Time Step in Rad Hydro

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Aalto-yliopisto

Another one of special issue

GAFD Special issue on “Physics and Algorithms of the Pencil Code”

The time step constraint in radiation hydrodynamics*

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Equations

$$\frac{\mathrm{D}\ln\rho}{\mathrm{D}t} = -\boldsymbol{\nabla}\cdot\boldsymbol{u},$$

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\boldsymbol{g} + \frac{\rho\kappa}{c}\boldsymbol{F}_{\mathrm{rad}} + \boldsymbol{\nabla}\cdot\boldsymbol{\tau},$$

$$\rho T\frac{\mathrm{D}s}{\mathrm{D}t} = \mathcal{H} - \boldsymbol{\nabla}\cdot\boldsymbol{F}_{\mathrm{rad}} + \boldsymbol{\tau} : \boldsymbol{\nabla}\boldsymbol{U},$$

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}I = -\kappa\rho\,(I - S), \quad \boldsymbol{F}_{\mathrm{rad}} = \int_{4\pi}\hat{\boldsymbol{n}}I\,\mathrm{d}\Omega, \quad \boldsymbol{\nabla}\cdot\boldsymbol{F}_{\mathrm{rad}} = \int_{4\pi}(I - S)\,\mathrm{d}\Omega,$$

A RADIATION TRANSFER SOLVER FOR ATHENA USING SHORT CHARACTERISTICS

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Since $\nu \propto \chi$, this constraint is most stringent where $\chi_i \Delta x_i \sim 1$ in which case $\delta t_{\rm rd} = \min(1/\nu_i)$. Assuming $\delta t_C \simeq \min(\Delta x_i/a_i)$, this implies that

$$\frac{\delta t_{\rm rd}}{\delta t_C} \propto \min({\rm Bo}). \qquad (43)$$

Hence, whenever the Bo number in any grid zone of the domain is less than unity, the maximum allowed time step

choices of ϵ. The agreement between the numeric and analytic solutions is quite good overall, but tends to be poorest at low optical depths. For fixed resolution, the discrepancies with the analytic solution tend to increase as ϵ decreases and the domain deviates more strongly from LTE. The accuracy

$dt_{rad}=dx/c_{rad}$

Simulations of stellar convection with CO5BOLD

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The numerical time step is not only limited by the CFL condition. In addition, Δt must be smaller than the characteristic radiative time scale $\tau_{\rm rad}$ that rules the decay of local temperature perturbations at the smallest possible spatial scale (wavenumber $k_0 = 10\pi/H_p$). To a good approximation, $\tau_{\rm rad}$ can be calculated as:

$$\tau_{\rm rad}(r) = \frac{c_v}{16\sigma\kappa T^3}\left(1 + 3\frac{\rho^2\kappa^2}{k^2}\right) = \frac{1}{\chi}\left(\frac{1}{3\rho^2\kappa^2} + k^{-2}\right), \tag{10}$$

which is valid in both optically thick and thin regions [86,87]. As illustrated in Fig. 3 (right), $\tau_{\rm rad}(k_0)$ reaches a sharp local minimum of $\approx$0.2 s close to the optical surface. The time step of the numerical simulation is thus set by the radiative time scale, $\Delta t < 0.2$ s, and the total number of required time steps is $N_t = t_{\rm sim}/\Delta t \approx 10^5$. Assuming for reference a processor that can update $N_c = 10^6$ grid cells per CPU second, the total CPU time required for this standard simulation would be

Comparison

$$\delta t = \min\left(C_{\rm rad}^{\rm thick}\frac{\delta z^2}{\chi D} + C_{\rm rad}^{\rm thin}\frac{\ell}{c_\gamma},\, C_{\rm CFL}\frac{\delta z}{c_{\rm s}},\, C_{\rm visc}\frac{\delta z^2}{\nu D}\right)$$

$$\delta t \neq \min\left(C_{\rm rad}^{\rm thick}\frac{\delta z^2}{\chi D},\, C_{\rm rad}^{\rm thin}\frac{\ell}{c_\gamma},\, C_{\rm CFL}\frac{\delta z}{c_{\rm s}},\, C_{\rm visc}\frac{\delta z^2}{\nu D}\right)$$

Cooling time

Linearising equation (3) about a hydrostatic homogeneous equilibrium solution with $\boldsymbol{u} = \boldsymbol{0}$, $T = \text{const}$, and $\rho = \text{const}$, and assuming the solution to be proportional to $e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\lambda t}$, where $\boldsymbol{k}$ is the wavevector, we find for the cooling or decay rate λ the expression (Unno & Spiegel 1966)

$$\lambda = \frac{c_\gamma}{\ell}\,\frac{k^2\ell^2/3}{1+k^2\ell^2/3} = \frac{c_\gamma k^2\ell/3}{1+k^2\ell^2/3} = \frac{\chi k^2}{1+k^2\ell^2/3}, \tag{4}$$

where $k = |\boldsymbol{k}|$ is the wavenumber,

$$c_\gamma = 16\sigma_{\mathrm{SB}}T^3/\rho c_p \tag{5}$$

is the characteristic velocity of photon diffusion (Barekat & Brandenburg 2014), and $\chi = c_\gamma\ell/3$ is the radiative diffusivity. The quantity c_γ is related to the radiative relaxation time ℓ/c_γ (equivalent to q^{-1} of Unno & Spiegel 1966). It is smaller than the speed of light by roughly the ratio of radiative to thermal energies. The expression in equation (4) has been obtained under the Eddington approximation and deviates only slightly from the exact expression obtained by Spiegel (1957), which can be written as

$$\lambda_{\mathrm{exact}} = \frac{c_\gamma}{\ell}\left(1-\frac{1}{k\ell}\cot^{-1}\frac{1}{k\ell}\right) = \frac{c_\gamma}{\ell}\left(1-\frac{\tan^{-1}k\ell}{k\ell}\right) \lesssim \lambda. \tag{6}$$

The largest value of the ratio $\lambda/\lambda_{\mathrm{exact}}$ is 1.29 at $k\ell = 2.53$.

