Hidden Conformal Symmetry from the Killing Tower

Victoria Martin

University of Iceland

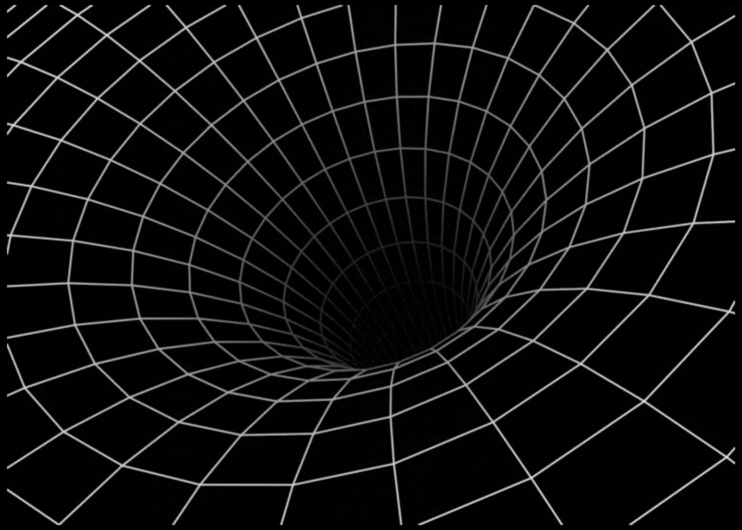
22 November 2021

2110.10723 with Cindy Keeler and Alankrita Priya

2112.xxxxxx with Valentina Giangreco M. Puletti

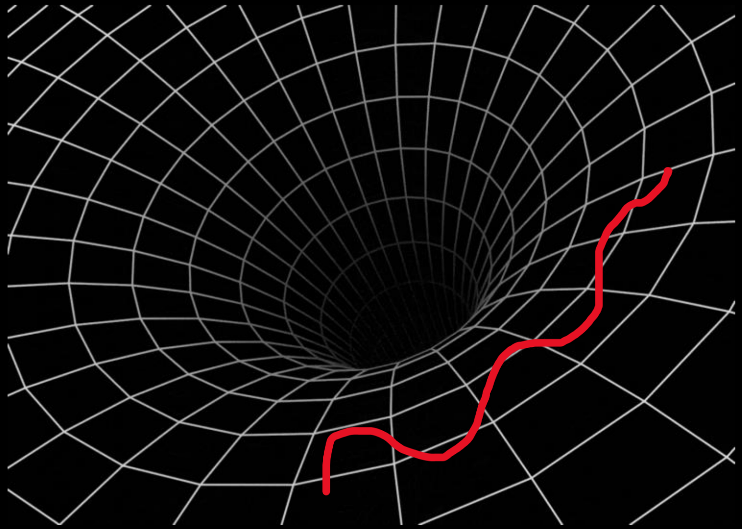
Isometries vs Hidden Symmetries

Isometries: symmetries of the metric generated by Killing vectors



$$\ell^a \qquad \nabla_{(a}\ell_{b)} = 0 \qquad \ell^a p_a = C$$

"Hidden": symmetries of the dynamics generated by Killing tensors



$$k^{ab} \qquad \nabla_{(a}k_{bc)} = 0 \qquad k^{ab} p_a p_b = C$$

Why are hidden symmetries interesting?

1) If a spacetime admits a tower of Killing tensors, you are *guaranteed:*

- Separation of equations of motion, like Klein-Gordon and Hamilton-Jacobi equations.
- Complete integrability of geodesic motion.

k^{ab} ✓ $SL(2,\mathbb{R})$ ✗

Comprehensive review: Frolov, Krtouš, Kubizňák 1705.05482

2) Hidden symmetries facilitate potential holographic dualities in "novel" contexts, such as a Kerr/CFT correspondence *away from extremality.*

k^{ab} ✗ $SL(2,\mathbb{R})$ ✓

Castro, Maloney, Strominger 1004.0996

What do we do?

- We construct the hidden conformal symmetry of non-extremal Kerr/CFT from the Killing tensors of the separability formalism.
- This construction allows us to generalize certain aspects of hidden conformal symmetry to general dimension and cosmological constant.

