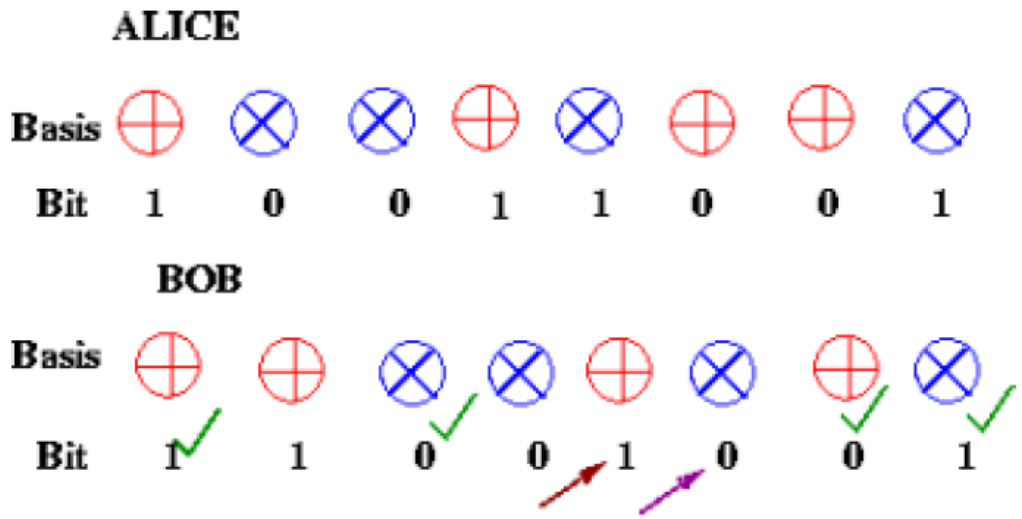


- Entanglement-based key distribution: [Ekert, PRL 67, 661 (1991)]

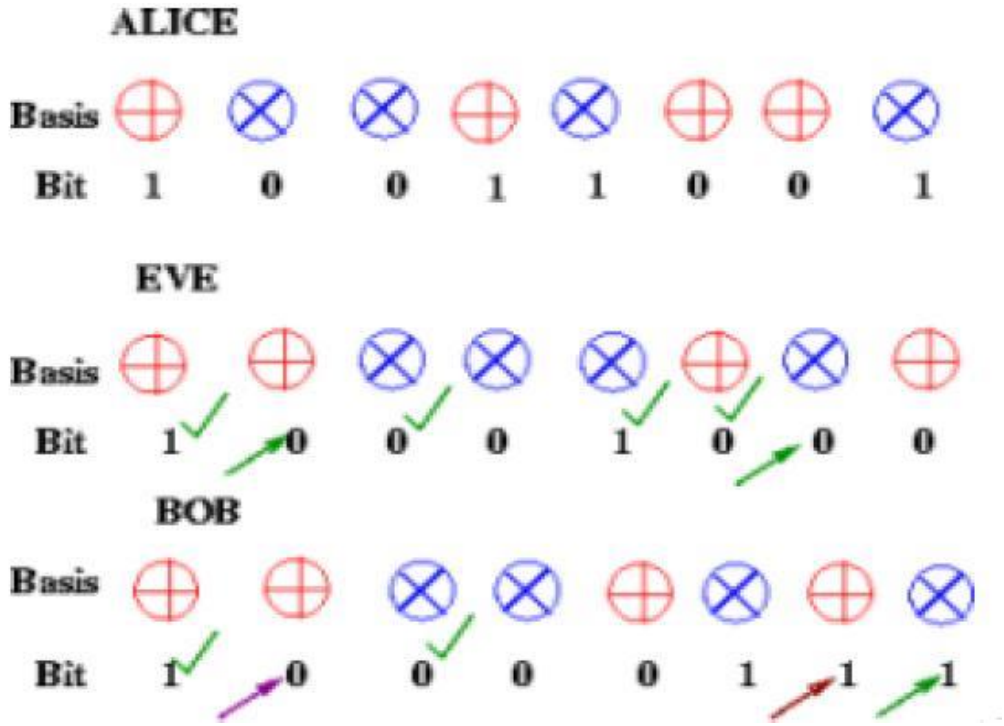

Sender
+
×
0
0
1
1
0
11
Eavesdropper
0
1
50%
Received
0
0
1
50%
0
–
1
⋮
Shared Key
0
–
0
⋮
Error rate : 50% × 50% = 25%

BB84 Protocol

- No Eve



- With Eve



If Eve is present, the probability that Alice and Bob can not tell is $(0.25)^N$ after they compare N raw key's value!

BB84 Security

	one-way communication	two-way communication
Upper bound	14.6%	25%
Lower bound	11.0%	18.9%

- All the error rates are brought by the eavesdropping
- When the error rate is lower than the lower bound, we can utilize some classical cryptography method to let Eve know nothing about the key
- If the error rate is higher than the upper bound, the key is insecure

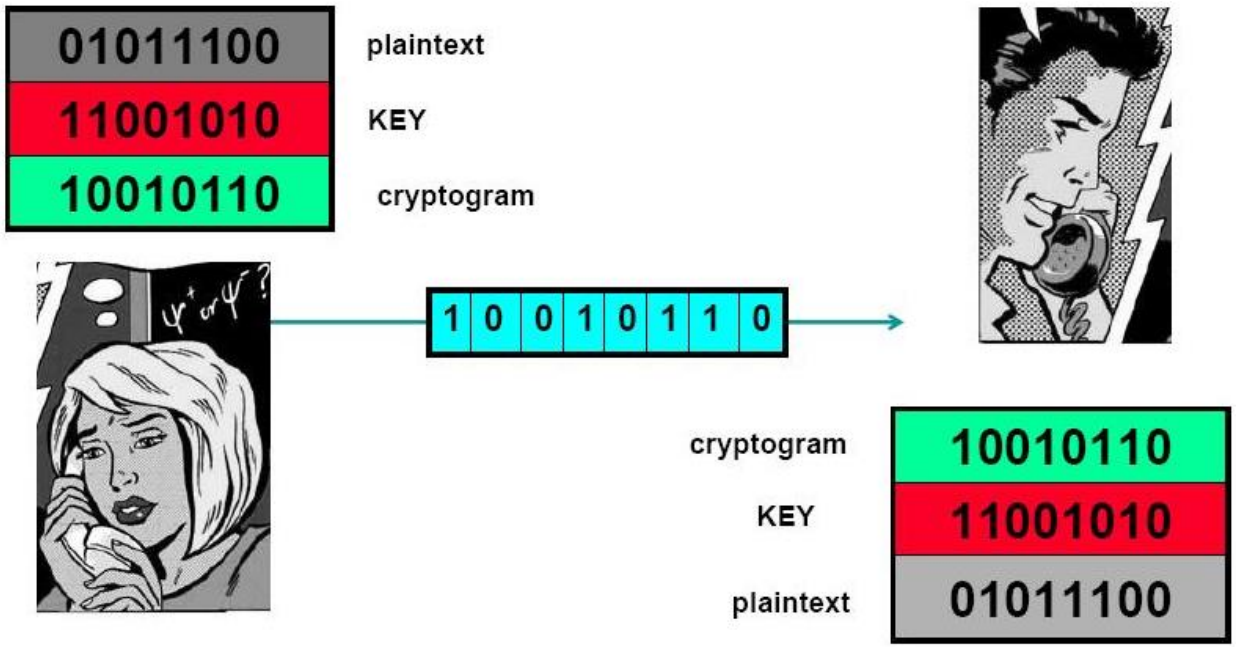
Gottesman and Lo, IEEE TIT 49, 457 (2003)

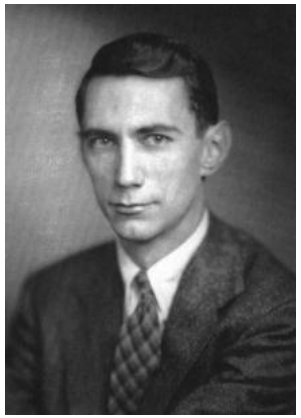
Perfect Cipher in Principle

QKD ➡ Secure key

\+

One-time pad



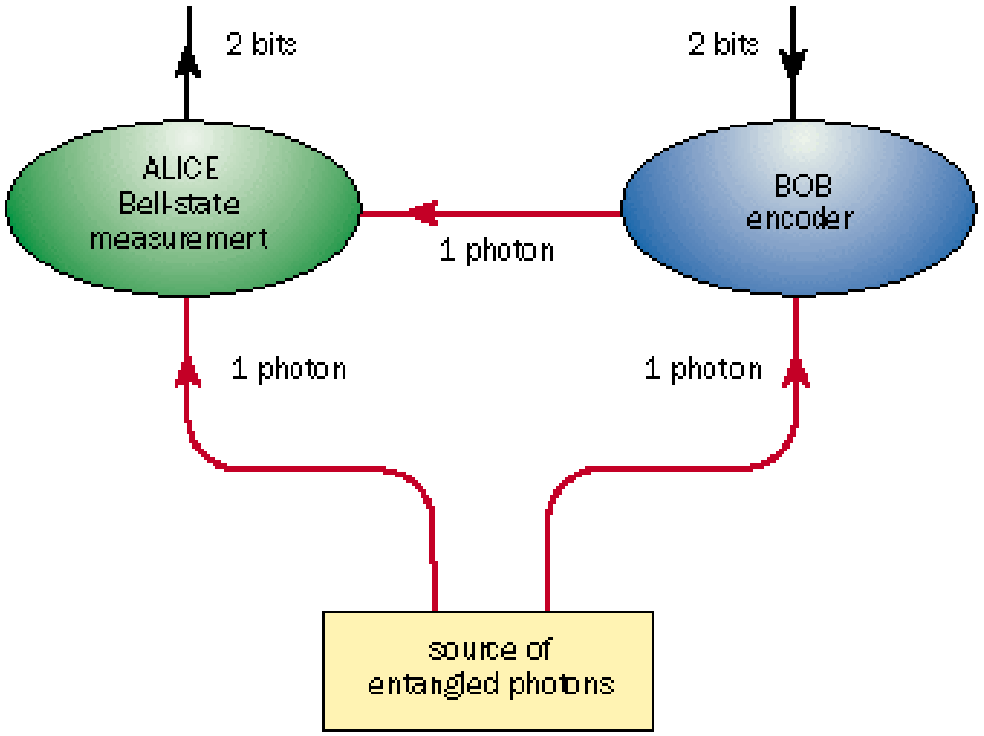


- First Discovered by Gilbert Vernam
- Security Proved by Claude Shannon [Bell Syst. Tech. J,28,656 (1949)]

Unconditional security!

Dense Coding

Transmit two bits of information by sending one photon



Transformations between 4 Bell states:

$$|\Phi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_2 + |V\rangle_1|V\rangle_2)$$

$$|\Psi^+\rangle_{12} = \sigma_{X1}|\Phi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2 + |V\rangle_1|H\rangle_2)$$

$$|\Phi^-\rangle_{12} = \sigma_{Z1}|\Phi^+\rangle_{12}\frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_2 - |V\rangle_1|V\rangle_2)$$

$$|\Psi^-\rangle_{12} = -i\sigma_{Y1}|\Phi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2 + |V\rangle_1|H\rangle_2)$$

Bennett & Wiesner, PRL 69, 2881 (1992)

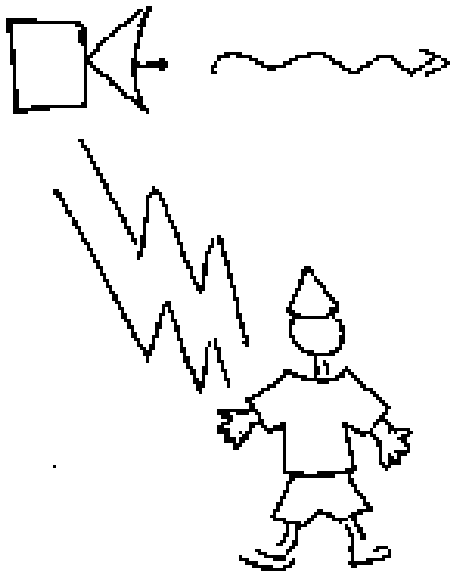
Dense Coding

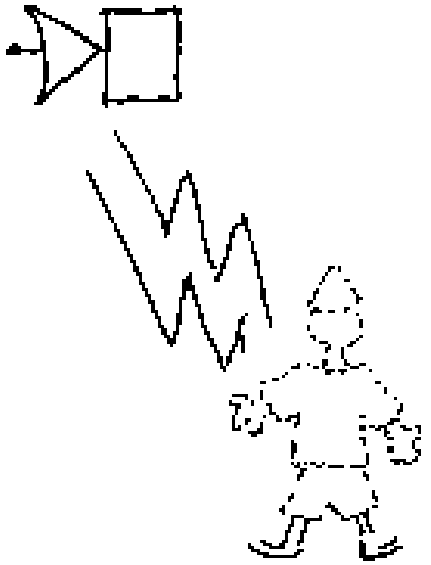
1. Alice and Bob share an entangled photon pair in the state of $|\Phi^+\rangle_{12}$
2. Bob chooses one of the four unitary transformation on his photon. The information of which choice is 2 bit.

e. g. $00: I \quad 01: \sigma_Z \quad 10: \sigma_X \quad 11: -i\sigma_Y$

3. Bob sends his photon to Alice
4. Alice does a joint Bell-state measurement (BSM) on the photon from Bob and her photon.
5. With the measurement result, she can know Bob's unitary transformation and achieve the 2 bit information.

Quantum Teleportation





- Classical physics

  Scanning and reconstructing

- Quantum physics

  Principle of quantum measurement forbidden extracting all the information from an unknown quantum state!