Comparison

$$\delta t = \min\left(C_{\rm rad}^{\rm thick}\frac{\delta z^2}{\chi D} + C_{\rm rad}^{\rm thin}\frac{\ell}{c_\gamma},\ C_{\rm CFL}\frac{\delta z}{c_{\rm s}},\ C_{\rm visc}\frac{\delta z^2}{\nu D}\right)$$

$$\delta t \neq \min\left(C_{\rm rad}^{\rm thick}\frac{\delta z^2}{\chi D},\ C_{\rm rad}^{\rm thin}\frac{\ell}{c_\gamma},\ C_{\rm CFL}\frac{\delta z}{c_{\rm s}},\ C_{\rm visc}\frac{\delta z^2}{\nu D}\right)$$

In direct contrast with the above calculation, we quote the radiative time step constraint used by Davis et al. (2012),

$$\delta t_{\rm rad}^{\rm Athena} \propto \min({\rm Bo})\ \min(\delta z/c_{\rm s})\ ; \qquad (10)$$

see their equations (29) and (43), where ${\rm Bo} = 16\, c_{\rm s}/c_\gamma$ is the local Boltzmann number and $c_{\rm s}$

Blow-up in opt thin part

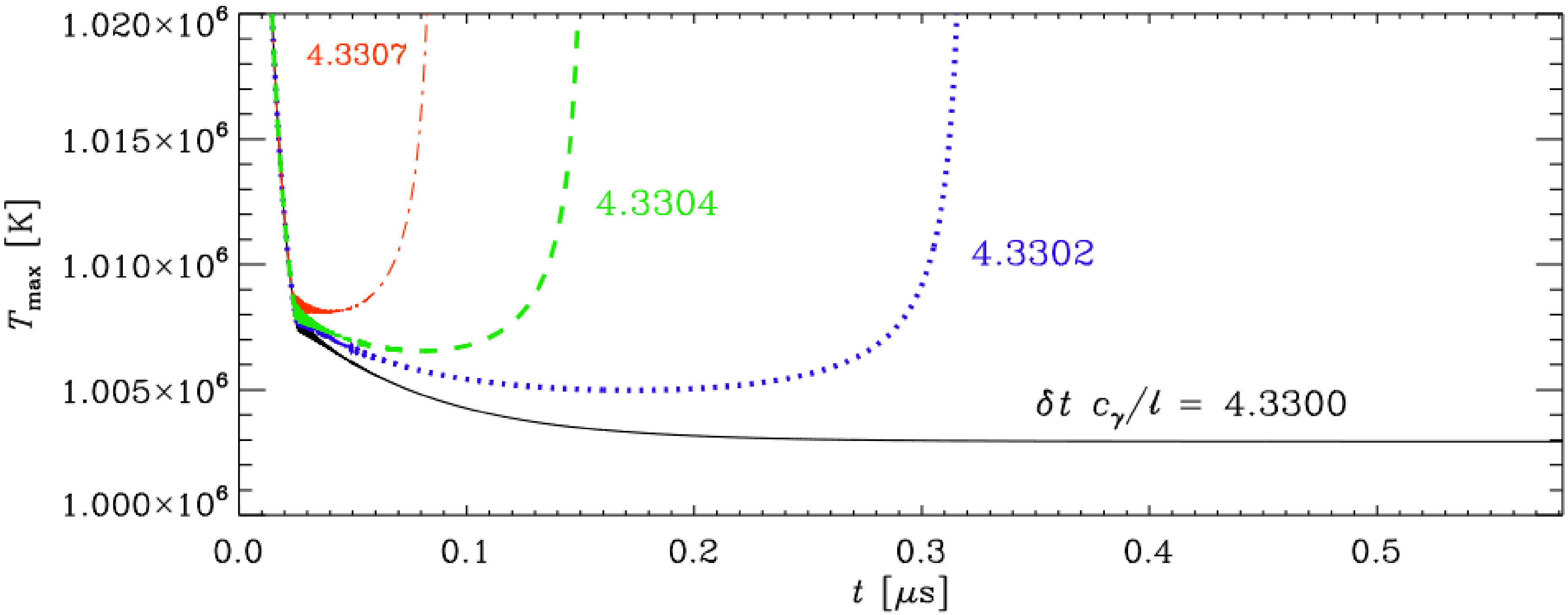


Figure 1. $T_{\rm max}(t)$ for different values of $\delta t\, c_\gamma/\ell$, for the unstratified model of section 3.3.

