

Slope-limited diffusion

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1 Background

We want to stabilize convection simulations and being able to run high stratifications and low diffusivities. Other interest include dynamo-corona models or CME ejections. The derivation followings roughly the one of Rempel et al. (2009) and Rempel, (2014). However it is based on the Piyali's implementation for density in `special/solar_corona.f90`.

2 Derivation

First we show the general derivation for variable \mathbf{u} . The idea is to define a diffusion, which is as large as it need to be to resolve the slope, but as small as possible. In the momentum/velocity equation have for the diffusion

$$\frac{D\mathbf{u}}{Dt} = \dots - \nabla^{2\text{nd}} \cdot \mathcal{F}_{\text{sld}}, \quad (1)$$

where we use the 2nd order divergence and do the derivative for each component i separately.

$$[\nabla^{2\text{nd}} \cdot \mathcal{F}_{\text{sld}}]_i = \sum_j \partial_j^{2\text{nd}} \mathcal{F}_{ij} = \frac{f_{xk+1/2} - f_{xk-1/2}}{\Delta x} + \frac{f_{yk+1/2} - f_{yk-1/2}}{\Delta y} + \frac{f_{zk+1/2} - f_{zk-1/2}}{\Delta z}, \quad (2)$$

where the diffusive fluxes are given by

$$f_{k+1/2} = -\frac{1}{2} c_{k+1/2}^{\text{sld}} Q_{k+1/2} (u_{k+1/2}^R - u_{k+1/2}^L) \quad (3)$$

with c^{sld} being the characteristic speed. $u_{k+1/2}^R$ and $u_{k+1/2}^L$ are the right and left side of the surface of cell $k + 1/2$. The u^R and u^L are defined via

$$u_{k-1/2}^L = u_{k-1} + \Delta u_{k-1} \quad (4)$$

$$u_{k-1/2}^R = u_k - \Delta u_k \quad (5)$$

$$u_{k+1/2}^L = u_k + \Delta u_k \quad (6)$$

$$u_{k+1/2}^R = u_{k+1} - \Delta u_{k+1}, \quad (7)$$

with

$$\Delta u_k = \text{minmod}(u_k - u_{k-1}, u_{k+1} - u_k), \quad (8)$$

where the minmod is defined as

$$\text{minmod}(a, b) = \text{sgn}(1/2, a) \max(0.0, \min(|a|, \text{sgn}(1.0, a)b)) \quad (9)$$

this means:

$$a > 0, b > 0 : \text{minmod}(a, b) = +\frac{1}{2} \max(0.0, \min(|a|, |b|)) \quad (10)$$

$$a < 0, b > 0 : \text{minmod}(a, b) = -\frac{1}{2} \max(0.0, \min(|a|, |b|)) \quad (11)$$

$$a > 0, b < 0 : \text{minmod}(a, b) = +\frac{1}{2} \max(0.0, \min(|a|, |b|)) \quad (12)$$

$$a < 0, b < 0 : \text{minmod}(a, b) = -\frac{1}{2} \max(0.0, \min(|a|, |b|)) \quad (13)$$

The flux is controlled by

$$Q_{k+1/2} = (\min(1.0, h_{\text{sld}} \mathcal{R}_{k+1/2}))^{n_{\text{sld}}}, \quad (14)$$

h_{sld} controles, what is the maximum slope still allowed. $h_{\text{sld}} = \infty$ represent the Lax-Friedrichs-scheme. The lower h_{sld} , the less diffusive the simulation is, usually we use $h_{\text{sld}} = 1$ to 2. n_{sld} can be also used to further reduce the slope. All **red** quantities are input parameter, which can be set in the code.

$$\mathcal{R}_{k+1/2} = \frac{|u_{k+1/2}^R - u_{k+1/2}^L|}{|u_{k+1}^R - u_k^L|} \quad (15)$$

For region, where

$$\frac{|u_{k+1/2}^R - u_{k+1/2}^L|}{|u_{k+1}^R - u_k^L|} \geq \frac{1}{h_{\text{sld}}}, \quad (16)$$

the diffusion maximal with a diffusivity of $\frac{1}{2}c_{k+1/2}^{\text{sld}}\Delta x$. For region, where

$$\frac{|u_{k+1/2}^R - u_{k+1/2}^L|}{|u_{k+1}^R - u_k^L|} < \frac{1}{h_{\text{sld}}}, \quad (17)$$

the diffusion is reduced. This scheme is implemented for \mathbf{u} , \mathbf{A} , $\ln \rho$, ρ , s , T and $\ln T$.

