# Topological theories and Moduli:

## A Heterotic Case Study

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Ref's: ArXiv: 2110.06549, 1904.01027, 1806.08367, 1710.06865, 1709.06974, 1704.08717, 1607.03473, 1509.08724, 1409.3347, 1402.1725

## Overview:

- Introduction: Stringy moduli in general
- Heterotic compactifications
    - Why heterotic?
- Compactifications to 4D:
    - Hull-Strominger System
    - Moduli: Complex manifold + Bundle
        - Atiyah extension
    - Moduli: Including Anomaly
        - Heterotic extension
- Higher order deformations
    - Topological theory?

- Removing Spurious degrees of freedom
  - Rinse and repeat...

- Compactifications to 3D (IF time):
  - Heterotic $G_2$-Structures
  - Infinitesimal moduli
  - Topological theory? Higher order def.'s?

- Conclusions / Outlook

## Stringy Moduli:

3 levels of understanding (stringy) moduli:

I) Infinitesimal massless spectrum:

- Geometry is described by BPS equations:

  - System of equations defining supersymmetric geometries:

$$\text{BPS} = 0$$

  - Infinitesimal deformation:

$$\delta(\text{BPS}) = 0 \rightsquigarrow \mathcal{D}\alpha = 0$$

- Identify differential $\mathcal{D}$, such that $\mathcal{D}^2 = 0$.

- Massless fields (infinitesimal moduli):

$$H^1_{\mathcal{D}}(Q) = \frac{\{\text{deformations } \alpha \text{ preserving } \mathcal{D}\alpha = 0\}}{\{\text{Infinitesimal symmetry transformations; } \alpha = \mathcal{D}\gamma\}}$$

- Moduli $\alpha$ are usually "1-forms" valued in some bundle/sheaf/... $Q$, naturally associated to the moduli problem.

- Usually elliptic $\Rightarrow$ Finite dimensional spectrum.

Exs: deformations of integrable complex structure:

$\mu$ : Beltrami differential $\rightsquigarrow$ $[\mu] \in H^{(0,1)}_{\bar\partial}(T^{(1,0)}X) \underset{X \text{ Calabi-Yau}}{\cong} H^{(2,1)}_{\bar\partial}(X)$.

II) Understand Geometry of moduli space $\mathcal{M}$:

- Structures on $\mathcal{M}$: Complex ?, Kähler ?, ...
- Higher order def.'s, obstructions ?, Yukawa couplings, superpotential, smooth directions, ...
- Finite deformations: Solve Maurer-Cartan equation in associated $L_\infty$-algebra:

$$\partial \alpha + \frac{1}{2}[\alpha,\alpha] \; (+\ldots) = 0$$

Exs: Finite def's of complex structure $\mu \in \Omega^{(0,1)}(T^{(1,0)}X)$ solve

MC-eq.: $\quad \bar{\partial}\mu + \frac{1}{2}[\mu,\mu] = 0$

$\rightsquigarrow$ Diff. graded Lie Algebra.

Tian - Todorov : $X$ Calabi - Yau (or $\partial\bar{\partial}$-lemma)

$\Rightarrow$ infinitesimal complex structure moduli are unobstructed.

III ) Understand Quantum moduli space :

- Non-perturbative effects (world-sheet instantons, dualities, ...)

associated to structure ...

- Compute Invariants : knot invariants (CS-theory), Donaldson - Thomas, Gromov - Witten, ....

- Find topological theory governing geometric structures

Examples:

| Structure | "Target space" | "World-sheet" |
|---|---|---|
| Complex structure | Kodaira-Spencer theory | Wittens B-model |
| Kähler structure | Kähler-Gravity | Wittens A-model |
| Various Gauge theories | Chern-Simons<br>Donaldson-Thomas (Hol. CS-theory) | Versions of topological open string |

- Theories are connected through dualities (E.g. Mirror Symmetry, open-closed duality, ...).