Time step vs dz

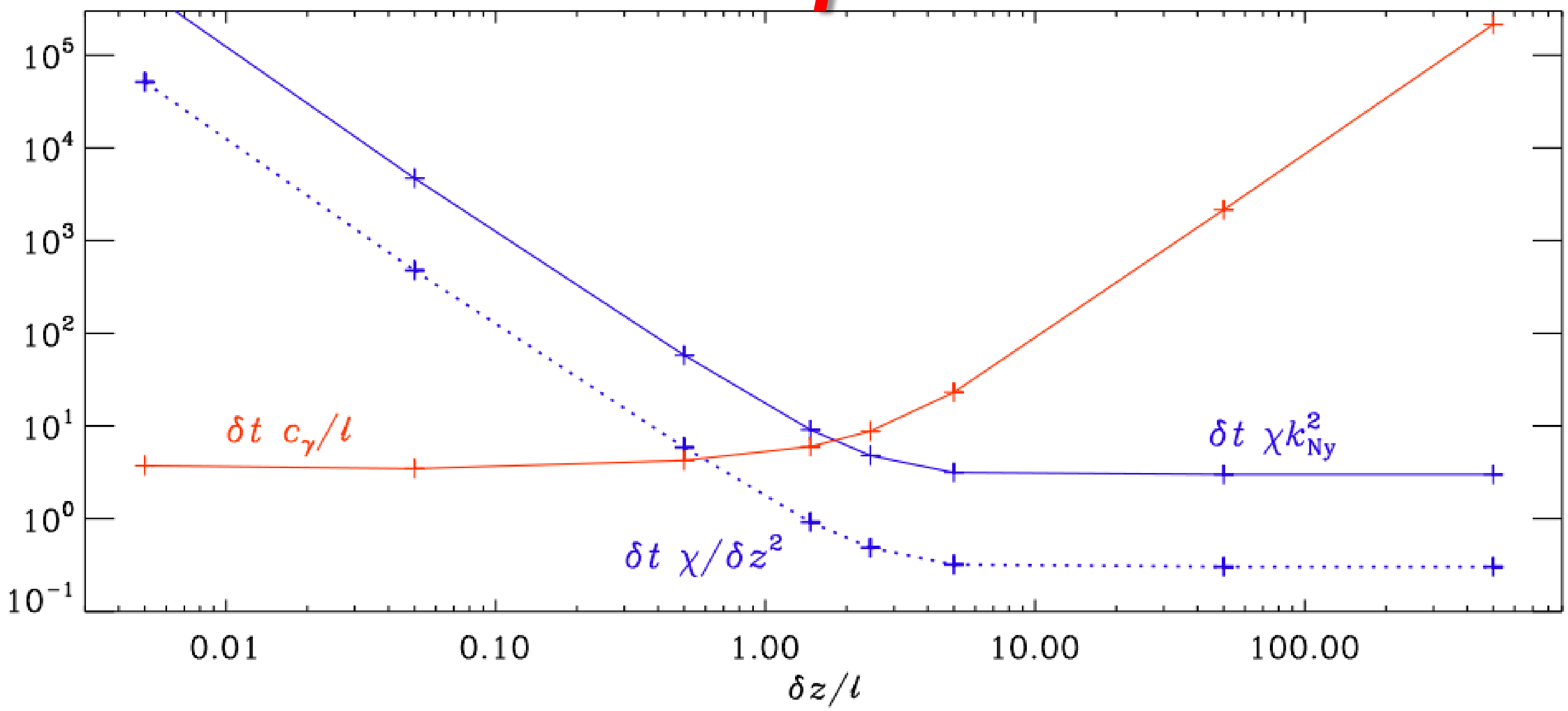


Figure 2. The maximum permissible time step δt versus $\delta z/\ell$, normalised by ℓ/c_γ (red) and by $(\chi k_{\rm Ny}^2)^{-1}$ (solid blue line), as well as $\delta z^2/\chi$ (dotted blue line), for the unstratified one-dimensional model of section 3.3. Note that $\delta t\,\chi k_{\rm Ny}^2 = \delta t\,\chi\pi^2/\delta z^2$.

Table 1. Values of $\delta t\,\chi/\delta z^2$ for the shortest permissible time step for given values of the number of dimensions D and the number of rays $n_{\rm ray}$ in the optically thick regime.

D	1	2	3	3	3
$n_{\rm ray}$	2	4	6	14	22
$\delta t\,\chi/\delta z^2$	0.375 ± 0.001	0.188 ± 0.001	0.127 ± 0.005	0.218 ± 0.005	0.291 ± 0.005