Quantum Teleportation

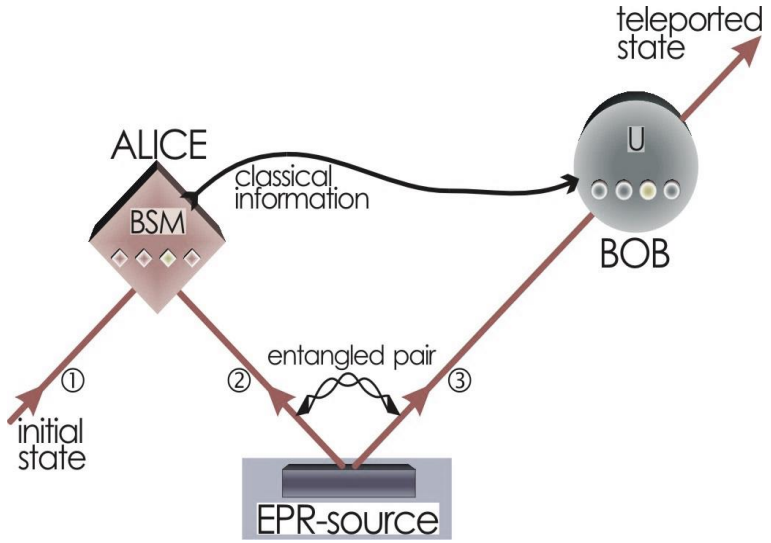
Initial state

$$|\Phi\rangle_1 = \alpha\,|H\rangle_1 + \beta\,|V\rangle_1$$

The shared entangled pair

$$|\Phi^+\rangle_{23} = \frac{1}{\sqrt{2}}\left(|H\rangle_2\,|H\rangle_3 + |V\rangle_2\,|V\rangle_3\right)$$

$$\begin{aligned}
|\Psi\rangle_{123} &= |\Phi\rangle_1 \otimes |\Phi^+\rangle_{23} \\
&= |\Phi^+\rangle_{12} \otimes \left(\alpha\,|H\rangle_3 + \beta\,|V\rangle_3\right) + \\
&\quad |\Phi^-\rangle_{12} \otimes \left(\alpha\,|H\rangle_3 - \beta\,|V\rangle_3\right) + \\
&\quad |\Psi^+\rangle_{12} \otimes \left(\alpha\,|V\rangle_3 + \beta\,|H\rangle_3\right) + \\
&\quad |\Psi^-\rangle_{12} \otimes \left(\alpha\,|V\rangle_3 - \beta\,|H\rangle_3\right)
\end{aligned}$$



BSM results on particles 1, 2	operations on particle 3
$\|\Phi^+\rangle_{12}$	I
$\|\Phi^-\rangle_{12}$	σ_Z
$\|\Psi^+\rangle_{12}$	σ_X
$\|\Psi^-\rangle_{12}$	$-i\sigma_Y$

Bennett *et al.*, PRL 73, 3801 (1993)

Quantum Teleportation

Though nowadays we can only teleport two-particle composite system……

Essential ingredient for distributed quantum information processing!

Part 3: Quantum Computation and Quantum Metrology

Quantum Parallelism

Bits		Qubits
0 or 1	V. S.	0 + 1
00, 01, 10 or 11		00 + 01 + 10 + 11
000, 001, 010……		000 + 001 + 010 + ……

Evaluating function f(x) for many different x simultaneously

$$U\frac{1}{\sqrt{2^N}}\sum_{i=0}^{2^N-1}|i\rangle|0\rangle = \frac{1}{\sqrt{2^N}}\sum_{i=0}^{2^N-1}|i\rangle|f(i)\rangle$$

Exponentially speedup!

This is what makes famous quantum algorithms, such as Shor's algorithm for factoring, or Grover's algorithm for searching

RSA Encryption and Factorizing

RSA public-key cryptosystem

- Produce a large integer N

$$m1 \times m2 = N, \text{ (with m1 and m2 primes)}$$

- N is made public available and is used as a key (x) to encrypt data
- m1 and m2 are the secret keys (k) enable one to decrypt the data

$$C = E_x(P)$$
$$P = D_k(C) = D_k(E_x(P))$$

X: Public Key; K: Private Key

P: Plain Text; E: Encryption; C: Ciphertext; D: Decryption

Riverst, Shamir and Adleman, MIT/LCS/TR-212, Jan. 1979

RSA Encryption and Factorizing

- To crack a code, a code breaker needs to factorize N
- The security of RSA based on the ease with which N can be calculated from m1 and m2, and the difficulty of calculating m1 and m2 from N

RSA Encryption and Factorizing

- Problem: given a number, what are its prime factors ?

e. g. a 129-digit odd number which is the product of two large primes,

1143816257578888676692357799761466120102182967212423625625618429357069352457338978305971236395870505898907514759929002687954354 1

=3490529510847650949147849619903898133417764638493387843990820577

x 32769132993266709549961988190834461413177642967992942539798288533

- Best factorizing algorithm requires sources that grow exponentially in the size of the number: $\exp\left(O\left(n^{1/3}\log^{2/3}n\right)\right)$, with n the length of N

Shor's Algorithm

Algorithms for quantum computation: discrete logarithms and factorizing

E.g. factor a 300-digit number with

- Classical THz computer: 10^{24} steps ➡ 150,000 years
- Quantum THz computer: 10^{10} steps ➡ 1 second!

Peter Shor

Foundations of Computer Science, 1994 Proceedings. 35th Annual Symposium

- Code-breaking can be done in minutes, not in millennia
- Public key encryption, based on factoring, will be vulnerable!

Deutsch-Jozsa Algorithm

Deutsch's problem: two types of functions f

Considering input n bits,

- Constant f: for all 2^n inputs, f=0 or f=1
- Balanced f: for 2^{n-1} inputs, f=0, for another 2^{n-1} inputs, f=1

Question: given a function f, whether is it constant or balanced ?

Classical deterministic algorithm: at most $2^{n-1}+1$ inquiries

- All outputs are the same ➡ constant
- At least 1 output is different from others ➡ balanced

Deutsch-Jozsa Algorithm

The simplest example:

(x=0 or 1)

Constant:

f(0)=1

f(1)=1

Balanced:

f(0)=0

f(1)=1

- Classical algorithm needs 2 inquiries
- Deutsch-Jozsa quantum algorithm:

  Assume f was mapped into a quantum oracle U satisfing

$$U|x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f(x)\rangle \quad \text{e. g.,} \quad U_C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad U_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Deutsch-Jozsa Algorithm

- Prepare two qubits input state $|\psi_i\rangle = |0\rangle|1\rangle$
- Perform Hadamard operation $|\psi_i\rangle \rightarrow \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)$
- Perform U

Key point: All possible inputs are in superposition state!

$$U|x\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = |x\rangle\left(\frac{|f(x)\rangle - |1 - f(x)\rangle}{\sqrt{2}}\right) = (-1)^{f(x)}|x\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Output state:

$$|\psi_o\rangle = \begin{cases} \pm\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right), & if\ f(0) = f(1) \\ \pm\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right), & if\ f(0) \neq f(1) \end{cases}$$

- Measure the first qubit on {+/-} basis: $|+\rangle$ ➡ constant f, $|-\rangle$ ➡ balanced f

Quantum algorithm only needs one inquiry

Deutsch-Jozsa Algorithm

Consider a more general case with n-bit inputs x: x=0, 1, 2, …, 2^n-1

- Prepare n+1 qubits input state $|\psi_i\rangle = |0\rangle^{\otimes n}|1\rangle$
- Perform Hadamard operation on all qubits $|\psi_i\rangle \rightarrow \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle\,|-\rangle$

The binary representation of x corresponds to values of each qubits, e. g.,

$$(|0\rangle + |1\rangle)\otimes(|0\rangle + |1\rangle) = |00\rangle + |01\rangle + |10\rangle + |11\rangle$$

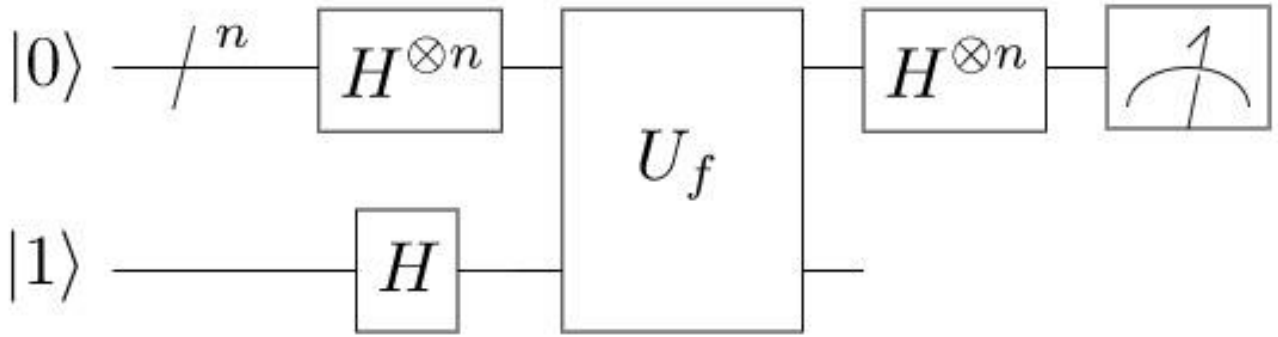
$$= |x = 0\rangle + |x = 1\rangle + |x = 2\rangle + |x = 3\rangle$$

- Perform $U \rightarrow \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle\,|-\rangle$

Deutsch-Jozsa Algorithm

- Measure the first n-qubit on {+/-} basis: ➡ if and only if the output is $|+\rangle^{\otimes n}$, f is constant

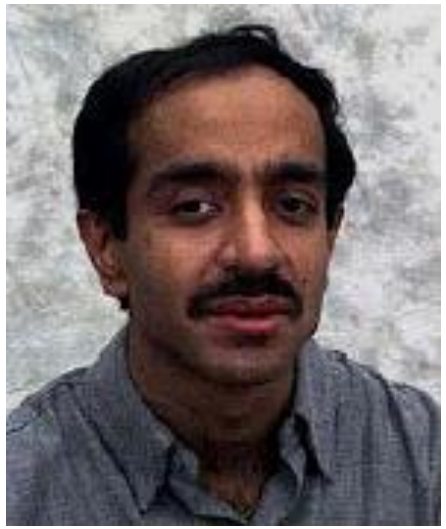
  Only needs one inquiry!



Deutsch and Jozsa, Proc. Royal Society London A 439, 553 (1992)

Deutsch problem is not a practically important problem, but Deutsch-Jozsa algorithm firstly demonstrated the superiority of quantum computation!

Grover's Search Algorithm



Lov Grover

How quickly can you find a needle in a haystack?

The simplest example:

Which one is equal to -1 in a database?

Serial	0	1	2	3	4	5
Value	1	1	-1	1	1	1

……

- Classically search
  - Sequentially try all N possibilities
  - Average search takes N/2 steps
- Quantum search
  - Simultaneously try all possibilities
  - Refining process reveals answer
  - Average search takes $N^{1/2}$ steps

Grover's Search Algorithm

- A databased is encoded with a N×N diagonal matrix R (rotate phase)

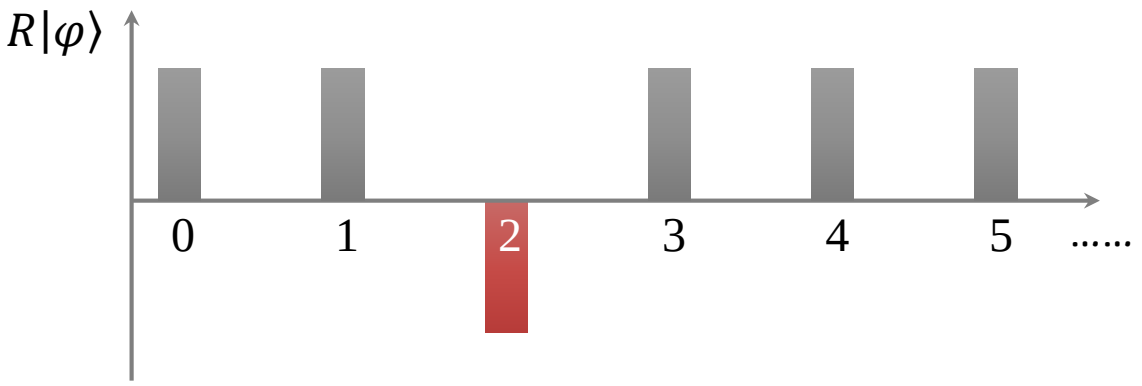
$$R = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad R|i\rangle = \begin{cases} -|i\rangle, i = x \\ |i\rangle, \text{otherwise} \end{cases}$$

The task is to find x

- Take a m-qubit register (2^m=N), and prepare the registers in an equal superposition state of all the states

$$|\varphi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N} |i\rangle$$

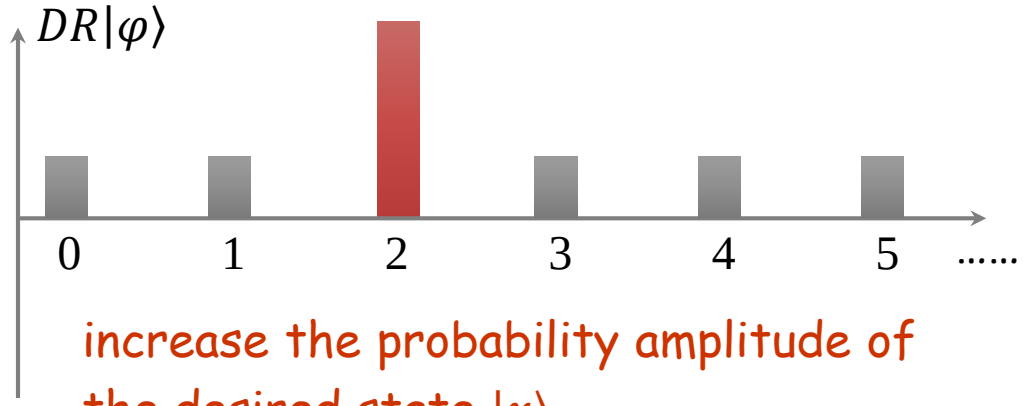
- Perform rotate phase matrix R on the register



Grover's Search Algorithm

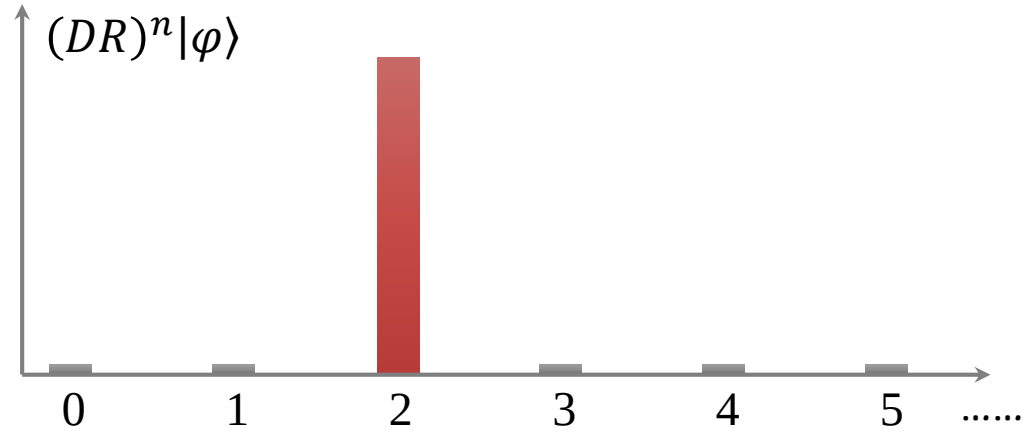
- Then perform diffusion operator D

$$D = \begin{bmatrix} -1+\frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & -1+\frac{2}{N} & & \frac{2}{N} \\ \vdots & & \ddots & \vdots \\ \frac{2}{N} & \frac{2}{N} & \cdots & -1+\frac{2}{N} \end{bmatrix}$$



increase the probability amplitude of the desired state $|x\rangle$

- Iterations of operators R and D



- Measure the register to get the specific state $|x\rangle$ with nearly unity probability

Grover's Search Algorithm

Formulas

- Initial state:

$$|\varphi\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N}|i\rangle = \frac{1}{\sqrt{N}}|x\rangle + \frac{1}{\sqrt{N}}\sum_{\substack{i=0 \\ (i\neq x)}}^{N-1}|i\rangle \equiv |\alpha\rangle + |\beta\rangle$$