# Why Heterotic?

- Naturally equipped with gauge-bundle
  - Great for model building.
  - Standard-Embedding (Candelas et al '85), and so on (LOT'S of work...)
  - BUT: Hard to stabilize moduli

    (Though moduli problem for generic solution is far from fully understood)

- Mathematically interesting:
  - Couplings between bundle and geometry.

- Playground for models of Generalised Geometry, DFT, ...

- Abundance of "Chern-Simons type" couplings:

⇝ Connections to Invariants and:

  - Knot invariants
  - Donaldson-Thomas invariants
  - Donaldson-Segal invariants
  - Kodaira-Spencer theory,
  - Combinations of all of the above

# Heterotic compactifications to 4D:

Often called Hull-Strominger solutions

$$M_{10} = M_4 \times X.$$

$\uparrow$ Minkowski

$X$: 6d manifold with $SU(3)$-structure $(\Psi, \omega)$:

$$\frac{i}{\|\Psi\|^2} \Psi \wedge \bar{\Psi} = \frac{1}{6} \omega \wedge \omega \wedge \omega \quad \& \quad \omega \wedge \Psi = 0 \iff X \text{ admits a spinor}$$

$\Psi$ defines (almost) complex structure $J$ w.r.t. which

$$\Psi \in \Omega^{(3,0)}_{\mathbb{C}}(X), \qquad \omega \in \Omega^{(1,1)}_{\mathbb{R}}(X).$$

Differential constraints from SUSY:

$$d(e^{-2\phi}\Psi) = 0 \rightsquigarrow X \text{ is complex}$$

($J$ is integrable)

dilaton

$$d(e^{-2\phi}\omega\wedge\omega) = 0$$

Conformally balanced constraint.

$$H = i(\partial - \bar{\partial})\omega = d_c\omega$$

Dolbeault operator: $\bar{\partial} : \Omega^{(p,q)}(X) \to \Omega^{(p,q+1)}(X)$.

$H$ is the heterotic NS-flux defined as

$$H = dB + \frac{\alpha'}{4}\left(\omega_{CS}(A) - \omega_{CS}(\tilde{\nabla})\right)$$

$A \in \Omega^1(\mathrm{End}(V))$ is the connection on a gauge-bundle $V$.

$\tilde{\nabla}$: Connection on $\mathrm{End}(TX)$. (Will come back to...)

$B \in \Omega^2(X)$. Heterotic Kalb-Ramond field.

Note: $B$ transforms non-trivially under gauge transformations ⇝ Green-Schwarz mechanism

⇝ $H$ is a gauge invariant globally defined 3-form.

Geometry and gauge sector are coupled through Bianchi Identity:

$$dH = -2i\partial\bar\partial\omega = \frac{\alpha'}{4}\left(\mathrm{tr}\,F\wedge F - \mathrm{tr}\,\tilde{R}\wedge\tilde{R}\right)$$

Note: Seems complicated, but plays a natural role in moduli problem.

$F = dA + A\wedge A$ curvature of $A$

$\tilde{R}$: Curvature of $\tilde{\nabla}$.

$F$ satisfies the hermitian Yang-Mills Constraints (Instanton):

$$F\wedge\Psi = 0, \qquad \omega\wedge\omega\wedge F = 0.$$

$\underbrace{F^{(0,2)}}_{=0} \rightsquigarrow$ Holomorphic connection.

Donaldson-Uhlenbeck-Yau / Li-Yau: $\Leftrightarrow$ (poly-) stable bundles.

## What is $\tilde{\nabla}$? It is field-dependant!

In the usual field choice:

- $\tilde{\nabla}$ is the Hull-connection $\nabla^-$:

$$\nabla^{\pm} = \nabla^{LC} \pm \frac{1}{2} H$$

$$\left( \begin{array}{ll} \nabla^+ : \text{Bismut/BPS-connection} & \nabla^+ \epsilon = 0 \\ & \epsilon : \text{internal spinor} \end{array} \right)$$

Note: The Hull-connection is an instanton modulo higher order $\alpha'$-corrections:

$$R(\nabla^-) \wedge \Omega = \omega \wedge \omega \wedge R(\nabla^-) = 0 + \mathcal{O}(\alpha').$$

Mathematicians: Want things to be exact.