Constraint in action

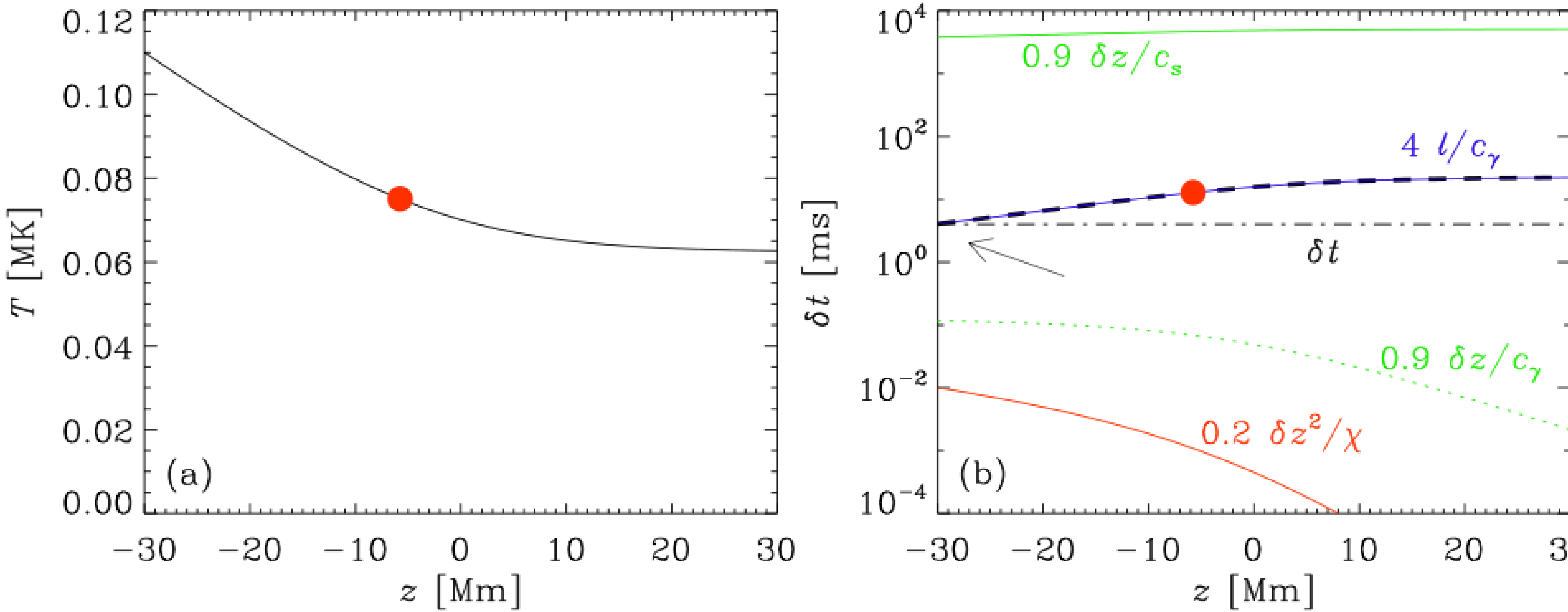
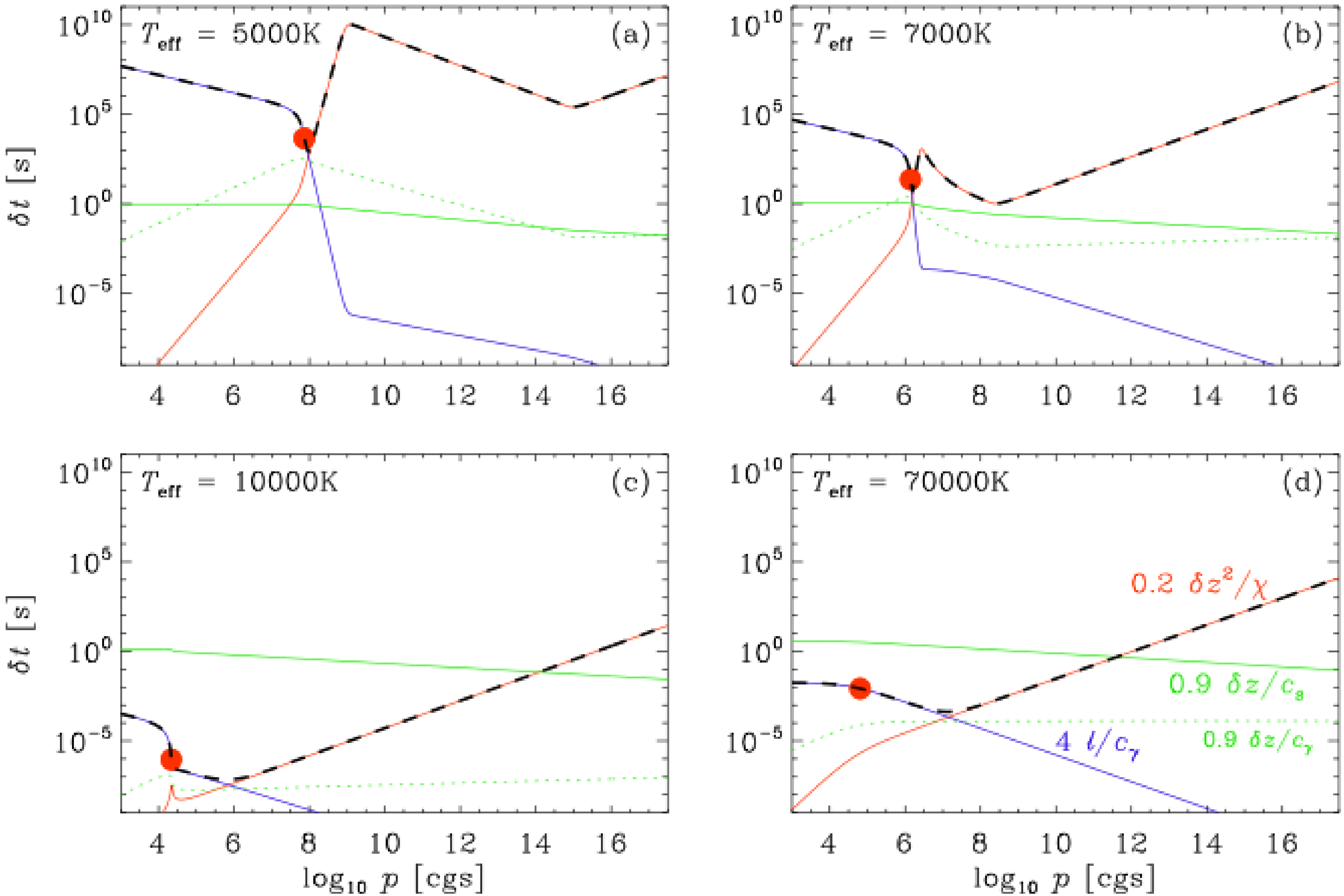