- Phase rotation R:

$$R|\varphi\rangle = -|\alpha\rangle + |\beta\rangle$$

- Diffusion operator D:

$$DR|\alpha\rangle = \left(1 - \frac{2}{N}\right)|\alpha\rangle - \frac{2}{N}|\beta\rangle$$

$$DR|\beta\rangle = \left(2 - \frac{2}{N}\right)|\alpha\rangle + \left(1 - \frac{2}{N}\right)|\beta\rangle$$

$$\Rightarrow \quad DR|\varphi\rangle = \left(3 - \frac{4}{N}\right)|\alpha\rangle + \left(1 - \frac{4}{N}\right)|\beta\rangle$$

Grover's Search Algorithm

Formulas

- After n iteration: $|\varphi_n\rangle = (DR)^n|\varphi\rangle \approx \sin\left(\frac{2n}{\sqrt{N}}\right)|x\rangle + \frac{\cos\left(\frac{2n}{\sqrt{N}}\right)}{\sqrt{N}} \sum_{\substack{i=0 \\ (i\neq x)}}^{N-1} |i\rangle$

- The probability to collapse into the x

$$P = \left|\sin\left(\frac{2n}{\sqrt{N}}\right)\right|^2$$

- Choose iteration steps

$$n = \left[\frac{\pi}{4}\sqrt{N}\right]$$

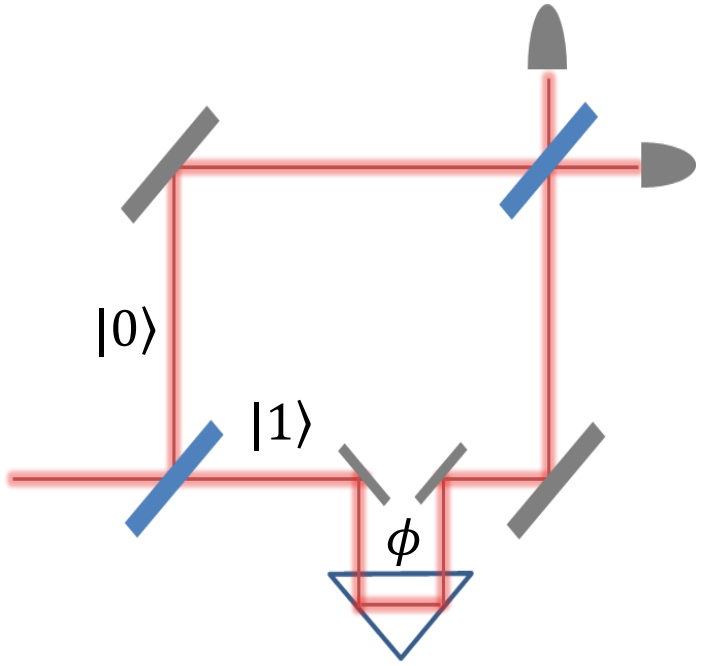
- The probability of failure:

$$1 - P \leq \cos^2\left(\frac{\pi}{2} - \frac{1}{\sqrt{N}}\right) \xrightarrow{N\to\infty} 0$$

Grover, PRL 79, 325 (1997)

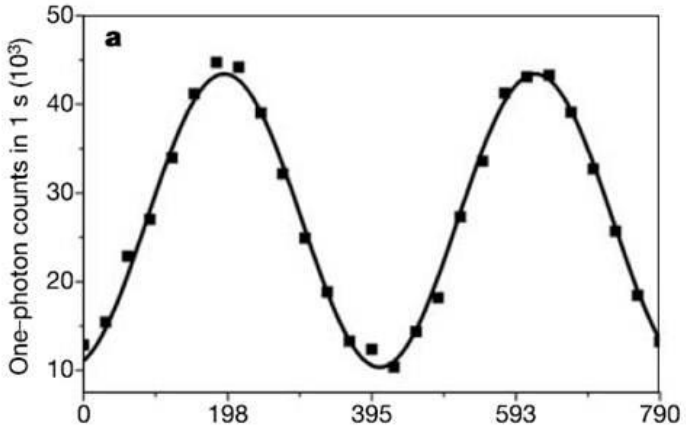
Quantum Metrology

Super-resolution with multi-particle entanglement



Single particle:

$$|\psi\rangle = |0\rangle + e^{i\phi}|1\rangle$$

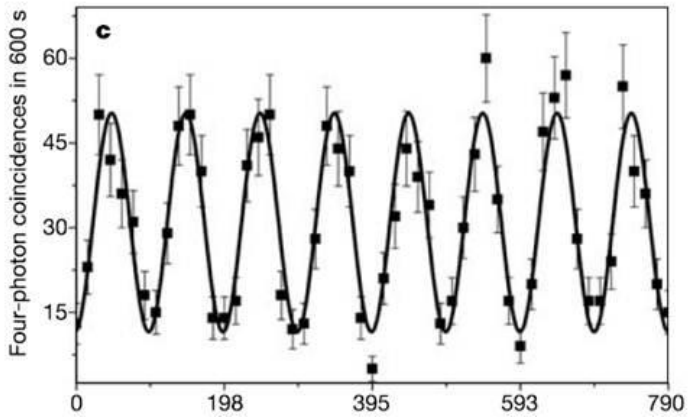


Phase uncertainty with N sampling: $1/\sqrt{N}$

V. S.

N-particle $N00N$ state

$$|\psi\rangle = |\underbrace{00\ldots0}_{N}\rangle + e^{iN\phi}|\underbrace{11\ldots1}_{N}\rangle$$



Phase uncertainty with same cost of resource N: $1/N$

Part 4: Quantum Repeaters and Quantum Error Correction

Noise Environment

Unavoidable interaction with environment and decoherence will happen

$$|0\rangle|E\rangle \xrightarrow{U(t)} |0\rangle|E_0(t)\rangle \qquad |1\rangle|E\rangle \xrightarrow{U(t)} |1\rangle|E_1(t)\rangle$$

- $|0\rangle, |1\rangle$represents the qubit state and $|E\rangle$represents the environment initial state, $U(t)$ is the joint unitary time evolution operator
- For arbitrary qubit state:

$$(\alpha_0|0\rangle + \alpha_1|1\rangle)|E\rangle \xrightarrow{U(t)} \alpha_0|0\rangle|E_0(t)\rangle + \alpha_1|1\rangle|E_1(t)\rangle \qquad \rho_q(t) = Tr_E \rho_{q+E} = \begin{bmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* \langle E_1|E_0\rangle \\ \alpha_1\alpha_0^* \langle E_0|E_1\rangle & |\alpha_1|^2 \end{bmatrix}$$

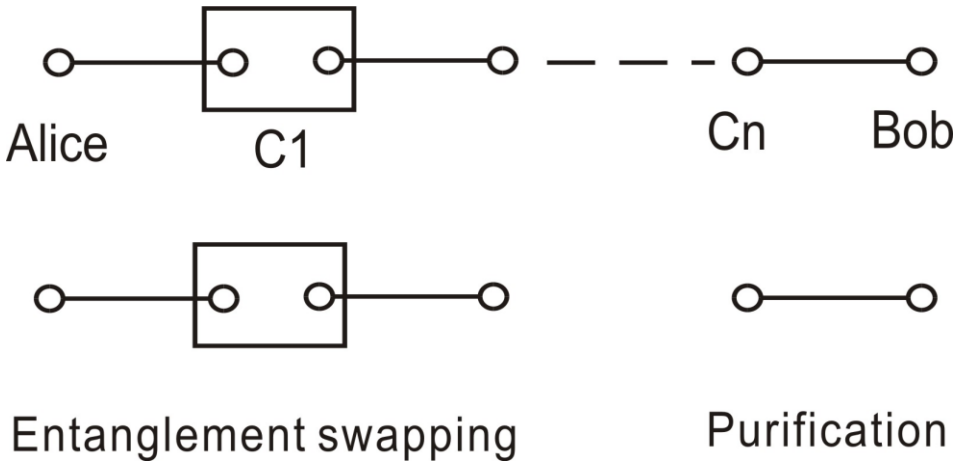
The off-diagonal element of the qubit density matrix will drop down with the rate $\langle E_0(t)|E_1(t)\rangle = e^{-\Gamma t}$

The maximally entangled state will be in some mixed state with a certain entanglement fidelity due to the process

Channel Loss

- Photon loss increases exponentially with channel length: $A \propto e^{-\Gamma L}$ (e. g., in commercial fiber $\Gamma = 0.2\mathrm{dB/km}$)
- For 1000 km commercial fiber, even with a perfect 10 GHz single-photon source and ideal detectors, only 0.3 photon can be transmitted on average per century!

Quantum repeater

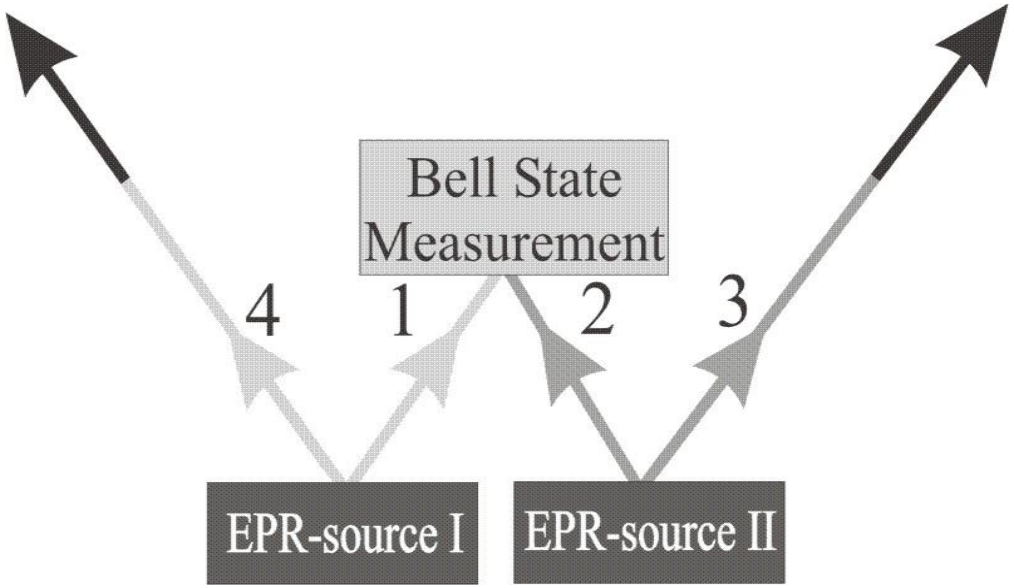


- Solution to photon loss: Entanglement swapping
- Solution to decoherence: Entanglement purification

Briegel *et al.*, PRL 81, 5932 (1998)

Entanglement Swapping

Entangling the remote particles which never interacted!

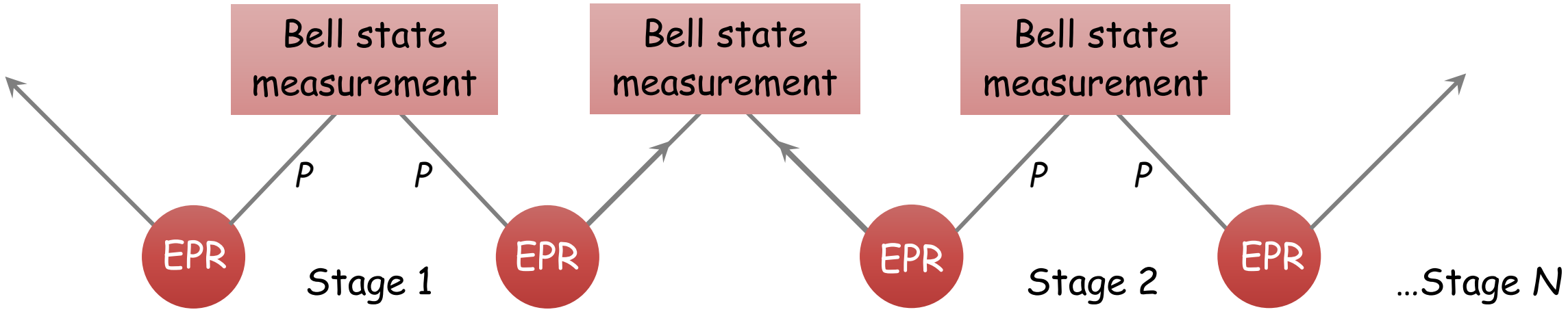


$$|\Psi\rangle_{1234} = |\Phi^+\rangle_{14} \otimes |\Phi^+\rangle_{23}$$

$$= |\Phi^+\rangle_{12} \otimes |\Phi^+\rangle_{34} + |\Phi^-\rangle_{12} \otimes |\Phi^-\rangle_{34} + |\Psi^+\rangle_{12} \otimes |\Psi^+\rangle_{34} + |\Psi^-\rangle_{12} \otimes |\Psi^-\rangle_{34}$$

Zukowski *et al.*, PRL 71, 4287 (1993)

Entanglement Swapping



Probability of transmission with channel loss: P^2

☒ Without entanglement swapping, the total cost in multi-stage is ~$1/P^{2N}$

☑ With entanglement swapping, the total cost is ~$1/P^2$

(assume that the emission probability of ERP sources is unity)

Entanglement Purification

Initially pure singlet state

$$|\Psi^-\rangle_{12} = \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2 - |V\rangle_1|H\rangle_2)$$

Mixed state:

$$M = \sum_{i=1}^{n} E_i|\Psi^-\rangle\langle\Psi^-|E_i^\dagger$$

Fidelity: $F = \langle\Psi^-|M|\Psi^-\rangle$

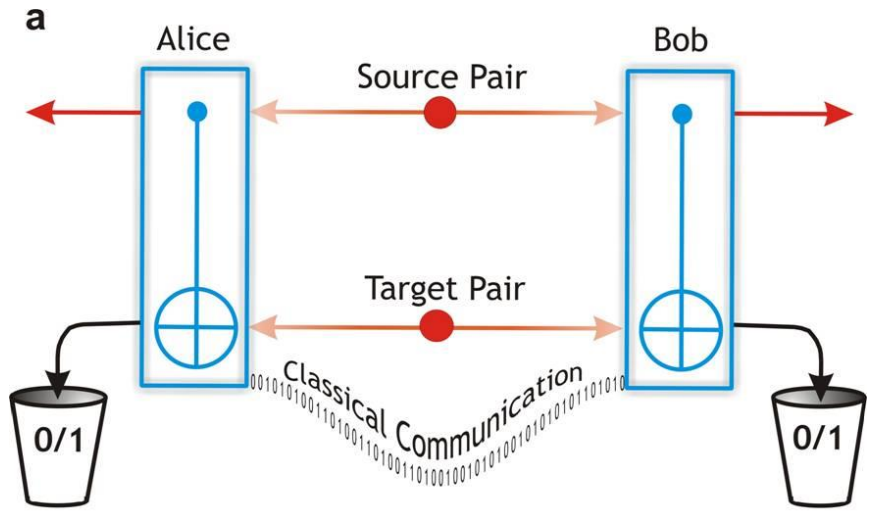
Goal: to extract from a large ensemble of low-fidelity M a small sub-ensemble with sufficiently high fidelity