k^{ab} ✓ $SL(2,\mathbb{R})$ ✗

k^{ab} ✗ $SL(2,\mathbb{R})$ ✓

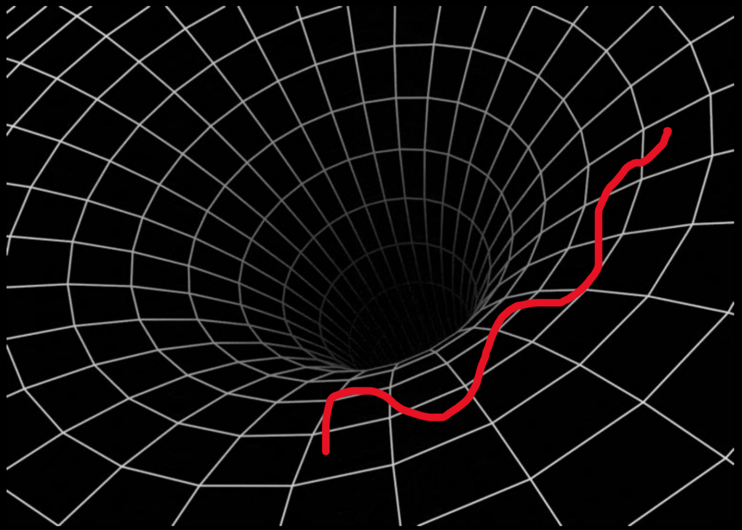
Comprehensive review: Frolov, Krtouš, Kubizňák 1705.05482

Castro, Maloney, Strominger 1004.0996

Hidden Symmetry from Killing Tower

Comprehensive review: Frolov, Krtouš, Kubizňák 1705.05482

"Hidden": symmetries of the dynamics generated by Killing tensors



k^{ab} $\quad \nabla_{(a}k_{bc)} = 0 \quad k^{ab}p_a p_b = C$

Why are Killing tensors and hidden symmetries interesting?

Separability of EOM!

Principal tensor: $\nabla_c h_{ab} = \frac{1}{D-1} g_{c[a,} \nabla_\mu h^\mu_{b]}$

Tower of Killing-Yano forms: $\boldsymbol{h}^{(j)} = \frac{1}{j!}\boldsymbol{h}^{\wedge j}$

$$k^{ab}_{(j)} = \frac{1}{N!} * h^{(j)a}{}_{c_1\ldots c_N} * h^{(j)bc_1\ldots c_N}$$

Hidden Conformal Symmetry: Kerr/CFT

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right) + \frac{\rho^2}{\Delta}dr^2 + \frac{\sin^2\theta}{\rho^2}\left(\left(r^2+a^2\right)d\phi - adt\right)^2 + \rho^2 d\theta^2$$

Extremal $a = M$

Near-horizon limit in metric $\hat{r} = \frac{r-M}{\lambda M}$ $\lambda \to 0$

$$ds^2 \propto \left[\frac{d\hat{r}^2}{\hat{r}^2} - \hat{r}^2 d\hat{t}^2 + F(\theta_0)(d\hat{\phi} + \hat{r}d\hat{t})^2\right]$$

$$F(\theta_0) = 1 \quad \to \quad SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$$

Non-extremal $a < M$

Near-horizon limit in EOM $\nabla^2\Phi = 0$ $\Phi = e^{i(m\phi - \omega t)}R(r)S(\theta)$

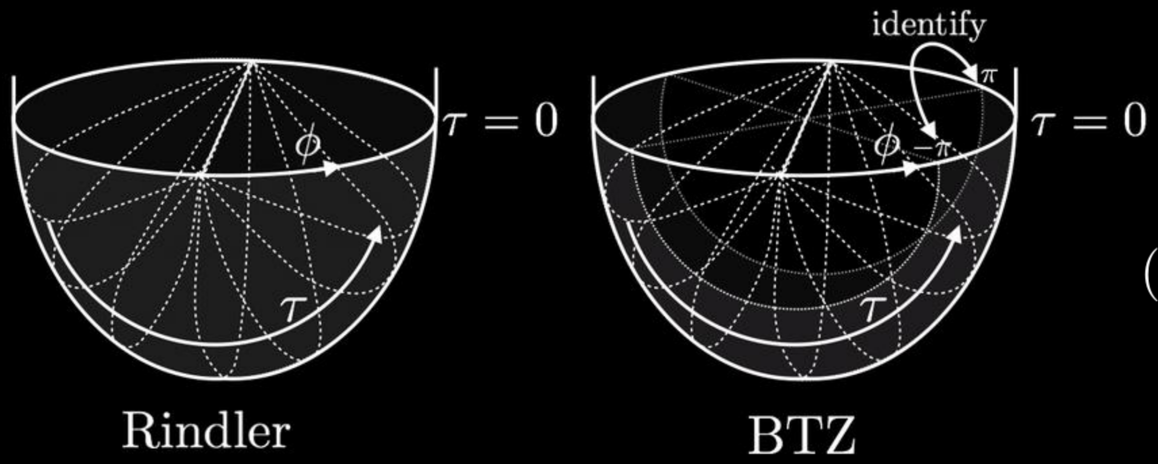
$$\left(\partial_r(\Delta\partial_r) + \alpha_+^2\frac{r_+ - r_-}{r - r_+} - \alpha_-^2\frac{r_+ - r_-}{r - r_-} + \cancel{n.s.}\right)R(r) = 0$$

$$[H_0, H_{\pm 1}] = \mp i H_{\pm 1} \qquad \mathcal{H}^2\Phi = \nabla^2_{R,near}\Phi$$

$$[H_{-1}, H_1] = -2iH_0$$

Guica, Hartman, Song, Strominger 0809.4266; Castro, Maloney, Strominger 1004.0996; Aggarwal, Castro, Detournay 1909.03137