2.1 Characteristic speed

The characteristic speed is related to the advection speeds in the system

$$c^{\text{sld}} = w_{\text{hydro}}^{\text{sld}} \sqrt{u_x^2 + u_y^2 + u_z^2} + w_{\text{energy}}^{\text{sld}} c_{\text{sound}} + w_{\text{magnetic}}^{\text{sld}} v_A, \quad (18)$$

where c_{sound} is the sound speed and v_A is the Alfven speed.

$$c_{k-1/2}^{\text{sld}} = \frac{c_{k-1}^{\text{sld}} + c_k^{\text{sld}}}{2} \quad (19)$$

Using this takes also care of the time step control as the advective speed are already taking care of.

2.2 Heating and other correction terms

2.2.1 Viscose heating

The viscose heat \mathcal{E}^ν is defined

$$\mathcal{E}_{ki}^\nu = \Sigma_{i,j} \frac{1}{2} f_{ki-1/2}^i(u^j) \frac{\rho_{ki} u_{ki}^j - \rho_{ki-1} u_{k-1}^j}{\Delta x_i} + \frac{1}{2} f_{ki+1/2}^i(u^j) \frac{\rho_{ki+1} u_{ki+1}^j - \rho_{ki} u_k^j}{\Delta x_i}, \quad (20)$$

where i is the component of \mathbf{u} and j the coordinate. One can also write as

$$\mathcal{E}^\nu / \rho = 1/\rho \Sigma_{i,j} \mathcal{F}_{ij} \partial_j^{2\text{nd}}(\rho u_i) \quad [\text{p}\% \text{visc_heat}] \quad (21)$$

$$\mathcal{E}_{ki}^\nu = \Sigma_{i,j} \frac{1}{2} \rho_{ki-1/2} f_{ki-1/2}^i(u^j) \frac{u_{ki}^j - u_{k-1}^j}{\Delta x_i} + \frac{1}{2} \rho_{ki+1/2} f_{ki+1/2}^i(u^j) \frac{u_{ki+1}^j - u_k^j}{\Delta x_i}, \quad [\text{Rempel et al.}] \quad (22)$$

This is then added as the other viscosity terms to the different energy equations.

2.2.2 Ohmic heating

The ohmic heating is defined analog to the usual way:

$$\mathcal{E}^\eta = \mathbf{J} \cdot (\nabla^{2\text{nd}} \cdot \mathcal{F}_{\text{sld}}(\mathbf{A})), \quad \text{normally :} [\mathbf{J} \cdot \mu_0 \eta \mathbf{J}] \quad (23)$$

2.2.3 Mass flux

If we use the slope-limited diffusion on the density, we have to compensate for the additional mass flux in the momentum and energy equation. This done in the same way as for the other mass diffusion scheme. `lmassdiff_fix=T` is always on with SLD.

2.3 4th order

In the Cartesian case one can also use the 4th order divergence for Equations (2) and (2) and the corresponding heating terms. Not implemented in spherical coordinates yet.

2.4 Alternative way for \mathbf{A} using SLD on \mathbf{B}

Instead of applying the slope-limited diffusion on \mathbf{A} , one can also apply it on \mathbf{B} . Not implemented in spherical coordinates yet. Instead of

$$\frac{D\mathbf{A}}{Dt} = \dots - \nabla^{2\text{nd}} \cdot \mathcal{F}_{\text{sld}}(\mathbf{A}), \quad (24)$$

we do

$$\frac{D\mathbf{A}}{Dt} = \dots + \hat{\mathbf{e}} \times \mathcal{F}_{\text{sld}}(\mathbf{B}), \quad (25)$$

where $\hat{\mathbf{e}}$ is the unit vector. One can also understand this by writing $\mathcal{F}_{ij}(\mathbf{A}) = \partial_j^{\text{sld}} A_i$ and $\mathcal{F}_{ij}(\mathbf{B}) = \partial_j^{\text{sld}} B_i$. The equations above read

$$\frac{DA_i}{Dt} = \dots - \partial_j \partial_j^{\text{sld}} A_i \quad (26)$$

$$\frac{DA_i}{Dt} = \dots + \epsilon_{ijk} \partial_k^{\text{sld}} B_j \quad (27)$$

The corresponding heating is:

$$\mathcal{E}^\eta = \mathbf{J} \cdot (\hat{\mathbf{e}} \times \mathcal{F}_{\text{sld}}(\mathbf{B})), \quad (28)$$

There might be a problem with the gauge for both ways.