Scheme for Entanglement Purification

- Random bilateral Pauli rotation on each photon in the states M ➡ change arbitrary mixed state into Werner state: [Werner, PRA 40, 4277 (1989)]

$$W_F = F|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1-F}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-|$$

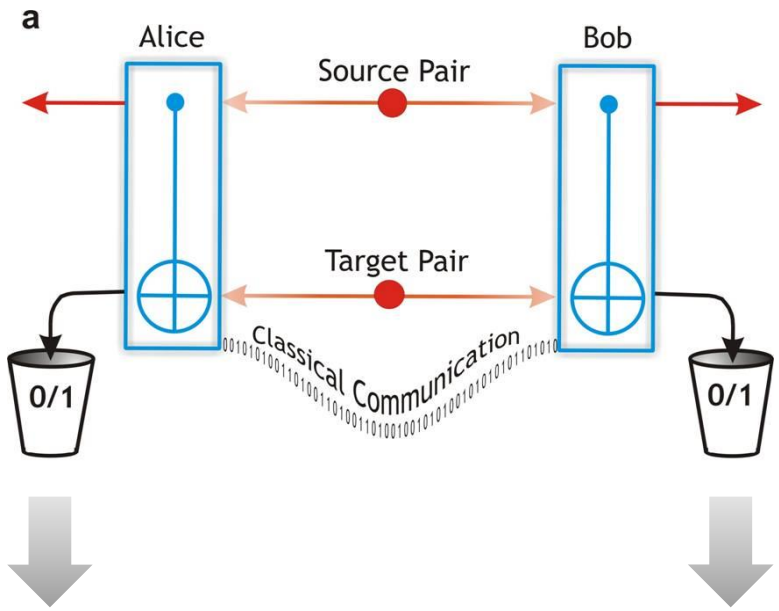
- For two same pairs of Werner states, we consider them as source pair and target pair respectively
- A unilateral σ_Y is performed on each of the two pairs: $|\Psi^{\pm}\rangle \leftrightarrow |\Phi^{\mp}\rangle$



i. e., states with a large component ($F > 1/2$) of $|\Phi^+\rangle$, and equal components of the other three Bell states

Scheme for Entanglement Purification

- Perform CNOT operation on source and target pairs:



- Measure target pair in {H/V} basis, keep the unmeasured source pair when measuring results are same

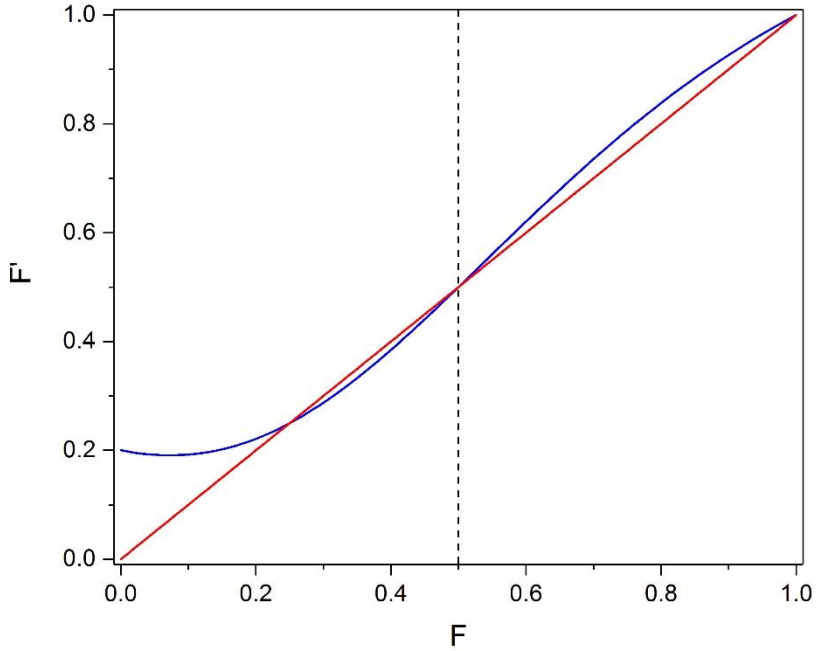
Probability	Before		After	
	Source	Target	Source	Target
F^2	$\|\Phi^+\rangle$	$\|\Phi^+\rangle$	$\|\Phi^+\rangle$	$\|\Phi^+\rangle$
$F(1-F)/3$	$\|\Phi^-\rangle$	$\|\Phi^+\rangle$	$\|\Phi^-\rangle$	$\|\Phi^+\rangle$
$F(1-F)/3$	$\|\Psi^+\rangle$	$\|\Phi^+\rangle$	$\|\Psi^+\rangle$	$\|\Psi^+\rangle$
$F(1-F)/3$	$\|\Psi^-\rangle$	$\|\Phi^+\rangle$	$\|\Psi^-\rangle$	$\|\Psi^+\rangle$
$(1-F)^2/9$	$\|\Psi^+\rangle$	$\|\Psi^+\rangle$	$\|\Psi^+\rangle$	$\|\Phi^+\rangle$
$(1-F)^2/9$	$\|\Psi^-\rangle$	$\|\Psi^+\rangle$	$\|\Psi^-\rangle$	$\|\Phi^+\rangle$
$F(1-F)/3$	$\|\Phi^+\rangle$	$\|\Psi^+\rangle$	$\|\Phi^+\rangle$	$\|\Psi^+\rangle$
$(1-F)^2/9$	$\|\Phi^-\rangle$	$\|\Psi^+\rangle$	$\|\Phi^-\rangle$	$\|\Psi^+\rangle$
$F(1-F)/3$	$\|\Phi^+\rangle$	$\|\Phi^-\rangle$	$\|\Phi^-\rangle$	$\|\Phi^-\rangle$
$(1-F)^2/9$	$\|\Phi^-\rangle$	$\|\Phi^-\rangle$	$\|\Phi^+\rangle$	$\|\Phi^-\rangle$
$(1-F)^2/9$	$\|\Psi^+\rangle$	$\|\Phi^-\rangle$	$\|\Psi^-\rangle$	$\|\Psi^-\rangle$
$(1-F)^2/9$	$\|\Psi^-\rangle$	$\|\Phi^-\rangle$	$\|\Psi^+\rangle$	$\|\Psi^-\rangle$
$(1-F)^2/9$	$\|\Psi^+\rangle$	$\|\Psi^-\rangle$	$\|\Psi^-\rangle$	$\|\Phi^-\rangle$
$(1-F)^2/9$	$\|\Psi^-\rangle$	$\|\Psi^-\rangle$	$\|\Psi^+\rangle$	$\|\Phi^-\rangle$
$F(1-F)/3$	$\|\Phi^+\rangle$	$\|\Psi^-\rangle$	$\|\Phi^-\rangle$	$\|\Psi^-\rangle$
$(1-F)^2/9$	$\|\Phi^-\rangle$	$\|\Psi^-\rangle$	$\|\Phi^+\rangle$	$\|\Psi^-\rangle$

Scheme for Entanglement Purification

- After that, the component of $|\Phi^+\rangle\langle\Phi^+|$ of the target pair will be

$$F' = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2} > F, \qquad \text{when } F > \frac{1}{2}$$

- Equivalently, the fidelity $\langle\Psi^-|M|\Psi^-\rangle$ is equal to F'
- Via several this kind processes, we can purify a general mixed state into a highly entangled state



Bennett *et al.*, PRL 76, 722 (1996)

Quantum error correction

Analogy between classical error correction:

Goal: store an unknown single bit for a time t

Errors:

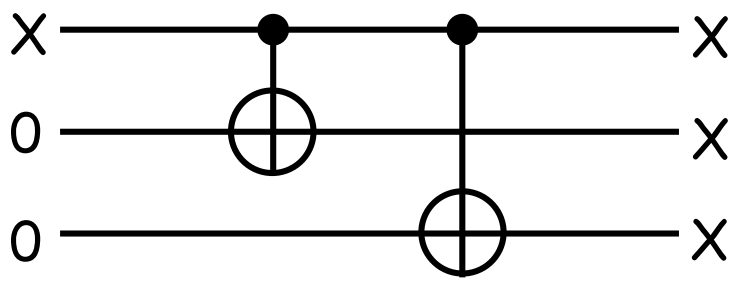
- In a time interval τ one error occurs with the probability P_τ
- Only one type of error: bit flips $0 \to 1, 1 \to 0$
- Suppose errors cause each physical bit to be flipped independently

Classical Error Correction Code

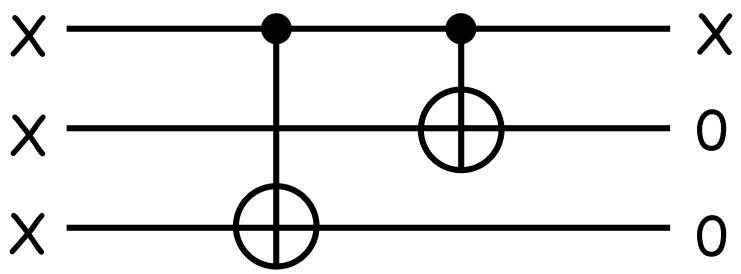
- Correct the errors by using a "redundant coding", e.g. :

Physical bits $000 \rightarrow$ Logical bit 0

Physical bits $111 \rightarrow$ Logical bit 1



Network for encoding



Network for decoding

Classical Error Correction Code

After the errors occur

- Probability of no errors: $(1 - P_\tau)^3$
- Probability of error in one bit: $3P_\tau(1 - P_\tau)^2$
- Probability of error in two bits: $3P_\tau^2(1 - P_\tau)$
- Probability of error in three bits:P_τ^3

Classical Error Correction Code

Correction for a long time

To keep the state for a very long time t:

Correct errors as frequently as possible

- Consider $P_\tau = c\tau$ for time τ sufficiently short
- Divide t in N intervals of duration $\tau = t/N$
- After the time t:

$$P_t^c = \left[1 - 3\left(\frac{ct}{N}\right)^2 + 2\left(\frac{ct}{N}\right)^3\right] \to 1, when\ N \gg 3(ct)^3$$

Zeno effect!

Classical Error Correcting Codes

- Decode the logical bits by taking the majority answer of the three bits and correct the encoded bits

$$000 \rightarrow 000 \qquad 111 \rightarrow 111$$
$$001 \rightarrow 000 \qquad 011 \rightarrow 111$$
$$010 \rightarrow 000 \qquad 101 \rightarrow 111$$
$$100 \rightarrow 000 \qquad 110 \rightarrow 111$$

- The correct state with a probability

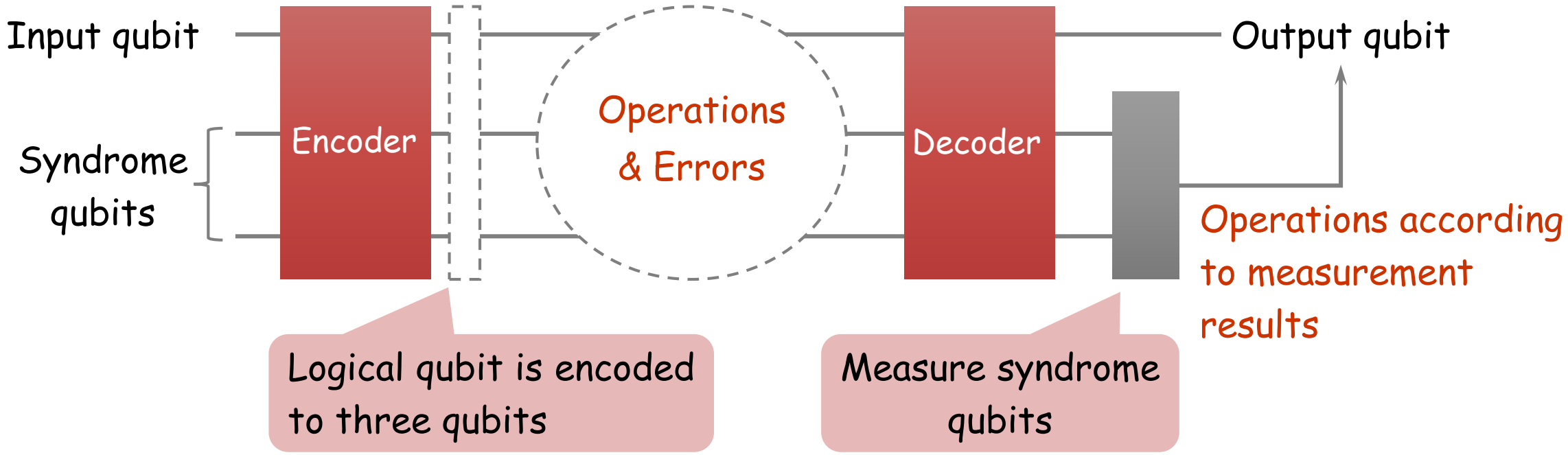
$$P_\tau^c = (1 - P_\tau)^3 + 3P_\tau(1 - P_\tau)^2 = 1 - 3P_\tau^2 + 2P_\tau^3$$

Quantum Errors

- ☒ Measurement of error destroys superpositions
- ☒ No-cloning theorem prevents repetition
- ☒ Multiple types of errors
  - Bit flip (σ_X): $\sigma_X(\alpha|0\rangle + \beta|1\rangle) \rightarrow \alpha|1\rangle + \beta|0\rangle$
  - Phase flip (σ_Z): $\sigma_Z(\alpha|0\rangle + \beta|1\rangle) \rightarrow \alpha|0\rangle - \beta|1\rangle$
  - Mixed (σ_Y): $-i\sigma_Y(\alpha|0\rangle + \beta|1\rangle) \rightarrow \alpha|1\rangle - \beta|0\rangle$

Example of Quantum Error Correction

A 3-bit quantum error correction scheme uses an encoder and a decoder circuit

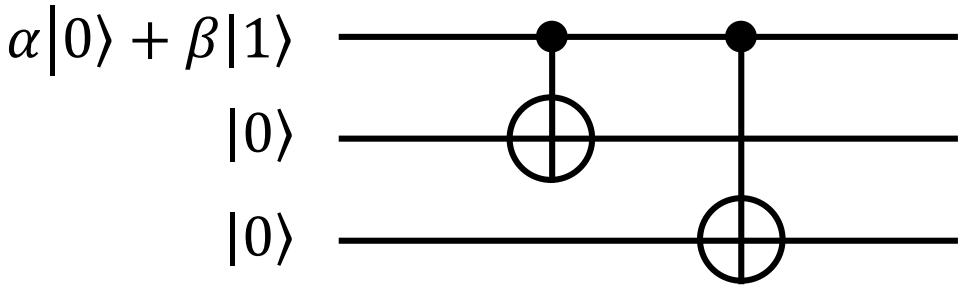


Any correction must be done without looking at the output

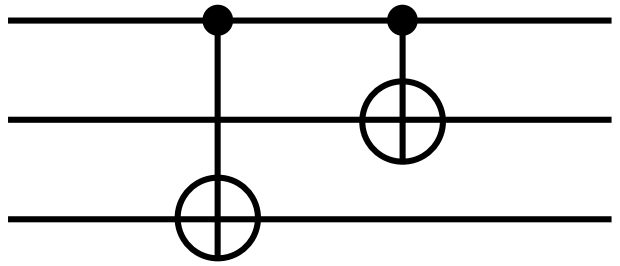
3-qubit Error Correction

Similar to classical error correction $|0\rangle \rightarrow |000\rangle \quad |1\rangle \rightarrow |111\rangle$