Benefits: - Can trust mathematical theorems.
- Things are "cleaner".

Drawback: Physics/SUGRA is perturbative in $\alpha'$.

Common "trick":

$\rightsquigarrow$ Work with slightly modified but exact system at $\mathcal{O}(\alpha')$.

$\Rightarrow$ Keep $\tilde{\nabla}$ arbitrary, but require it to satisfy it's own instanton condition:

$$\Omega \wedge \tilde{R} = \omega \wedge \omega \wedge \tilde{R} = 0$$ on the nose!

Drawback: Have to deal with extra spurious modes $K \in \Omega^1(\mathrm{End}(TX))$ (def.'s of $\tilde{\nabla}$).

- These can be interpreted as field-redef.'s.

BUT: - Coupled with the other "real" moduli in quite nontrivial ways.

⇝ Hard to disentangle!

- At some point we want to quantise (See Point III) above).

⇝ Spurious d.o.f. are not real...

How do we quantise?

Benefit: Much cleaner "Classical" moduli story.

# Recap: "Classical" Heterotic Moduli

## Topological Toy Model:

Chern-Simons: $S_{CS} = \int_{M_3} \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A^3\right)$

EOM: $F(A) = dA + A \wedge A = 0.$

These are the Classical Backgrounds.

Infinitesimal moduli: $d\delta A + [A, \delta A] = 0$

or $d_A \delta A = 0$ where $d_A^2 = F(A) = 0$

Modulo gauge transf: $\delta A = d_A \gamma$

$$\Rightarrow \quad [\delta A] \in H^1_{d_A}(M_3).$$

Finite deformations: $\quad d_A \delta A + \frac{1}{2}[\delta A, \delta A] = 0$

Maurer-Cartan equation.

Alternatively: Deform action $S_{CS}$ around a classical background ($F(A_0) = 0$)

$$\Rightarrow \quad S_{CS}(\delta A) = \int_{M_3} tr\left(\delta A \, d_{A_0} \delta A + \frac{1}{3} \delta A^3\right)$$

The deformed action has EOM:

$$d_{A_0} \delta A + \frac{1}{2}[\delta A, \delta A] = 0$$

MC-equation!

## Heterotic moduli:

Fields: $y = (x, \kappa, \alpha, \mu) \in \Omega^{(0,1)}(Q)$

$Q = \widetilde{T^{*(1,0)}}X \oplus \mathfrak{g} \oplus \widetilde{T^{(1,0)}}X$, $\mathfrak{g} = \mathrm{End}(TX) \oplus \mathrm{End}(V)$

That is:

$\mu \in \Omega^{(0,1)}(\widetilde{T^{(1,0)}}X) \rightsquigarrow$ def. complex structure

$\alpha \in \Omega^{(0,1)}(\mathrm{End}(V)) \rightsquigarrow$ def. gauge connection

$x \in \Omega^{(0,1)}(\widetilde{T^{*(1,0)}}X) \rightsquigarrow$ "Hermitian def.'s"

$\kappa \in \Omega^{(0,1)}(\mathrm{End}(TX)) \rightsquigarrow$ Spurious fields (Field-redef.'s ...)

Here $\widetilde{T^{(1,0)}}X$ denote divergence-free fields:

Also true in Kodaira-Spencer

$$\mu \in \widetilde{\Omega^*(T^{(1,0)}X)} \iff \nabla_a \mu^a = 0$$

and $\widetilde{T^{*(1,0)}}X = T^{*(1,0)}X / \partial\text{-exact}$:

$$x_a \sim x_a + \partial_a b, \quad b \in \Omega^{(0,1)}(X).$$

Superpotential: $W = \int_X (H + i\,d\omega) \wedge \Omega$

F-terms

$\Omega = e^{-2\phi}\Phi$

Do a holomorphic deformation $\Delta$ of parameters from a supersymmetric solution:

$W \to W + \Delta W$, where

$$\Delta W = S(q) = \int_X \left( \langle q, \bar{D} q \rangle + \frac{1}{3} \langle q, [q, q] \rangle \right) \wedge \Omega$$

$\langle\,,\rangle$ : Natural pairing on $Q$: $\quad q_1, q_2 \in \Omega^*(Q)$

$\Rightarrow \langle q_1, q_2\rangle = \mu_1^a x_{2a} + \mu_2^a x_{1a} + \mathrm{tr}(\alpha_1 \alpha_2) + \mathrm{tr}(\kappa_1 \kappa_2)$.

$\bar{D}$ : Natural differential on $Q$. $\bar{D}^2 = 0$

Upper-triangular : Defines $Q$ as a double extension.

$\rightsquigarrow$ Apply Homological algebra and long exact sequences to compute cohomologies.

Note :

$$\bar{D}^2 = 0 \iff \left\{\begin{array}{l} - \; X \text{ is complex} \\ - \; A \text{ and } \tilde{D} \text{ are holomorphic} \\ - \; \text{The Bianchi Identity is satisfied} \end{array}\right\} \text{"F-terms"}$$

Note: $\bar{D}$ is also locally trivialisable:

$$\bar{D} = G \bar{\partial} G^{-1}.$$

Akin to going to twisted frame in generalised geometry.

Upshot: Can use check-cohomology (Dolbeault theorem, ...).

$\rightsquigarrow$ Algebraic Geometry applies!

$[\ ,\ ]$: Natural holomorphic Courant-type bracket on $Q$.

- Satisfies Leibniz-rule w.r.t. $\bar{D}$:

$$\bar{D}[\psi_1,\psi_2] = [\bar{D}\psi_1,\psi_2] + (-1)^{|\psi_1|}[\psi_1,\bar{D}\psi_2].$$

EOM for action $S(\psi)$:

$$\bar{D}\psi + \frac{1}{2}[\psi,\psi] = 0 \qquad (*)$$

Heterotic Maurer-Cartan equation.

Actually: Susy also requires $S(\psi)=0$.

$\Rightarrow$ Can show that MC-eq $(*)$ with $S(\psi)=0$ are equivalent to the MC-eq. of an $L_3$-algebra.

Infinitesimally: $\bar{D}y = 0$

Modulo (complexified) gauge transf: $\bar{D}\gamma$ , $\gamma \in \Omega^0(Q)$

$\rightsquigarrow [y] \in H^{(0,1)}_{\bar{D}}(Q)$

What about D-terms ?

$$d(e^{-2\phi}\omega^2) = 0$$
$$\omega^2 \wedge F = 0$$
$$\omega^2 \wedge \tilde{R} = 0.$$

$\rightsquigarrow$ conf. balanced + Yang-Mills conditions.

Toy-model: Holomorphic bundle on fixed Calabi-Yau:

$$F^{(0,2)} = 0 \qquad (F\text{-term})$$

$$\omega \wedge \omega \wedge F = 0 \qquad (D\text{-term})$$

Deform F-term: $\quad \bar{\partial}_A \alpha = 0 \,, \quad \alpha \in \Omega^{(0,1)}(\mathrm{End}(V))$

Deform D-term: $\quad \partial_A(\omega \wedge \omega \wedge \alpha) = 0 \Leftrightarrow \bar{\partial}_A^{\dagger} \alpha = 0.$

$\Rightarrow$ actual moduli are given by harmonic representatives in $H^{(0,1)}_{\bar{\partial}_A}(\mathrm{End}(V))$.

- Similarly, Imposing heterotic D-terms pick out a particular "harmonic" representative of $H_{\bar{\partial}}(Q)$.

- Can further show that a (perturbative) solution to the MC-eq. can be chosen so as to also solve the D-terms.