Figure 3. (a) Temperature stratification from the Pencil Code simulation of a hot star with $T_{\rm eff} = 69000K$ (see section 4.1). (b) z dependence of various time step constraints: $\delta t_{\rm rad}^{\rm thick}$ (red solid line), $\delta t_{\rm rad}^{\rm thin}$ (blue solid line), $\delta t_{\rm rad}$ (black dashed line), $\delta t_{\rm s}$ (green solid line), δt_γ (green dotted line), and the empirically determined maximum permissible time step δt (black dot-dashed line). The red dot denotes the photosphere. The arrow points to the location where the minimum of $4\ell/c_\gamma$ coincides with δt and is therefore constraining the time step. All time steps are in milliseconds.

Across HR diagram



Constraint in discs: cold

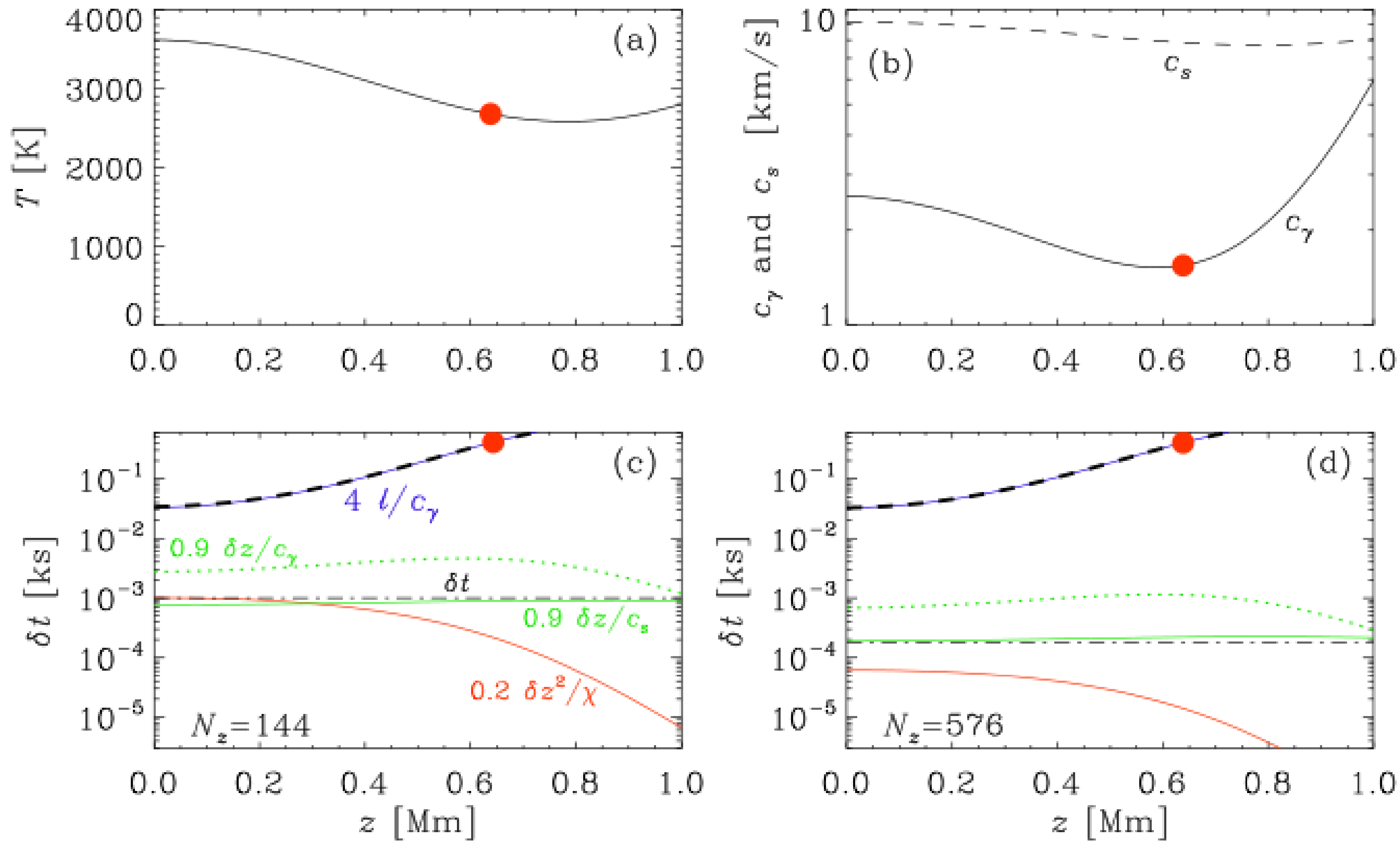


Figure 7. Vertical profiles of T (a), and c_γ and c_s (b). $\delta t_{\rm rad}^{\rm thin}$ (blue lines), $\delta t_{\rm rad}^{\rm thick}$ (red lines), their sum $\delta t_{\rm rad}$ (thick dashed lines), δt_s (green solid lines), and δt (black dot-dashed lines) for the cold disc model discussed in section 4.3, with N_z = 144 (c) and N_z = 576 (d). All time steps are in kiloseconds.