Superposition $\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$



Quantum circuit for encoding



Decoder looks just like the encoder

3-qubit Error Correction

All the possible error conditions

Error	Decoded		Correction
$\alpha\|000\rangle + \beta\|111\rangle$	$\alpha\|000\rangle + \beta\|100\rangle = (\alpha\|0\rangle + \beta\|1\rangle)$	$\|00\rangle$	
$\alpha\|100\rangle + \beta\|011\rangle$	$\alpha\|111\rangle + \beta\|011\rangle = (\alpha\|1\rangle + \beta\|0\rangle)$	$\|11\rangle$	Flip the top qubit
$\alpha\|010\rangle + \beta\|101\rangle$	$\alpha\|010\rangle + \beta\|110\rangle = (\alpha\|0\rangle + \beta\|1\rangle)$	$\|10\rangle$	
$\alpha\|001\rangle + \beta\|110\rangle$	$\alpha\|001\rangle + \beta\|101\rangle = (\alpha\|0\rangle + \beta\|1\rangle)$	$\|01\rangle$	

After decoding, the states of syndrome qubits are orthogonal enabling to distinguish which qubit is the error occurred on

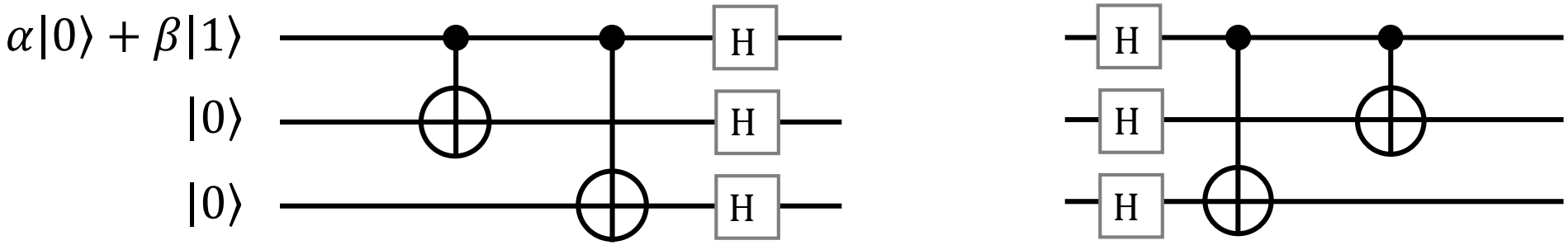
Phase-flip Error Correction

Phase flip: $\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle - \beta|1\rangle$ Represent with {+/-} basis $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

$(\alpha+\beta)|+\rangle + (\alpha-\beta)|-\rangle \rightarrow (\alpha+\beta)|-\rangle + (\alpha-\beta)|+\rangle$ ➡ Bit flip error in {+/-} basis

Similar to bit flip error correction, a logical qubit is encoded with

$|0\rangle \rightarrow |{+}{+}{+}\rangle \quad |1\rangle \rightarrow |{-}{-}{-}\rangle$



Encoder Decoder

Shor's 9 Qubits Error Correcting Code

Concatenated code

- Bit flip error correction: $|0\rangle \rightarrow |000\rangle \quad |1\rangle \rightarrow |111\rangle$

Consider $|000\rangle$ as $|0'\rangle$, $|111\rangle$ as $|1'\rangle$

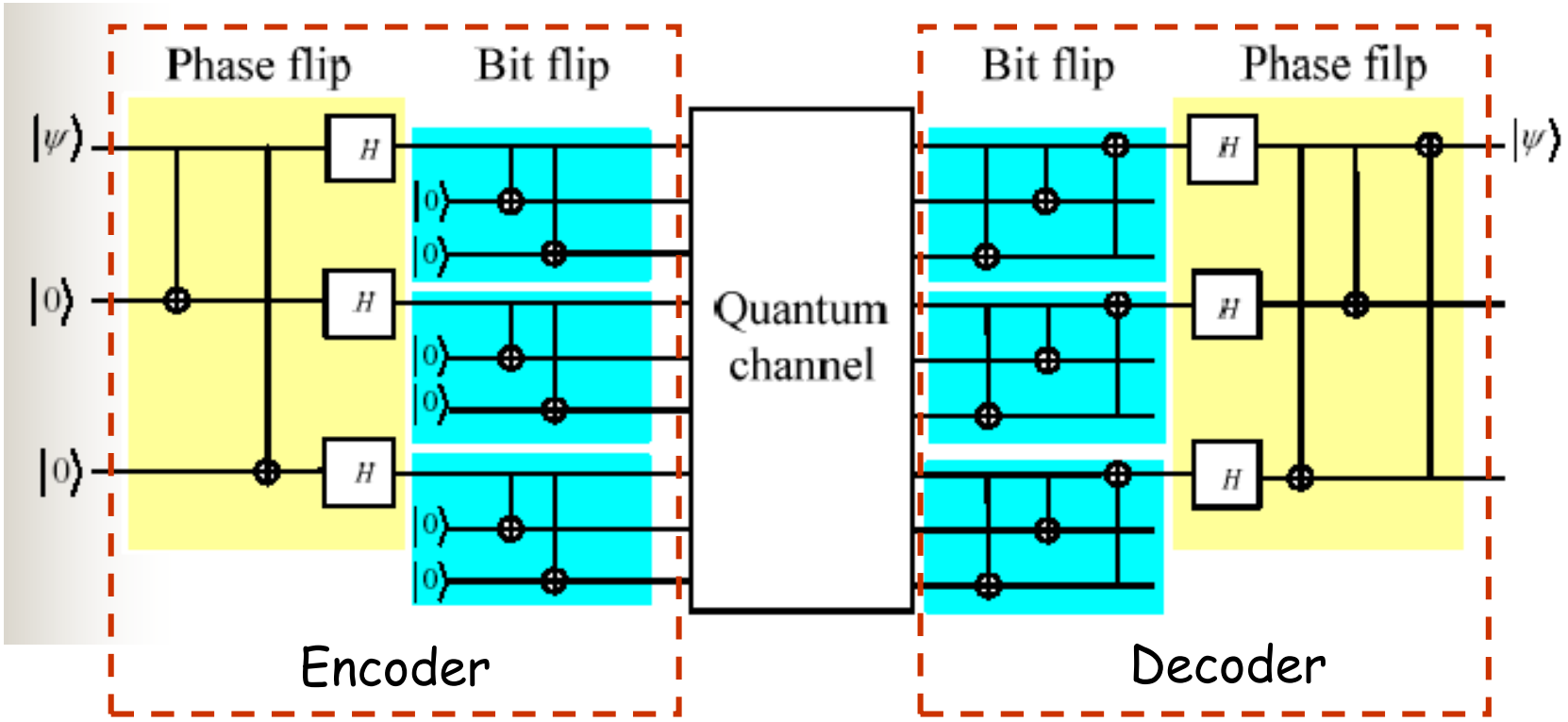
- Phase flip error correction: $|0'\rangle \rightarrow |+'\,+'\,+'\rangle \quad |1'\rangle \rightarrow |-'\,-'\,-'\rangle$

Corrects both bit flip and phase flip errors!

$$|0\rangle \rightarrow \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3} \qquad |1\rangle \rightarrow \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 3}$$

Shor, PRA 52, 2493 (1995)

Shor's 9 Qubits Error Correcting Code



- General single-qubit error: $E = c_0 I + c_1 \sigma_Z + c_1 \sigma_X + c_3 \sigma_X \sigma_Z$
- Each term will be represented with orthogonal states of syndrome bits ➡ If a code can correct both bit flip and phase flip errors, it can correct arbitrary single-qubit error!

More Efficient Code

The Steane (CSS) code

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{1}{\sqrt{8}}[|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle$$
$$+ |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle]$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{1}{\sqrt{8}}[|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle$$
$$+ |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle]$$

Calderbank and Shor, PRA 54, 1098 (1996)

Steane, Proc. R. Soc. Lodon A 452, 2551 (1996)

What is The Minimum Number to Encode 1 Qubit?

Argument

- Encode 1 logical qubit using n physical qubits
- n-1 syndrome bits ➡ them can represent 2^{n-1} states at most
- n qubits ➡ 3n possible errors and also the case of no errors

n must satisfies: $3n + 1 \leq 2^{n-1}$

The minimum number in theory is 5

5 Qubits Error Correcting Code

$$|0\rangle \xrightarrow{encode} \frac{1}{\sqrt{8}}\big(|00000\rangle - |01111\rangle - |10011\rangle + |11100\rangle + |00110\rangle + |01001\rangle + |10101\rangle + |11010\rangle\big)$$

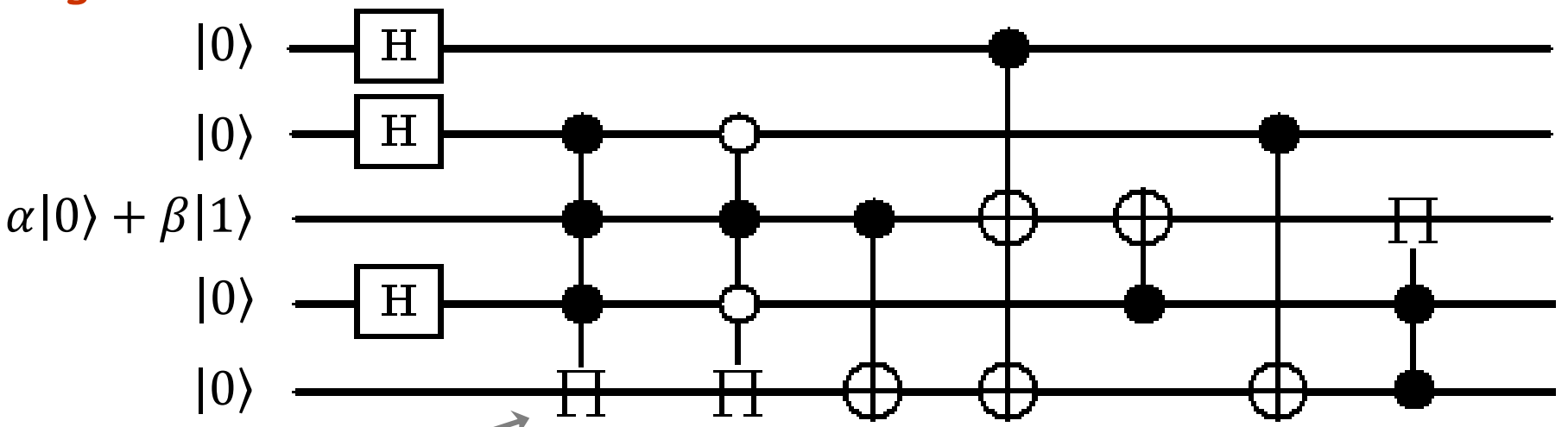
$$|1\rangle \xrightarrow{encode} \frac{1}{\sqrt{8}}\big(|11111\rangle - |10000\rangle + |01100\rangle - |00011\rangle + |11001\rangle + |10110\rangle - |01010\rangle - |00101\rangle\big)$$

Laflamme *et al.*, PRL 77, 198 (1996)

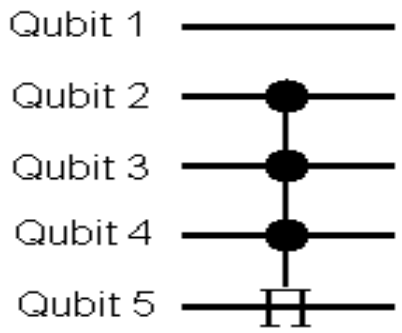
5 Qubits Error Correcting Code

$$\alpha|0\rangle+\beta|1\rangle \xrightarrow{encode} \frac{\alpha}{\sqrt{8}}\big(|0_1 0_2 0_3 0_4 0_5\rangle-|0_1 1_2 1_3 1_4 1_5\rangle-|1_1 0_2 0_3 1_4 1_5\rangle+|1_1 1_2 1_3 0_4 0_5\rangle$$
$$+|0_1 0_2 1_3 1_4 0_5\rangle+|0_1 1_2 0_3 0_4 1_5\rangle+|1_1 0_2 1_3 0_4 1_5\rangle+|1_1 1_2 0_3 1_4 0_5\rangle\big)$$
$$+\frac{\beta}{\sqrt{8}}\big(|1_1 1_2 1_3 1_4 1_5\rangle-|1_1 0_2 0_3 0_4 0_5\rangle+|0_1 1_2 1_3 0_4 0_5\rangle-|0_1 0_2 0_3 1_4 1_5\rangle$$
$$+|1_1 1_2 0_3 0_4 1_5\rangle+|1_1 0_2 1_3 1_4 0_5\rangle-|0_1 1_2 0_3 1_4 0_5\rangle-|0_1 0_2 1_3 0_4 1_5\rangle\big)$$