# Heterotic Quantum Moduli (part III):

Recall Holomorphic Chern-Simons:

$$S(\alpha) = \int_X \mathrm{tr}\left(\alpha \wedge \bar\partial \alpha + \frac{1}{3}\alpha^3\right) \wedge \Omega, \quad \alpha \in \Omega^{(0,1)}(\mathrm{End}(V)).$$

Partition function $\quad Z = \int \mathcal{D}\alpha \, e^{-S(\alpha)}$

Generating functional for DT-invariants!

- (quasi-) topological in nature.
  (Wall-crossings, etc.)

Note: $S(\alpha)$ is part of $S(g)$.

Very tempting: Can we make sense of

$$(*) \quad Z = \int \mathcal{D}g \, e^{-S(g)} \quad ??$$

Note: The path integral $(*)$ is also over all geometric d.o.f., so any heterotic invariants extracted ought to be truly topological!

Catch: $\mathcal{D}g = \mathcal{D}\mu \, \mathcal{D}\alpha \, \mathcal{D}x \, \mathcal{D}\kappa$

Not physical!
(Ignore at your peril...)

Potential Solutions:

- Integrate out $\kappa$ using Lagrange-multiplier techniques?

- Embed story into larger duality covariant framework (DFT, ...) ?
  ( K ⇝ Field redef.'s)

- Redo moduli story without introducing K ?

## Moduli without $K \in \Omega^{(0,1)}(\mathrm{End}(TX))$

⇝ Deform Hull connection as function of fields. $(\tilde{\nabla} = \nabla^-)$

Aim: Look for something Mathematically exact.

- Retain mathematical theorems
- Physically correct modulo $\mathcal{O}(\alpha'^2)$.

Note: $X$ compact:

$\rightsquigarrow$ Consistency in the SUGRA framework requires $X$ to be Calabi-Yau at zeroth order in $\alpha'$.

(Without this assumption we have not (yet) found a Mathematically exact description.)

New moduli: $y = (\mu, \alpha, x) \in \Omega^{(0,1)}(\tilde{Q})$

$\tilde{Q} = \widetilde{T^{*(1,0)}} X \oplus \mathrm{End}(V) \oplus \widetilde{T^{(1,0)}} X$

Modulo $\mathcal{O}(\alpha'^2)$ (in physics), we get

Infinitesimal moduli: $\bar{D} y = 0$ Exact equation!

Note: $\bar{D}^2 = 0$ on the nose, and

$$\bar{D}_y = \begin{array}{cl} \left( \begin{array}{c} \bar\partial_X + \mathcal{H}(\alpha,\mu) \\ \bar\partial_\alpha + \mathcal{F}(\mu) \\ \bar\partial_\mu \end{array} \right) & \left. \begin{array}{l} \widetilde{T^{*(1,0)}X} \\ \mathrm{End}(V) \\ \widetilde{T^{(1,0)}X} \end{array} \right\} \tilde{Q} \end{array}$$

Here the Extension / Atiyah maps are:

$$\mathcal{F}(\mu) = F_a \wedge \mu^a \qquad F_a = F_{a\bar b}\, dz^{\bar b} \quad \text{(Atiyah map)}$$

$$\mathcal{H}(\alpha,\mu)_b = H_{bd} \wedge \mu^d + \frac{\alpha'}{2} \mathrm{tr}(\alpha \wedge F_b) - \alpha' R^{LC}{}_b{}^c{}_d \nabla^{LC}_c \mu^d$$

Here $R^{LC}$ is the curvature of the Ricci-flat Calabi-Yau LC-connection, $\nabla^{LC}$.

- Modulo (complex) gauge $\rightsquigarrow$ $[y] \in H^{(0,1)}_{\bar D}(\tilde{Q})$

- D-terms: Again pick harmonic representative $y$.

Hodge-theory: Harmonic forms in 1-1 correspondence with $H^{(0,1)}_{\bar{D}}(\tilde{Q})$

$\rightsquigarrow$ Sufficient to look at $H^{(0,1)}_{\bar{D}}(\tilde{Q})$.

Note: $\bar{D}$ is still upper-triangular.