Constraint in discs: hot

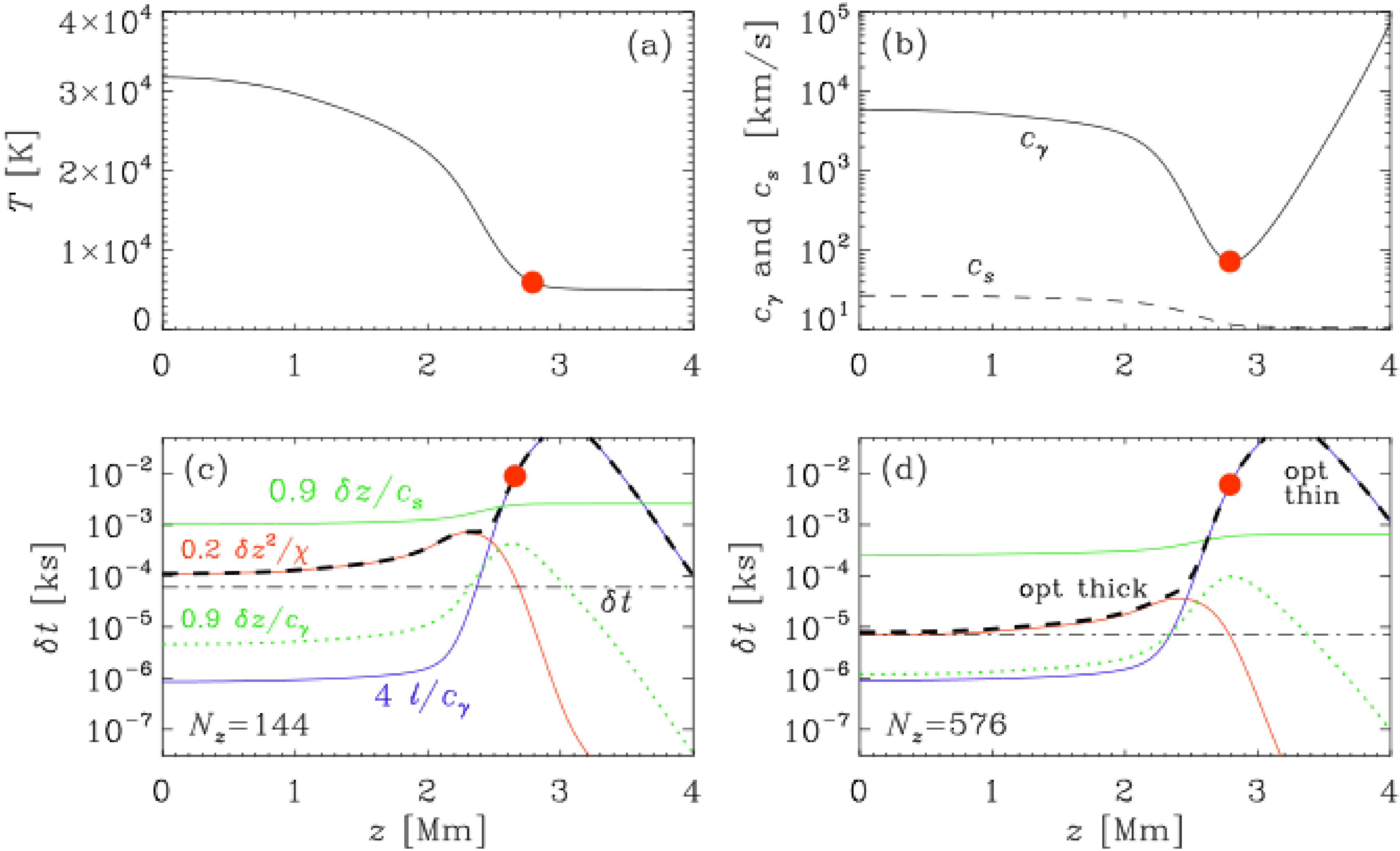


Figure 8. Same as figure 7, but for the hot disc model discussed in section 4.3. All time steps are in kiloseconds.