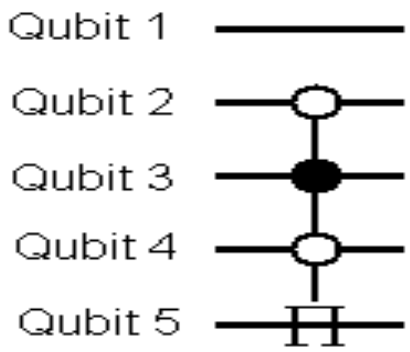
Encoding circuit



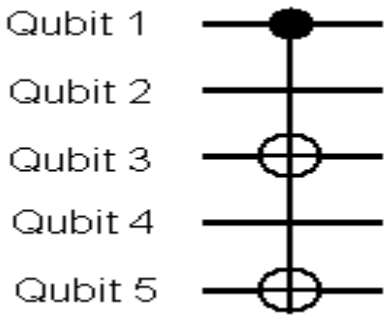
Rules of Shifting Phase and Flip Bit



If qubits 2,3 and 4 are '1', shift the phase



If qubits 2 and 4 is '0' and qubit 3 is '1', shift the phase



If qubit 1 is '1', flip qubits 3 and 5

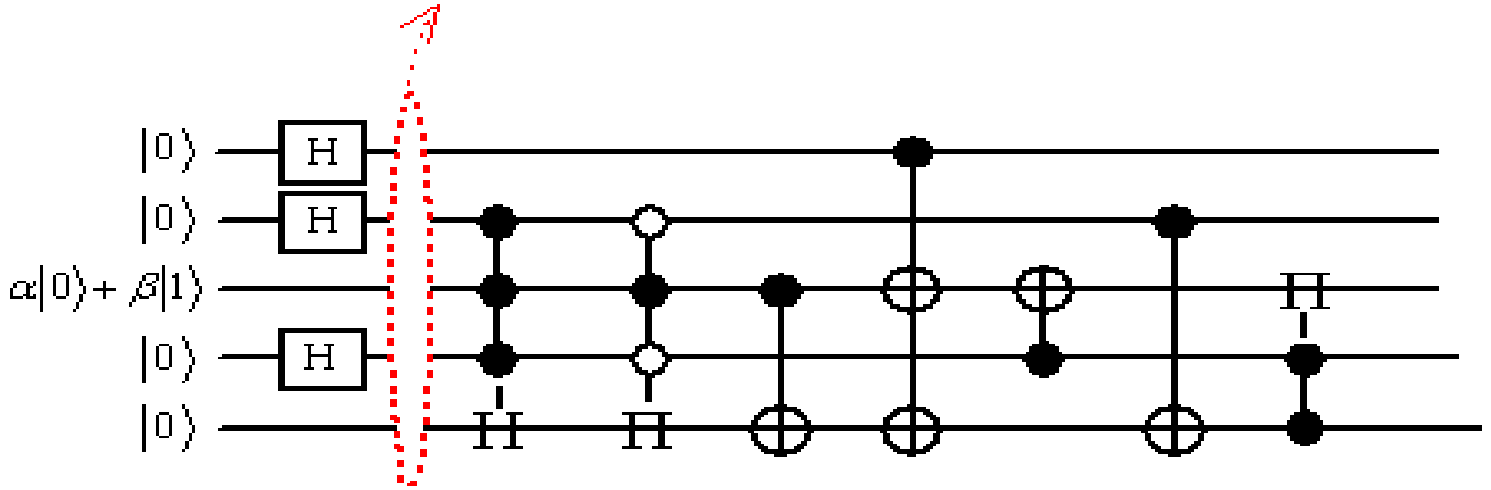
Signal after Hadamards

Hadamard

$$\left(\frac{(|0_1\rangle+|1_1\rangle)}{\sqrt{2}}\right)\left(\frac{(|0_2\rangle+|1_2\rangle)}{\sqrt{2}}\right)(\alpha|0_3\rangle+\beta|1_3\rangle)\left(\frac{(|0_4\rangle+|1_4\rangle)}{\sqrt{2}}\right)|0_5\rangle$$

$$=\frac{1}{\sqrt{4}}(|0_1 0_2\rangle+|0_1 1_2\rangle+|1_1 0_2\rangle+|1_1 1_2\rangle)(\alpha|0_3\rangle+\beta|1_3\rangle)\frac{1}{\sqrt{2}}(|0_4 0_5\rangle+|1_4 0_5\rangle)$$

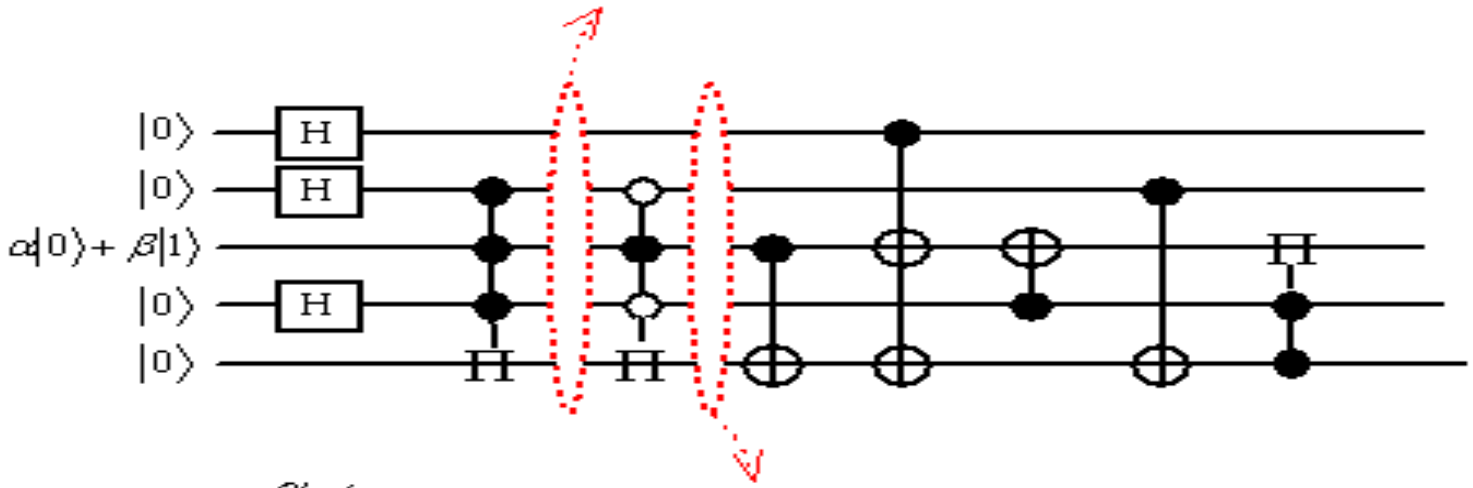
$$=\frac{\alpha}{\sqrt{8}}(|0_1 0_2 0_3 0_4 0_5\rangle+|0_1 1_2 0_3 0_4 0_5\rangle+|1_1 0_2 0_3 0_4 0_5\rangle+|1_1 1_2 0_3 0_4 0_5\rangle$$
$$+|0_1 0_2 0_3 1_4 0_5\rangle+|0_1 1_2 0_3 1_4 0_5\rangle+|1_1 0_2 0_3 1_4 0_5\rangle+|1_1 1_2 0_3 1_4 0_5\rangle)$$
$$+\frac{\beta}{\sqrt{8}}(|0_1 0_2 1_3 0_4 0_5\rangle+|0_1 1_2 1_3 0_4 0_5\rangle+|1_1 0_2 1_3 0_4 0_5\rangle+|1_1 1_2 1_3 0_4 0_5\rangle$$
$$+|0_1 0_2 1_3 1_4 0_5\rangle+|0_1 1_2 1_3 1_4 0_5\rangle+|1_1 0_2 1_3 1_4 0_5\rangle+|1_1 1_2 1_3 1_4 0_5\rangle)$$



Step-by-step Analysis of Encoding Circuit

$$\frac{\alpha}{\sqrt{8}}(|0_1 0_2 0_3 0_4 0_5\rangle+|0_1 1_2 0_3 0_4 0_5\rangle+|1_1 0_2 0_3 0_4 0_5\rangle+|1_1 1_2 0_3 0_4 0_5\rangle$$
$$+|0_1 0_2 0_3 1_4 0_5\rangle+|0_1 1_2 0_3 1_4 0_5\rangle+|1_1 0_2 0_3 1_4 0_5\rangle+|1_1 1_2 0_3 1_4 0_5\rangle)$$
$$+\frac{\beta}{\sqrt{8}}(|0_1 0_2 1_3 0_4 0_5\rangle+|0_1 1_2 1_3 0_4 0_5\rangle+|1_1 0_2 1_3 0_4 0_5\rangle+|1_1 1_2 1_3 0_4 0_5\rangle$$
$$+|0_1 0_2 1_3 1_4 0_5\rangle-|0_1 1_2 1_3 1_4 0_5\rangle+|1_1 0_2 1_3 1_4 0_5\rangle-|1_1 1_2 1_3 1_4 0_5\rangle)$$

Shifting phase when 2,3,4 are "1"



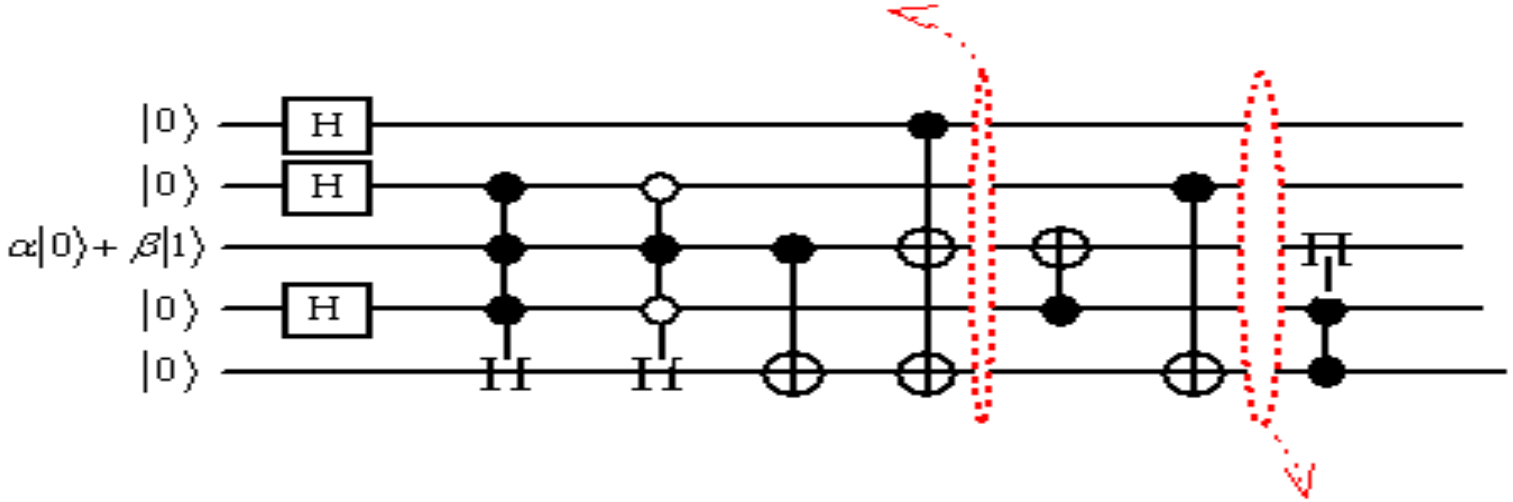
$$\frac{\alpha}{\sqrt{8}}(|0_1 0_2 0_3 0_4 0_5\rangle+|0_1 1_2 0_3 0_4 0_5\rangle+|1_1 0_2 0_3 0_4 0_5\rangle+|1_1 1_2 0_3 0_4 0_5\rangle$$
$$+|0_1 0_2 0_3 1_4 0_5\rangle+|0_1 1_2 0_3 1_4 0_5\rangle+|1_1 0_2 0_3 1_4 0_5\rangle+|1_1 1_2 0_3 1_4 0_5\rangle)$$
$$+\frac{\beta}{\sqrt{8}}(-|0_1 0_2 1_3 0_4 0_5\rangle+|0_1 1_2 1_3 0_4 0_5\rangle-|1_1 0_2 1_3 0_4 0_5\rangle+|1_1 1_2 1_3 0_4 0_5\rangle$$
$$+|0_1 0_2 1_3 1_4 0_5\rangle-|0_1 1_2 1_3 1_4 0_5\rangle+|1_1 0_2 1_3 1_4 0_5\rangle-|1_1 1_2 1_3 1_4 0_5\rangle)$$

Shifting phase when 2,4 are "0" and 3 is "1"

Step-by-step Analysis of Encoding Circuit

$$\frac{\alpha}{\sqrt{8}}\left(|0_1 0_2 0_3 0_4 0_5\rangle + |0_1 1_2 0_3 0_4 0_5\rangle + |1_1 0_2 1_3 0_4 1_5\rangle + |1_1 1_2 1_3 0_4 1_5\rangle\right.$$
$$\left. + |0_1 0_2 0_3 1_4 0_5\rangle + |0_1 1_2 0_3 1_4 0_5\rangle + |1_1 0_2 1_3 1_4 1_5\rangle + |1_1 1_2 1_3 1_4 1_5\rangle\right)$$
$$+\frac{\beta}{\sqrt{8}}\left(-|0_1 0_2 1_3 0_4 1_5\rangle + |0_1 1_2 1_3 0_4 1_5\rangle - |1_1 0_2 0_3 0_4 0_5\rangle + |1_1 1_2 0_3 0_4 0_5\rangle\right.$$
$$\left. + |0_1 0_2 1_3 1_4 1_5\rangle - |0_1 1_2 1_3 1_4 1_5\rangle + |1_1 0_2 0_3 1_4 0_5\rangle - |1_1 1_2 0_3 1_4 0_5\rangle\right)$$

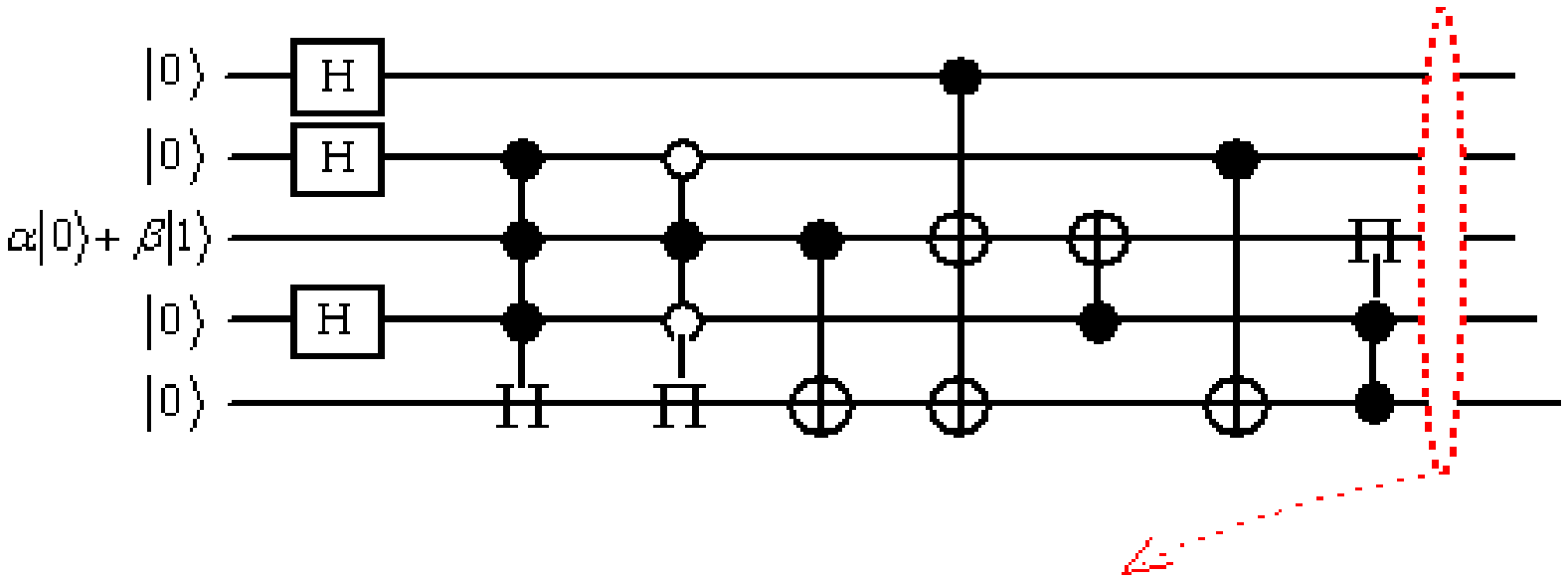
Flipping bits 5 when bit 3 is "1" and then flipping bit 3 and 5 when bit 1 is "1"



$$\frac{\alpha}{\sqrt{8}}\left(|0_1 0_2 0_3 0_4 0_5\rangle + |0_1 1_2 0_3 0_4 1_5\rangle + |1_1 0_2 1_3 0_4 1_5\rangle + |1_1 1_2 1_3 0_4 0_5\rangle\right.$$
$$\left. + |0_1 0_2 1_3 1_4 0_5\rangle + |0_1 1_2 1_3 1_4 1_5\rangle + |1_1 0_2 0_3 1_4 1_5\rangle + |1_1 1_2 0_3 1_4 0_5\rangle\right)$$
$$+\frac{\beta}{\sqrt{8}}\left(-|0_1 0_2 1_3 0_4 1_5\rangle + |0_1 1_2 1_3 0_4 0_5\rangle - |1_1 0_2 0_3 0_4 0_5\rangle + |1_1 1_2 0_3 0_4 1_5\rangle\right.$$
$$\left. + |0_1 0_2 0_3 1_4 1_5\rangle - |0_1 1_2 0_3 1_4 0_5\rangle + |1_1 0_2 1_3 1_4 0_5\rangle - |1_1 1_2 1_3 1_4 1_5\rangle\right)$$