Conformal Coordinates



$$(\tau, r, \phi) \quad \rightarrow \quad (w, \bar{w}, z) \quad \rightarrow \quad ds^2 = \frac{1}{z^2}\left(dz^2 + dw d\bar{w}\right)$$

Kerr

$$w^+ = f(r)e^{t_R}$$
$$w^- = f(r)e^{-t_L}$$
$$y = g(r)e^{(t_R - t_L)/2}$$

$$(t, r, \phi, \theta_0) \quad \rightarrow \quad (w^+, w^-, y) \quad \rightarrow \quad ds^2 \propto \frac{1}{y^2}\left(dy^2 + F(\theta_0)dw^+dw^-\right) + \ldots$$

Image adapted from Goto, Takayanagi 1704.00053

Conformal Coordinates

Kerr

$$w^+ = f(r)e^{t_R}$$
$$w^- = f(r)e^{-t_L}$$
$$y = g(r)e^{(t_R - t_L)/2}$$

$$(t, r, \phi, \theta_0) \quad \rightarrow \quad (w^+, w^-, y) \quad \rightarrow \quad ds^2 \propto \frac{1}{y^2}\left(dy^2 + F(\theta_0)dw^+dw^-\right) + ...$$

Build hidden conformal symmetry generators, modeled after AdS3 Killing vectors:

$$H_1 = i\partial_+ \qquad H_0 = i(w^+\partial_+ + \frac{1}{2}y\partial_y) \qquad H_{-1} = i((w^+)^2\partial_+ + w^+y\partial_y - y^2\partial_-)$$
$$\bar{H}_1 = i\partial_- \qquad \bar{H}_0 = i(w^-\partial_- + \frac{1}{2}y\partial_y) \qquad \bar{H}_{-1} = i((w^-)^2\partial_- + w^-y\partial_y - y^2\partial_+)$$

$$\mathcal{H}^2\Phi = \nabla^2_{R,near}\Phi$$

- This has been done for Kerr and the 5D Myers-Perry black hole.
- Killing tensors are known for all Kerr-NUT-(A)dS black holes.
- We build Casimir from Killing tensors as a step toward generalizing the notion of hidden conformal symmetry and learn some interesting things along the way.

Kerr-NUT-(A)dS geometry in canonical coordinates

$$ds^2 = \sum_{\mu=1}^{n} \left[\frac{U_\mu}{X_\mu} dx_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{n-1} A_\mu^{(j)} d\psi_j \Big)^2 \right] + \epsilon \frac{c}{A^{(n)}} \Big(\sum_{k=0}^{n} A^{(k)} d\psi_k \Big)^2$$

Chen, Lu, Pope 0604125

x_μ: radial and longitudinal directions $\sim (r, \theta s)$. In particular, $x_n = ir$.
ψ_k: Killing directions $\sim (t, \phi s)$

Here $D = 2n+\epsilon$, so $\epsilon = 0$ for even D and $\epsilon = 1$ for odd D.

$X_r = -\Delta$

Killing vectors and Killing tensors:

$$l_{(j)} = \partial_{\psi_j}$$

$$k_{(j)} = \sum_{\mu=1}^{n} A_\mu^{(j)} \left[\frac{X_\mu}{U_\mu} \partial_{x_\mu}^2 + \frac{U_\mu}{X_\mu} \Big(\sum_{k=0}^{n-1+\epsilon} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} \partial\psi_k \Big)^2 \right] + \epsilon \frac{A_{(j)}}{A_{(n)}} \partial{\psi_n}^2$$

Conserved quantities:

$$-il_{(j)}^a \nabla_a \Phi = L_j \Phi$$

$$-\nabla_a k_{(j)}^{ab} \nabla_b \Phi = K_j \Phi$$

Separated Klein-Gordon in canonical coordinates

Separation ansatz: $$\Phi = \prod_{\mu=1}^{n} R_\mu \prod_{k=0}^{n-1+\epsilon} \exp(iL_k\psi_k)$$

EOM: $$X_\mu R_\mu'' + \left(X_\mu' + \frac{\epsilon X_\mu}{x_\mu}\right) R_\mu' + \frac{\chi_\mu}{X_\mu} R_\mu = 0, \qquad \chi_\mu = X_\mu \sum_{j=0}^{n-1+\epsilon} K_j(-x_\mu^2)^{n-1-j} - \left[\sum_{j=0}^{n-1+\epsilon} L_j(-x_\mu^2)^{n-1-j}\right]^2$$