Deardorff here too

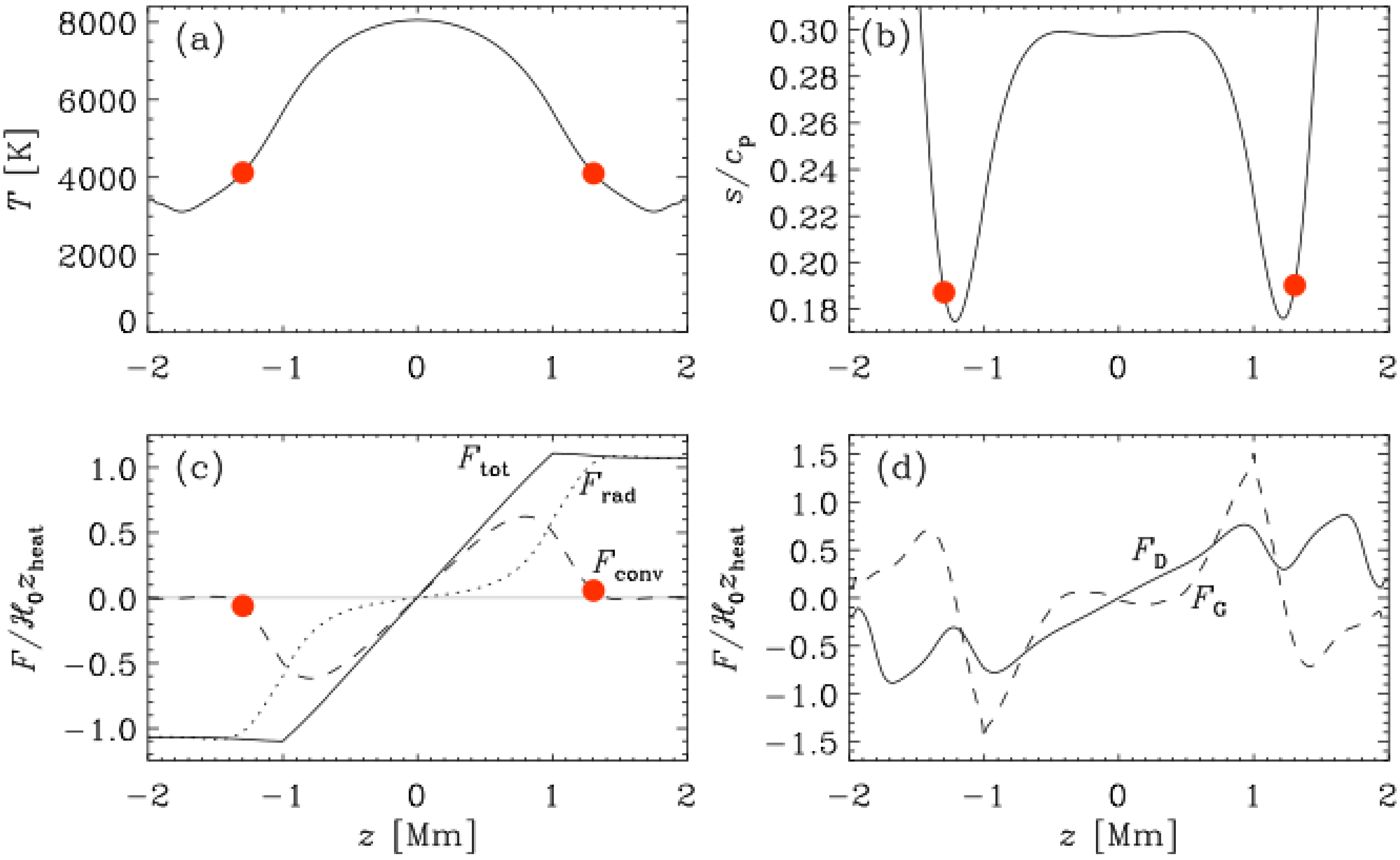


Figure 9. Temperature (a) and specific entropy (b) along with the various fluxes normalised by $\mathcal{H}_0 z_{\rm heat}$ (c) and (d), for a model with $\Sigma = 7.2 \times 10^{-7}\,{\rm g\,cm^{-3}\,Mm}$ and $\mathcal{H}_0 = 10^{-5}\,{\rm g\,cm^{-3}\,km^3\,s^{-3}\,Mm^{-1}}$.