Flipping bits 3 and 5 with "1" in bits 2 and 4

Step-by-step Analysis of Encoding Circuit

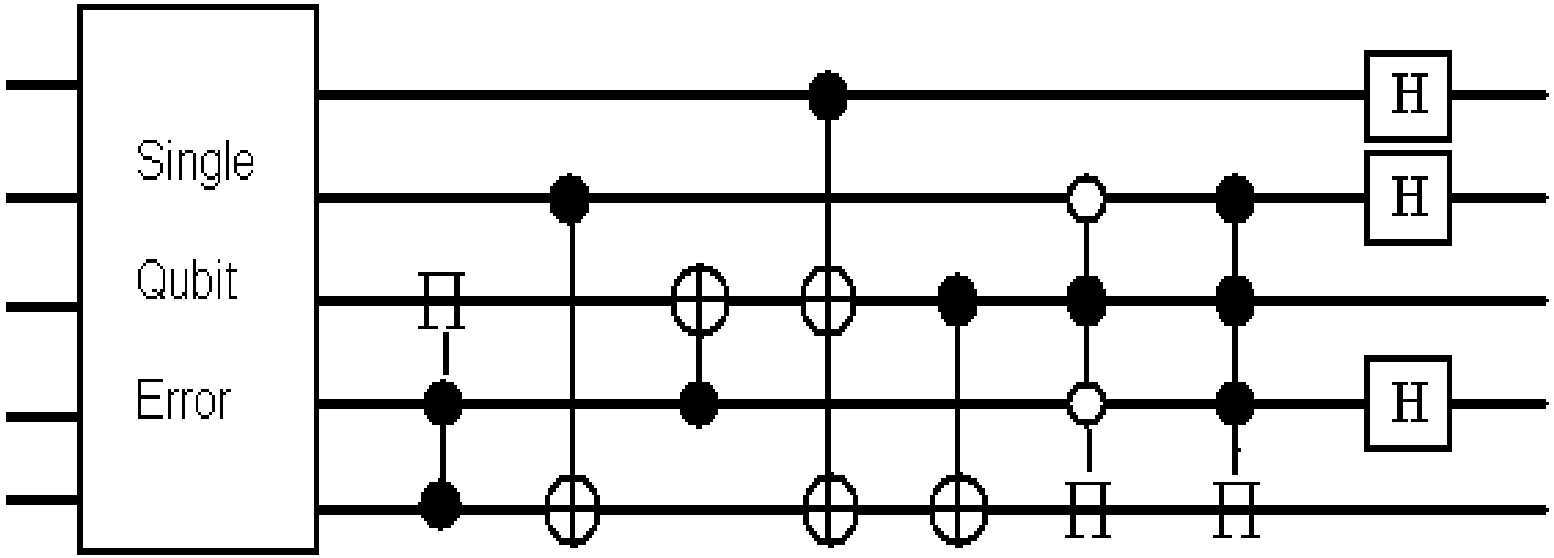


$$\frac{\alpha}{\sqrt{8}}\left(|0_1 0_2 0_3 0_4 0_5\rangle+|0_1 1_2 0_3 0_4 1_5\rangle+|1_1 0_2 1_3 0_4 1_5\rangle+|1_1 1_2 1_3 0_4 0_5\rangle\right.$$
$$\left.+|0_1 0_2 1_3 1_4 0_5\rangle-|0_1 1_2 1_3 1_4 1_5\rangle-|1_1 0_2 0_3 1_4 1_5\rangle+|1_1 1_2 0_3 1_4 0_5\rangle\right)$$
$$+\frac{\beta}{\sqrt{8}}\left(-|0_1 0_2 1_3 0_4 1_5\rangle+|0_1 1_2 1_3 0_4 0_5\rangle-|1_1 0_2 0_3 0_4 0_5\rangle+|1_1 1_2 0_3 0_4 1_5\rangle\right.$$
$$\left.-|0_1 0_2 0_3 1_4 1_5\rangle-|0_1 1_2 0_3 1_4 0_5\rangle+|1_1 0_2 1_3 1_4 0_5\rangle+|1_1 1_2 1_3 1_4 1_5\rangle\right)$$

Shifting phase in data bit when bits 4 and 5 are "1"

Step-by-step Analysis of Encoding Circuit

- Assuming at most 1 qubit error and the error is just as likely to affect any qubit
- The decoding circuit is the encoding circuit in reverse:



Example: Error is Phase and Bit Flip on 3rd Qubit

Assume encoded qubit damaged such that:

$$\frac{a}{\sqrt{8}}\left(|0_1 0_2 0_3 0_4 0_5\rangle - |0_1 1_2 1_3 1_4 1_5\rangle - |1_1 0_2 0_3 1_4 1_5\rangle + |1_1 1_2 1_3 0_4 0_5\rangle \right.$$
$$\left. + |0_1 0_2 1_3 1_4 0_5\rangle + |0_1 1_2 0_3 0_4 1_5\rangle + |1_1 0_2 1_3 0_4 1_5\rangle + |1_1 1_2 0_3 1_4 0_5\rangle\right)$$
$$+ \frac{b}{\sqrt{8}}\left(|1_1 1_2 1_3 1_4 1_5\rangle - |1_1 0_2 0_3 0_4 0_5\rangle + |0_1 1_2 1_3 0_4 0_5\rangle - |0_1 0_2 0_3 1_4 1_5\rangle \right.$$
$$\left. + |1_1 1_2 0_3 0_4 1_5\rangle + |1_1 0_2 1_3 1_4 0_5\rangle - |0_1 1_2 0_3 1_4 0_5\rangle - |0_1 0_2 1_3 0_4 1_5\rangle\right)$$

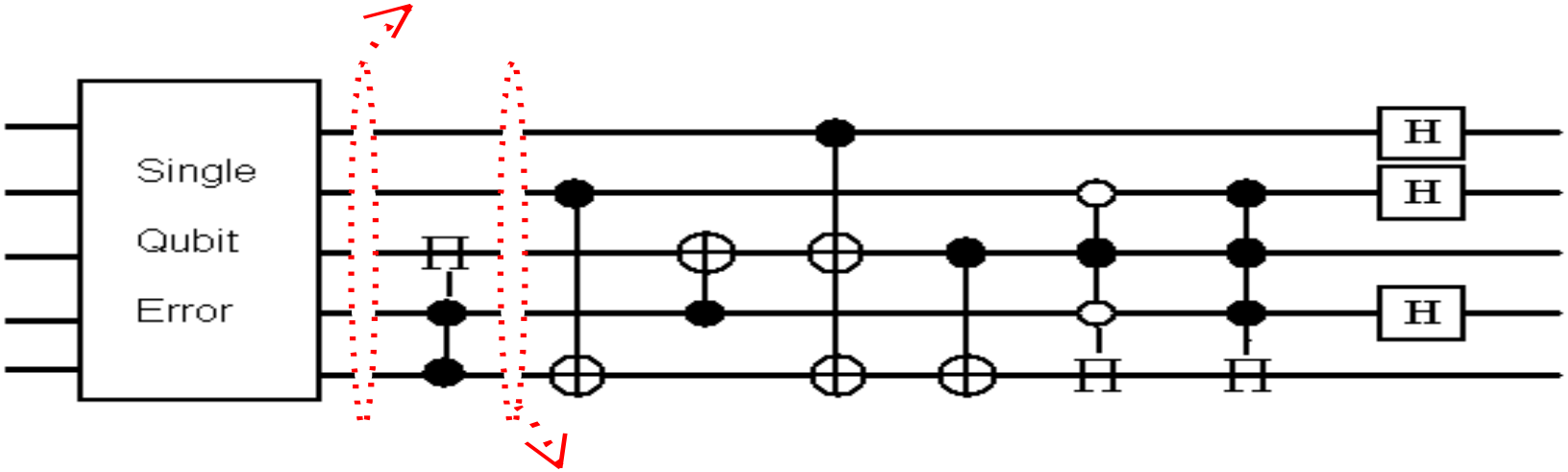
↓ Phase and bit flip on 3rd qubit $(-i\sigma_y)$

$$\frac{a}{\sqrt{8}}\left(|0_1 0_2 1_3 0_4 0_5\rangle + |0_1 1_2 0_3 1_4 1_5\rangle - |1_1 0_2 1_3 1_4 1_5\rangle - |1_1 1_2 0_3 0_4 0_5\rangle \right.$$
$$\left. - |0_1 0_2 0_3 1_4 0_5\rangle + |0_1 1_2 1_3 0_4 1_5\rangle - |1_1 0_2 0_3 0_4 1_5\rangle + |1_1 1_2 1_3 1_4 0_5\rangle\right)$$
$$+ \frac{b}{\sqrt{8}}\left(-|1_1 1_2 0_3 1_4 1_5\rangle - |1_1 0_2 1_3 0_4 0_5\rangle - |0_1 1_2 0_3 0_4 0_5\rangle - |0_1 0_2 1_3 1_4 1_5\rangle \right.$$
$$\left. + |1_1 1_2 1_3 0_4 1_5\rangle - |1_1 0_2 0_3 1_4 0_5\rangle - |0_1 1_2 1_3 1_4 0_5\rangle + |0_1 0_2 0_3 0_4 1_5\rangle\right)$$

Continuation of Error Analysis in Decoder

$$\frac{\alpha}{\sqrt{8}}(|0_1 0_2 1_3 0_4 0_5\rangle + |0_1 1_2 0_3 1_4 1_5\rangle - |1_1 0_2 1_3 1_4 1_5\rangle - |1_1 1_2 0_3 0_4 0_5\rangle$$
$$- |0_1 0_2 0_3 1_4 0_5\rangle + |0_1 1_2 1_3 0_4 1_5\rangle - |1_1 0_2 0_3 0_4 1_5\rangle + |1_1 1_2 1_3 1_4 0_5\rangle)$$
$$+ \frac{\beta}{\sqrt{8}}(- |1_1 1_2 0_3 1_4 1_5\rangle - |1_1 0_2 1_3 0_4 0_5\rangle - |0_1 1_2 0_3 0_4 0_5\rangle - |0_1 0_2 1_3 1_4 1_5\rangle$$
$$+ |1_1 1_2 1_3 0_4 1_5\rangle - |1_1 0_2 0_3 1_4 0_5\rangle - |0_1 1_2 1_3 1_4 0_5\rangle + |0_1 0_2 0_3 0_4 1_5\rangle)$$

Phase and bit flip on 3rd qubit



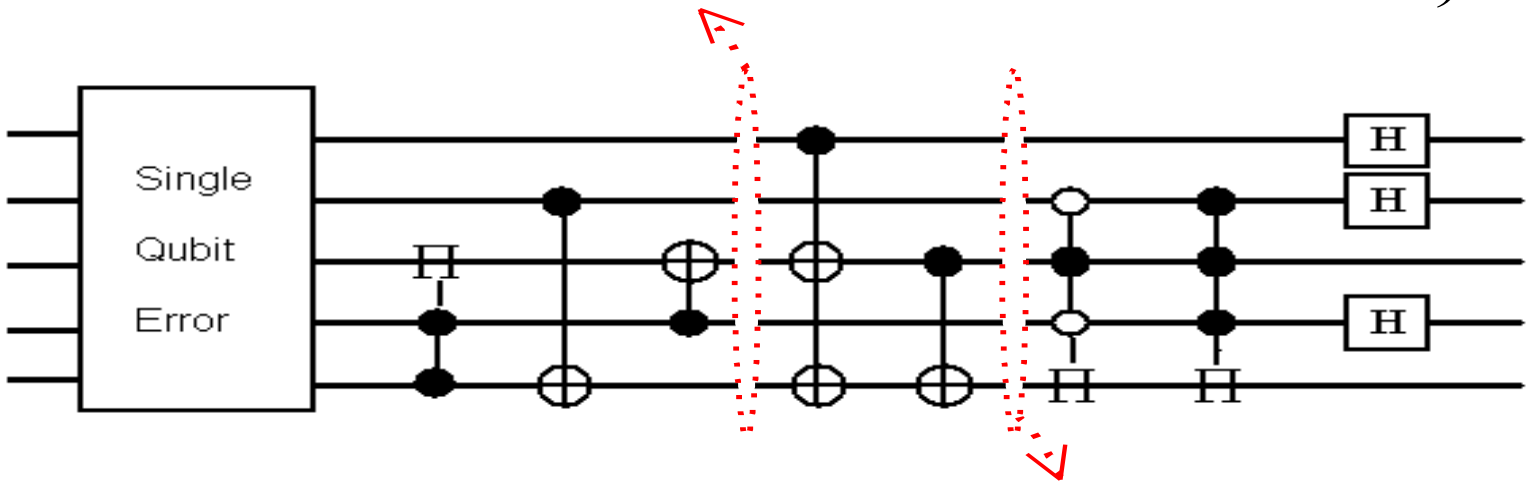
$$\frac{\alpha}{\sqrt{8}}(|0_1 0_2 1_3 0_4 0_5\rangle - |0_1 1_2 0_3 1_4 1_5\rangle + |1_1 0_2 1_3 1_4 1_5\rangle - |1_1 1_2 0_3 0_4 0_5\rangle$$
$$- |0_1 0_2 0_3 1_4 0_5\rangle + |0_1 1_2 1_3 0_4 1_5\rangle - |1_1 0_2 0_3 0_4 1_5\rangle + |1_1 1_2 1_3 1_4 0_5\rangle)$$
$$+ \frac{\beta}{\sqrt{8}}(+ |1_1 1_2 0_3 1_4 1_5\rangle - |1_1 0_2 1_3 0_4 0_5\rangle - |0_1 1_2 0_3 0_4 0_5\rangle + |0_1 0_2 1_3 1_4 1_5\rangle$$
$$+ |1_1 1_2 1_3 0_4 1_5\rangle - |1_1 0_2 0_3 1_4 0_5\rangle - |0_1 1_2 1_3 1_4 0_5\rangle + |0_1 0_2 0_3 0_4 1_5\rangle)$$