2.5 Spherical coordinates

For scalar quantities Equation (2) reads:

$$\nabla^{2\text{nd}} \cdot \mathcal{F}_{\text{sld}} = \Sigma_j \partial_j^{2\text{nd}} \mathcal{F}_j \quad (29)$$

$$= \frac{r_{xk+1/2}^2 f_{xk+1/2} - r_{xk+1/2}^2 f_{xk-1/2}}{r_{xk}^2 \Delta r} \quad (30)$$

$$+ \frac{\sin(\theta_{yk+1/2}) f_{yk+1/2} - \sin(\theta_{yk-1/2}) f_{yk-1/2}}{r \sin(\theta_{yk}) \Delta \theta} \quad (31)$$

$$+ \frac{f_{zk+1/2} - f_{zk-1/2}}{r \sin(\theta) \Delta \phi}, \quad (32)$$

For vector quantities Equation (2) reads differently, for this we first look at the divergence of a tensor in spherical coordinates in general:

$$[\nabla \cdot \mathbf{T}]_r = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 T_{rr}) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta} (\sin(\theta) T_{\theta r}) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} (\sin(\theta) T_{\phi r}) - \frac{T_{\theta\theta} + T_{\phi\phi}}{r} \quad (33)$$

$$[\nabla \cdot \mathbf{T}]_\theta = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 T_{r\theta}) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta} (\sin(\theta) T_{\theta\theta}) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} (\sin(\theta) T_{\phi\theta}) + \frac{T_{\theta r}}{r} - \frac{\cot(\theta) T_{\phi\phi}}{r} \quad (34)$$

$$[\nabla \cdot \mathbf{T}]_\phi = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 T_{r\phi}) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta} (\sin(\theta) T_{\theta\phi}) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} (\sin(\theta) T_{\phi\phi}) + \frac{T_{\phi r}}{r} + \frac{\cot(\theta) T_{\phi\theta}}{r} \quad (35)$$

Now applying this to our diffusion tensor:

$$[\nabla^{2\text{nd}} \cdot \mathcal{F}_{\text{sld}}]_r = \frac{r_{xk+1/2}^2 f_{xk+1/2}^r - r_{xk+1/2}^2 f_{xk-1/2}^r}{r_{xk}^2 \Delta r} \quad (36)$$

$$+ \frac{\sin(\theta_{yk+1/2}) f_{yk+1/2}^r - \sin(\theta_{yk-1/2}) f_{yk-1/2}^r}{r \sin(\theta_{yk}) \Delta \theta} \quad (37)$$

$$+ \frac{f_{zk+1/2}^r - f_{zk-1/2}^r}{r \sin(\theta) \Delta \phi} \quad (38)$$

$$- \frac{f_{yk+1/2}^\theta + f_{yk-1/2}^\theta}{2r} - \frac{f_{zk+1/2}^\phi + f_{zk-1/2}^\phi}{2r} \quad (39)$$

$$[\nabla^{2\text{nd}} \cdot \mathcal{F}_{\text{sld}}]_{\theta} = \frac{r_{xk+1/2}^2 f_{xk+1/2}^{\theta} - r_{xk+1/2}^2 f_{xk-1/2}^{\theta}}{r_{xk}^2 \Delta r} \quad (40)$$

$$+ \frac{\sin(\theta_{yk+1/2}) f_{yk+1/2}^{\theta} - \sin(\theta_{yk-1/2}) f_{yk-1/2}^{\theta}}{r \sin(\theta_{yk}) \Delta \theta} \quad (41)$$

$$+ \frac{f_{zk+1/2}^{\theta} - f_{zk-1/2}^{\theta}}{r \sin(\theta) \Delta \phi} \quad (42)$$