Time step in turb discs

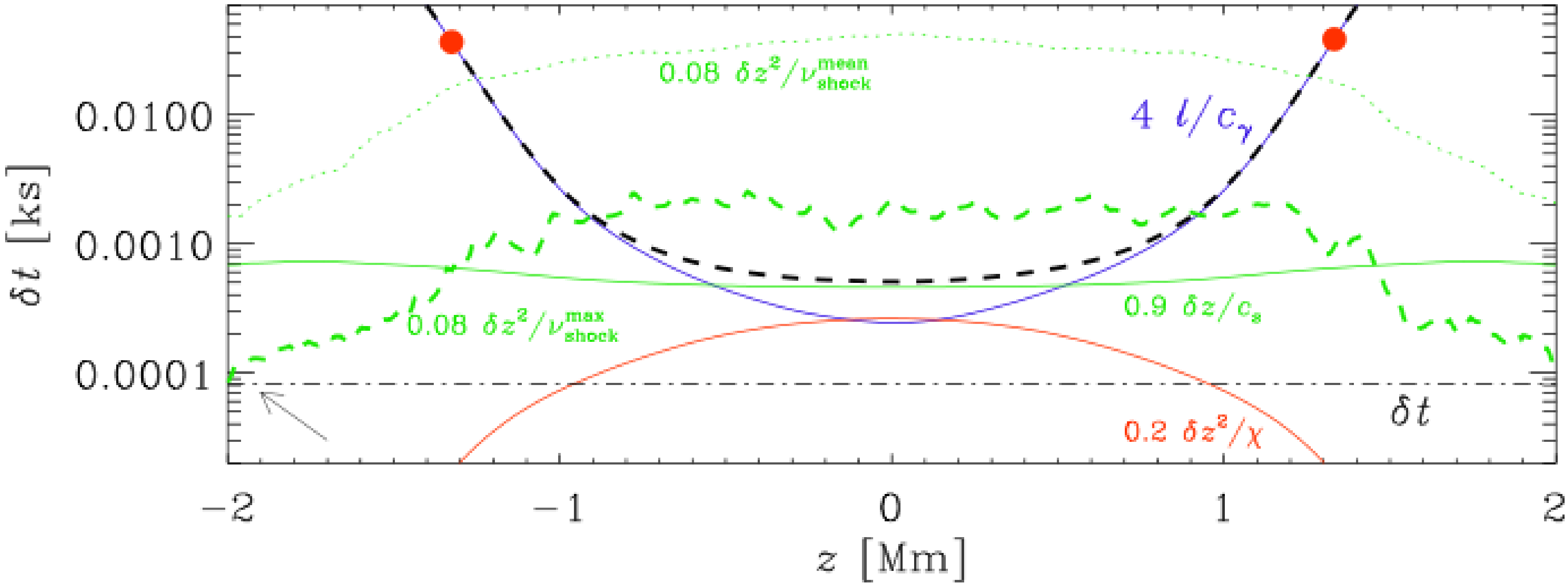


Figure 10. z dependence of the various time steps for the three-dimensional convective accretion disc simulation with $\Sigma = 7 \times 10^{-6}\,\mathrm{g\,cm^{-3}\,Mm}$ and $\mathcal{H}_0 = 10^{-5}\,\mathrm{g\,cm^{-3}\,km^3\,s^{-3}\,Mm^{-1}}$. $\delta t_{\rm rad}^{\rm thin}$ (blue solid line), $\delta t_{\rm rad}^{\rm thick}$ (red solid line), their sum $\delta t_{\rm rad}$ (black dashed line), $\delta t_{\rm s}$ (green solid line), the time step due to the maximum shock viscosity ($0.08\delta z^2/\nu_{\rm shock}^{\rm max}$; green dashed line), the time step due to mean shock viscosity ($0.08\delta z^2/\nu_{\rm shock}^{\rm mean}$; green dotted line), and the empirical time step δt (black dot-dashed line). All time steps are in kiloseconds. The arrow indicates the location from where the limiting time step constraint originates.

Time step in turb discs

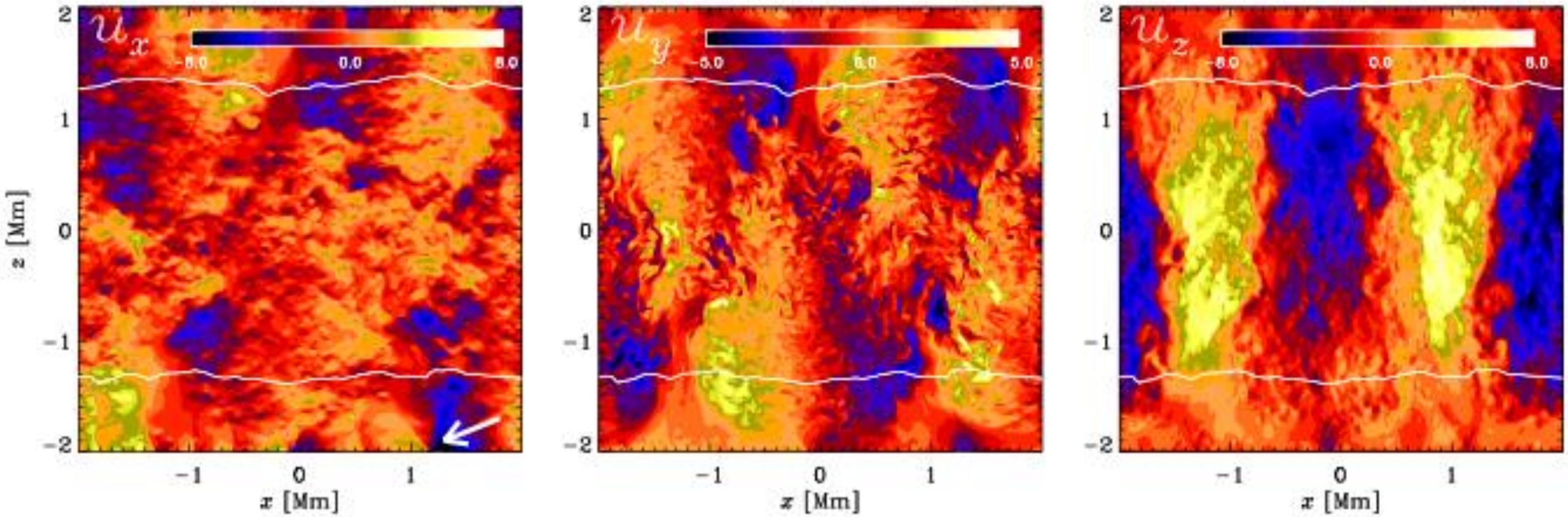


Figure 11. xz cross sections of u_x, u_y, and u_z (in $\mathrm{km\,s^{-1}}$) for the three-dimensional convective accretion disc simulation with $\Sigma = 7 \times 10^{-6}\,\mathrm{g\,cm^{-3}\,Mm}$ and $\mathcal{H}_0 = 10^{-5}\,\mathrm{g\,cm^{-3}\,km^3\,s^{-3}\,Mm^{-1}}$. The white lines show the $\tau = 1$ surfaces and the arrow points to the strongest shock near $z = -2\,\mathrm{Mm}$.

Conclusions

- dtrad(thin) ~ 4
- dtrad(thick) ~ 0.2
- Different from standard Courant step
- Deardorff and other pleasures