Shift phase when bits 4 and 5 are "1"

Continuation of Error Analysis in Decoder

$$\frac{\alpha}{\sqrt{8}}(|0_1 0_2 1_3 0_4 0_5\rangle - |0_1 1_2 1_3 1_4 0_5\rangle + |1_1 0_2 0_3 1_4 1_5\rangle - |1_1 1_2 0_3 0_4 1_5\rangle$$
$$- |0_1 0_2 1_3 1_4 0_5\rangle + |0_1 1_2 1_3 0_4 0_5\rangle - |1_1 0_2 0_3 0_4 1_5\rangle + |1_1 1_2 0_3 1_4 1_5\rangle)$$
$$+ \frac{\beta}{\sqrt{8}}(+ |1_1 1_2 1_3 1_4 0_5\rangle - |1_1 0_2 1_3 0_4 0_5\rangle - |0_1 1_2 0_3 0_4 1_5\rangle + |0_1 0_2 0_3 1_4 1_5\rangle$$
$$+ |1_1 1_2 1_3 0_4 0_5\rangle - |1_1 0_2 1_3 1_4 0_5\rangle - |0_1 1_2 0_3 1_4 1_5\rangle + |0_1 0_2 0_3 0_4 1_5\rangle)$$

Flipping bit 3 when bit 4 is "1" and flipping bit 5 when bit 2 is "1"



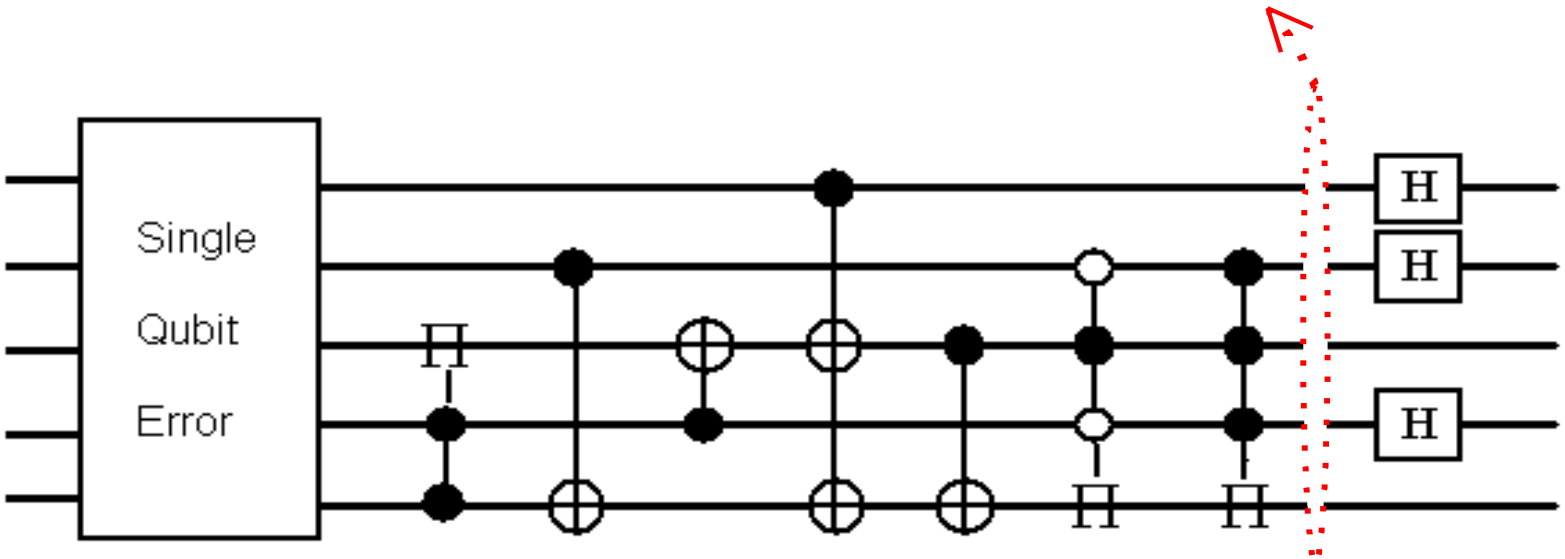
$$\frac{\alpha}{\sqrt{8}}(|0_1 0_2 1_3 0_4 1_5\rangle - |0_1 1_2 1_3 1_4 1_5\rangle + |1_1 0_2 1_3 1_4 1_5\rangle - |1_1 1_2 1_3 0_4 1_5\rangle$$
$$- |0_1 0_2 1_3 1_4 1_5\rangle + |0_1 1_2 1_3 0_4 1_5\rangle - |1_1 0_2 1_3 0_4 1_5\rangle + |1_1 1_2 1_3 1_4 1_5\rangle)$$
$$+ \frac{\beta}{\sqrt{8}}(+ |1_1 1_2 0_3 1_4 1_5\rangle - |1_1 0_2 0_3 0_4 1_5\rangle - |0_1 1_2 0_3 0_4 1_5\rangle + |0_1 0_2 0_3 1_4 1_5\rangle$$
$$+ |1_1 1_2 0_3 0_4 1_5\rangle - |1_1 0_2 0_3 1_4 1_5\rangle - |0_1 1_2 0_3 1_4 1_5\rangle + |0_1 0_2 0_3 0_4 1_5\rangle)$$

Flipping bits 3 and 5 when bit 1 is "1" and then flipping bit 5 when bit 3 is "1"

Continuation of Error Analysis in Decoder

$$\frac{\alpha}{\sqrt{8}}\left(-\left|0_1 0_2 1_3 0_4 1_5\right\rangle+\left|0_1 1_2 1_3 1_4 1_5\right\rangle+\left|1_1 0_2 1_3 1_4 1_5\right\rangle-\left|1_1 1_2 1_3 0_4 1_5\right\rangle\right.$$
$$\left.-\left|0_1 0_2 1_3 1_4 1_5\right\rangle+\left|0_1 1_2 1_3 0_4 1_5\right\rangle+\left|1_1 0_2 1_3 0_4 1_5\right\rangle-\left|1_1 1_2 1_3 1_4 1_5\right\rangle\right)$$
$$+\frac{\beta}{\sqrt{8}}\left(+\left|1_1 1_2 0_3 1_4 1_5\right\rangle-\left|1_1 0_2 0_3 0_4 1_5\right\rangle-\left|0_1 1_2 0_3 0_4 1_5\right\rangle+\left|0_1 0_2 0_3 1_4 1_5\right\rangle\right.$$
$$\left.+\left|1_1 1_2 0_3 0_4 1_5\right\rangle-\left|1_1 0_2 0_3 1_4 1_5\right\rangle-\left|0_1 1_2 0_3 1_4 1_5\right\rangle+\left|0_1 0_2 0_3 0_4 1_5\right\rangle\right)$$

Shifting phase on bit 5 when 2,3,4 are "1"



Continuation of Error Analysis in Decoder

Re-express equation to prepare for Hadamard transform:

$$\frac{\alpha}{\sqrt{8}}\left(-|0_1 0_2 1_3 0_4 1_5\rangle+|0_1 1_2 1_3 1_4 1_5\rangle+|1_1 0_2 1_3 1_4 1_5\rangle-|1_1 1_2 1_3 0_4 1_5\rangle\right.$$
$$\left.-|0_1 0_2 1_3 1_4 1_5\rangle+|0_1 1_2 1_3 0_4 1_5\rangle+|1_1 0_2 1_3 0_4 1_5\rangle-|1_1 1_2 1_3 1_4 1_5\rangle\right)$$
$$+\frac{\beta}{\sqrt{8}}\left(+|1_1 1_2 0_3 1_4 1_5\rangle-|1_1 0_2 0_3 0_4 1_5\rangle-|0_1 1_2 0_3 0_4 1_5\rangle+|0_1 0_2 0_3 1_4 1_5\rangle\right.$$
$$\left.+|1_1 1_2 0_3 0_4 1_5\rangle-|1_1 0_2 0_3 1_4 1_5\rangle-|0_1 1_2 0_3 1_4 1_5\rangle+|0_1 0_2 0_3 0_4 1_5\rangle\right)$$

$$=\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{\alpha}{\sqrt{4}}\left(-|0101\rangle+|1111\rangle-|0111\rangle+|1101\rangle\right)+\frac{\beta}{\sqrt{4}}\left(-|1001\rangle+|0011\rangle-|1011\rangle+|0001\rangle\right)\right)$$

$$=\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{\alpha}{\sqrt{2}}\left(-|101\rangle-|111\rangle\right)+\frac{\beta}{\sqrt{2}}\left(|011\rangle+|001\rangle\right)\right)$$

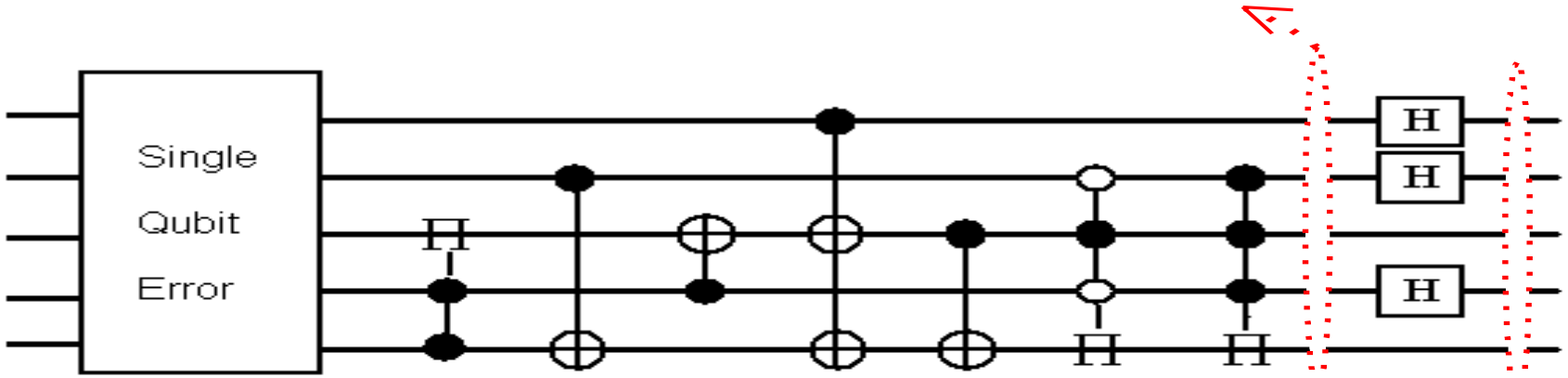
$$=\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(-\alpha|1\rangle+\beta|0\rangle\right)\left(\frac{|01\rangle+|11\rangle}{\sqrt{2}}\right)$$

$$=\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(-\alpha|1\rangle+\beta|0\rangle\right)\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(|1\rangle\right)$$

➡ Input to Hadamard

Continuation of Error Analysis in Decoder

$$\left(\frac{|0_1\rangle - |1_1\rangle}{\sqrt{2}}\right)\left(\frac{|0_2\rangle - |1_2\rangle}{\sqrt{2}}\right)\left(-\alpha|1_3\rangle + \beta|0_3\rangle\right)\left(\frac{|0_4\rangle + |1_4\rangle}{\sqrt{2}}\right)\left(|1_5\rangle\right)$$



$$|1_1\rangle|1_2\rangle\left(-\alpha|1_3\rangle + \beta|0_3\rangle\right)|0_4\rangle|1_5\rangle$$

- Qubits 1,2,4 and 5 are the syndrome bits which indicate the exact error that occurred and the current state of qubit 3:
- So apply a phase shift and a bit flip on qubit 3 to obtain the protected qubit $\alpha|0\rangle + \beta|1\rangle$

Syndromes Table after Decoding

Syndrome qubits: q1, q2, q4, q5 Output qubit: q3

Syndrome states	Error on					Output qubit
	q1	q2	q3	q4	q5	
0000	--					$\alpha\|0\rangle + \beta\|1\rangle$
0001		X				$\alpha\|0\rangle - \beta\|1\rangle$
0010				Z		$-\alpha\|0\rangle - \beta\|1\rangle$
0011					X	$-\alpha\|0\rangle - \beta\|1\rangle$
0100		Z				$-\alpha\|0\rangle - \beta\|1\rangle$
0101		Y				$\alpha\|0\rangle - \beta\|1\rangle$
0110	X					$-\alpha\|1\rangle - \beta\|0\rangle$
0111			X			$-\alpha\|1\rangle - \beta\|0\rangle$

Syndrome states	Error on					Output qubit
	q1	q2	q3	q4	q5	
1000	Z					$-\alpha\|0\rangle - \beta\|1\rangle$
1001				Y		$-\alpha\|1\rangle - \beta\|0\rangle$
1010			Z			$\alpha\|0\rangle - \beta\|1\rangle$
1011				X		$-\alpha\|1\rangle - \beta\|0\rangle$
1100					Z	$\alpha\|0\rangle - \beta\|1\rangle$
1101			Y			$-\alpha\|1\rangle + \beta\|0\rangle$
1110	Y					$-\alpha\|1\rangle - \beta\|0\rangle$
1111					Y	$-\alpha\|0\rangle + \beta\|1\rangle$

Realistic Issues

Recalling that these codes require $P_\tau \propto \tau \ll 1$ for time τ sufficiently short

☒ In realistic devices, τ cannot be infinite small

➡ threshold for tolerable error rate

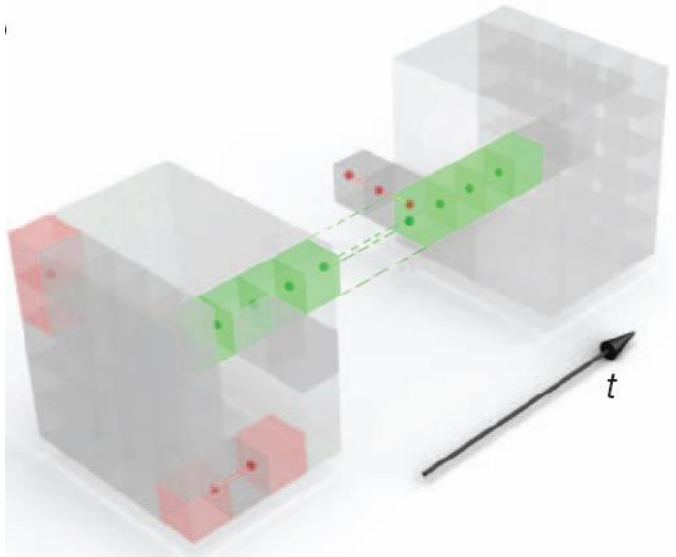
Highest threshold: 2.02×10^{-5} Extremely hard to achieve!

Spedalieri *et al.*, Quantum Inf. Comput. 9, 666 (2009)

☑ A possible solution: Topological error correction!

Raussendorf *et al.*, Ann. Phys. 321, 2242 (2003)

- Topological homology of 3D cluster state (encoding one logical qubit with 180 physical qubits)
- Relax the error threshold rate from 10^{-5} to 10^{-2}



Thanks for your attention!