$\rightsquigarrow$ Still defines $\tilde{Q}$ as a "double extension":

$\tilde{\mathcal{F}}$ defines an extension of $\widetilde{T^{(1,0)}X}$ by $\mathrm{End}(V)$ (the Atiyah extension):

$$0 \to \Omega^{(0,p)}(\mathrm{End}(V)) \to \Omega^{(0,p)}(Q_1) \to \Omega^{(0,p)}(\widetilde{T^{(1,0)}X}) \to 0$$

$\rightsquigarrow$ LES in cohomology:

$$\ldots \to H^{(0,1)}(\mathrm{End}(V)) \to H^{(0,1)}(Q_1) \to H^{(0,1)}(\widetilde{T^{(1,0)}X})$$

$$\xrightarrow{[\mathcal{F}]} H^{(0,2)}(\mathrm{End}(V)) \to \ldots$$

Connecting homomorphism (Atiyah map).

$$\Rightarrow \quad H^{(0,1)}(Q_1) \cong H^{(0,1)}(\mathrm{End}(V)) \oplus \ker([\mathcal{F}])$$

bundle moduli (assuming stable bundle)

$$[\mathcal{F}] : \quad H^{(0,1)}(T^{(1,0)}X) \to H^{(0,2)}(\mathrm{End}(V))$$

Complex structure moduli

$\mathcal{H}$ defines an "extension" of $Q_1$ by $T^{*(1,0)}X$:

$$0 \to \Omega^{(0,p)}_{\bar\partial}(\widetilde{T^{*(1,0)}X}) \to \Omega^{(0,p)}_{\bar D}(\tilde{Q}) \to \Omega^{(0,p)}_{\bar\partial_1}(Q_1) \to 0 \qquad (*)$$

Not algebraic!

BUT: Extension map $\mathcal{H}$ has a holomorphic derivative...

$$\mathcal{H}(\alpha,\mu)_b = H_{bd}\wedge\mu^d + \frac{\alpha'}{2}\mathrm{tr}(\alpha\wedge F_b) - \alpha' R^{LC}{}_b{}^c{}_d \nabla_c^{LC}\mu^d$$

⇝ $\bar{D}$ no longer a connection on $\tilde{Q}$... (?)

$f\in\Omega^0(X)$, $\alpha\in\Omega^{(0,q)}(\tilde{Q})$: $\bar{D}(f\alpha)\neq\bar{\partial}f\wedge\alpha+f\bar{D}\alpha$

Still: $\bar{D}^2=0$ and we have a short exact sequence of chain complexes (*).

⇝ long exact sequence in cohomology:

$$\ldots\to H^{(0,1)}_{\bar\partial}(\widetilde{T^{*(1,0)}X})\to H^{(0,1)}_{\bar D}(\tilde{Q})\to H^{(0,1)}_{\bar\partial_1}(Q_1)$$

$$\xrightarrow{[\mathcal{H}]} H^{(0,2)}_{\bar\partial}(\widetilde{T^{*(1,0)}X})\to\ldots$$

Connecting hom. ("Atiyah map").

$$\Rightarrow \quad H^1_{\bar{D}}(\tilde{Q}) \cong \underbrace{H^1(\widetilde{T^{*(1,0)}X})}_{\text{Hermitian moduli}} \oplus \ker([\mathcal{H}])$$

$$[\mathcal{H}] : \quad \underbrace{H^1(Q_1)}_{\substack{\text{complex structure} \\ +\ \text{bundle moduli.}}} \longrightarrow H^2(\widetilde{T^{*(1,0)}X})$$

Moduli story so far without field redefinitions ... (covered pt. I)).

## Recap / Outlook:

- Introducing spurious $\mathrm{End}(TX)$-valued d.o.f. allows for a "clean" heterotic moduli story.

BUT:
- These fields couple non-trivially to real fields.
- How do we quantise?

Solution (?): Redo moduli story without spurious d.o.f.

So far:

$\rightsquigarrow$ Inf. moduli cohomology $H^{(0,1)}_{\bar{D}}(\tilde{Q})$.