$$+ \frac{f_{yk+1/2}^r + f_{yk-1/2}^r}{2r} - \cot(\theta_{yk}) \frac{f_{zk+1/2}^{\phi} + f_{zk-1/2}^{\phi}}{2r} \quad (43)$$

$$[\nabla^{2\text{nd}} \cdot \mathcal{F}_{\text{sld}}]_{\phi} = \frac{r_{xk+1/2}^2 f_{xk+1/2}^{\phi} - r_{xk+1/2}^2 f_{xk-1/2}^{\phi}}{r_{xk}^2 \Delta r} \quad (44)$$

$$+ \frac{\sin(\theta_{yk+1/2}) f_{yk+1/2}^{\phi} - \sin(\theta_{yk-1/2}) f_{yk-1/2}^{\phi}}{r \sin(\theta_{yk}) \Delta \theta} \quad (45)$$

$$+ \frac{f_{zk+1/2}^{\phi} - f_{zk-1/2}^{\phi}}{r \sin(\theta) \Delta \phi} \quad (46)$$

$$+ \frac{f_{zk+1/2}^r + f_{zk-1/2}^r}{2r} + \cot(\theta_{zk}) \frac{f_{yk+1/2}^{\theta} + f_{yk-1/2}^{\theta}}{2r} \quad (47)$$

Correspondingly also the cylindrical coordinates are implemented.

3 How to use it

We need to set one auxiliary communicative variable, because we need the staggered values for different variables. For this set in

cparam.local:

```
MAUX CONTRIBUTION 1
COMMUNICATED AUXILIARIES 1
```

Then for the SLD using for different variables set in

run.in:

```
&run_pars
bcx='cop' for aux: copy boundary values to ghost points for characteristic speed
bcy='cop' for aux
bcz='cop' for aux

&hydro_run_pars
w_sldchar_hyd=1.0 pre-factor for characteristic speed contribution

&density_run_pars
idiff='density-slope-limited' activate SLD on density
h_sld_dens=2.0
nlf_sld_dens=1.0
div_sld_dens='2nd'

&entropy_run_pars
iheatcond='entropy-slope-limited': activate SLD on entropy
```

```

would be iheatcond='temperature-slope-limited' for temperature_idealgas.f90
h_sld_ene=2.0
nlf_sld_ene=1.0
div_sld_ene='2nd'
w_sldchar_ene=0.1 pre-factor for characteristic speed contribution

&viscosity_run_pars
ivisc='nu-slope-limited' activate SLD on velocity
h_sld_visc=2.0
nlf_sld_visc=1.0
div_sld_visc='2nd'

&magnetic_run_pars
iresistivity='eta-slope-limited' activate SLD on magnetic field
h_sld_magn=2.0
nlf_sld_magn=1.0
div_sld_magn='2nd'
lsld_bb=F: for 'T' it used the alternative way for SLD on  $\mathbf{B}$ . Need 3 extra auxiliary communicative
variables.
w_sldchar_mag=1.0 pre-factor for characteristic speed contribution

```

4 Implementation

Done only at the last sub-time step `llast`. Go to `viscosity.f90`, `magnetic.f90`, `density.f90`, `entropy.f90` or `temperature_idealgas.f90` and then to `sub.f90`. For characteristic speed check out `hydro.f90`, `magnetic.f90`, `entropy.f90` or `temperature_idealgas.f90` or the other energy modules.

4.1 Still missing

Because the diagnostics are written out in the first sub-time step `lfirst`, we cannot use any outputs for the heating and diffusive fluxes and forces.

5 Timing

As an example, we use a Cartesian convection run with 256x256x256 grid points with SSD, around 200 density contrast.

ρ	s	u	A	Wall clock time/timestep/meshpoint [microsec]
				1.446E-02
X				1.724E-02
X	X			1.925E-02
X	X	X		2.232E-02
X	X	X	X	2.578E-02
X	X	X	X (smoothing of JJ)	4.458E-02

6 Examples

6.1 Cartesian convection with SSD

6.2 Spherical convection with zero rotation

7 Issues

7.1 SLD for the A

7.2 Spherical coordinates for vector quantities: A and u