## Outlook:

- $\bar{D}$ has a holomorphic derivative. What does this mean?

Sidenote: $[\ ,\ ] : Q \times Q \to Q$ also involves hol. derivatives, and so does the cubic coupling $\langle g, [g, y] \rangle$.

- Is there a sense in which we should think of hol. derivatives $\nabla_a$ as "couplings"? (Higher Spin? SFT?)

- Is the story still "loc. trivialisable":

$$\bar{D} = G \bar{\partial} G^{-1} \quad ?$$

$\rightsquigarrow$ Define and use Check-cohomology, and methods of Algebraic Geometry?

- Higher order def.'s: Is it still a DGLA / $L_3$-algebra?

- Quantum moduli: Invariants, Anomalies, world-sheet description, dualities, ...

Sidenote: $\bar{D}$ forms part of a connection

$$\mathcal{D} = \bar{D} + D \quad \text{on } Q$$ (with spurious d.o.f.'s)

Heterotic system $\Rightarrow$ $\mathcal{D}$ is hermitian Yang-Mills!

$$F_{\mathcal{D}}^{(0,2)} = \bar{D}^2 = 0, \qquad \omega \wedge \omega \wedge F_{\mathcal{D}} = 0.$$

Li-Yau $\Rightarrow$ $Q$ is (poly-) stable!

Hint towards a heterotic Donaldson-Uhlenbeck-Yau theorem / Kobayashi-Hitchin correspondence.

# Heterotic compactifications to 3D:

Put the heterotic string on spacetimes:

$$M_{10} = M_3 \times X_7 \rightsquigarrow \text{heterotic } G_2\text{-system}$$

SUSY $\Rightarrow$ $X_7$ admits a $G_2$-structure: $\varphi \in \Omega^3(X_7)$

($G_2$-holonomy: $d\varphi = d * \varphi = 0$)

$\varphi \rightsquigarrow$ metric $g \rightsquigarrow * \varphi = \psi \in \Omega^4(X_7)$.

Heterotic $G_2$-structures are integrable. We can define differential complex:

$$0 \to \Omega^0(X_7) \xrightarrow{\check{d}} \Omega^1(X_7) \xrightarrow{\check{d}} \Omega^2_7(X_7) \xrightarrow{\check{d}} \Omega^3_1(X_7) \to 0$$

where $\check{d} = \pi \circ d$ Projection onto appropriate $G_2$ representation.

Note: This complex can be extended to bundle-valued forms, provided the connection is an instanton:

$$F \wedge \psi = 0 .$$

## Moduli:

Ignoring complications of spurious d.o.f's,

moduli: $g = (x, \alpha) \in \Omega^1(E)$, $E = TX_7 \oplus End(V)$.

- $x$ : Geometric d.o.f (metric + B-field)

- $\alpha$ : Bundle d.o.f.

$$\mathcal{D}^2 \wedge \varphi = 0$$

Heterotic $G_2$-System $\rightsquigarrow$ Instanton connection $\mathcal{D}$ on $E$.

$$\Rightarrow \quad 0 \to \Omega^0(E) \xrightarrow{\check{\mathcal{D}}} \Omega^1(E) \xrightarrow{\check{\mathcal{D}}} \Omega^2_7(E) \xrightarrow{\check{\mathcal{D}}} \Omega^3_1(E) \to 0$$

Moduli: $\quad \check{\mathcal{D}} y = 0 \Rightarrow [y] \in H^1_{\check{\mathcal{D}}}(E)$

modulo gauge transf.

BUT: $\check{\mathcal{D}}$ does NOT define $E$ as an extension!

- No SES or LES to compute cohomology.
- Moduli do not decouple, not even locally in moduli space.

⇝ Moduli cannot be separated into geometric and bundle moduli!!

Outlook:
- Higher order def.'s ...
- Topological theory?
- Invariants?

Sidenote: $G_2$-version of DT-invariants: Donaldson-Segal invariants.

- Less "topologically protected".
- agrees with heterotic story, suggesting generically coupled moduli.

THANK YOU !!