$\mathcal{H}^2\Phi = \nabla^2\Phi$	Goal: Use the Killing tower to propose a tensor equation that will recover this relationship for general dimension and cosmological constant

We do this in two (very involved) steps:

1) Generalize conformal coordinates

2) Figure out how to take the near horizon limit in the EOM

$$-H_0^a H_0^b + \frac{1}{2} H_1^a H_{-1}^b + \frac{1}{2} H_{-1}^a H_1^b = f(k_{(j)}^{ab}, \ell_{(j)}^a)$$

Generalizing conformal coordinates:

$$w^+ = f(r)e^{t_R}$$
$$w^- = f(r)e^{-t_L}$$
$$y = g(r)e^{(t_R - t_L)/2}$$

What we learn from matching: $\mathcal{H}^2\Phi = s\nabla^2\Phi$

For the radial functions: $f^2 + g^2 = C \quad \frac{f^2}{g^2} = \frac{e^I}{1-e^I} \quad e^I = \prod_{i=1}^{2N_{\epsilon,\sigma}} (r - r_i)^{\frac{2c_1}{r_i^\epsilon \Delta'(r_i)}}$

General D: Branch cuts force a near-***horizon*** limit $\quad c_1 = \frac{1}{2} r^\epsilon \Delta'(r_+) \quad$ Expand remaining terms near outer horizon

In general dimension, the near-region limit (ωr<<1, ωM<<1, sometimes called the "soft hair" limit – cf Haco, Hawking, Perry, Strominger 1810.01847 –) is **not enough** to match the Casimir to the Klein-Gordon operator. You really need (r-r+)<<1.

Tensor Equation

Shifted conserved quantities

"Error" term

$$-H_0^a H_0^b + \frac{1}{2} H_1^a H_{-1}^b + \frac{1}{2} H_{-1}^a H_1^b = -s \sum_{k=0}^{n-1} r^{2(n-1-k)} \left(-k_{(k)}^{ab} + \sum_{i=0}^{n+\epsilon-1} \sum_{i=0}^{n+\epsilon-1} Q_k^{ij} l_{(i)}^a l_{(j)}^b \right) + E^{ab}$$

4D

$$\tilde{Q}^{00} = 4M^2 + 2Mr + r^2, \qquad E^{ab} = \delta_t^a \delta_t^b \tilde{Q}^{00}$$

5D

$$E^{ab} = \frac{1}{4} \Bigg[\left(-a_1^2 - a_2^2 + 2M + r^2\right) \delta_t^a \delta_t^b + \frac{a_2(a_1^2 + r^2)(a_1^2 + r_-^2)(a_1^2 + r_+^2)}{a_1^2(r^2 - r_-^2)(r^2 - r_+^2)} \left(\delta_t^a \delta_{\phi_2}^b + \delta_{\phi_2}^a \delta_t^b\right)$$

$$\frac{a_2(a_1^2 + r_-^2)(a_1^2 + r_+^2)}{a_1(r^2 - r_-^2)(r^2 - r_+^2)} \left(\delta_{\phi_1}^a \delta_{\phi_2}^b + \delta_{\phi_2}^a \delta_{\phi_1}^b\right) + \frac{-a_1^4 + a_2^2 r^2 + a_1^2 a_2^2 + 2M a_1^2 - a_1^2 r^2}{(r^2 - r_-^2)(r^2 - r_+^2)} \delta_{\phi_2}^a \delta_{\phi_2}^b \Bigg]$$

"Monodromy" Parameter Results

Castro, Lapan, Maloney, Rodriguez: 1303.0759, 1304.3781; Aggarwal, Castro, Detournay: 1909.03137

Recall Kerr EOM: $$\left(\partial_r(\Delta\partial_r) + \alpha_+^2 \frac{r_+ - r_-}{r - r_+} - \alpha_-^2 \frac{r_+ - r_-}{r - r_-} + (r^2 + 2M(r + 2M))\omega^2\right) R(r) = K R(r)$$

Monodromy parameters for general D and cosmological constant:

$$\boxed{\alpha_\pm = i\frac{\sqrt{\chi(r_\pm)}}{\mathcal{P}^\pm(r_\pm)}}$$

$$\chi = X_r \sum_{j=0}^{n-1+\epsilon} K_j (r^2)^{n-1-j} - \left[\sum_{j=0}^{n-1+\epsilon} L_j (r^2)^{n-1-j}\right]^2$$

Interesting thermodynamic connection

Wald gr-qc/9307038

Killing vectors: $\zeta^\pm = \kappa_\pm(\partial_t + \Omega_\pm \partial_\phi)$

BH entropy: $S_\pm = 2\pi \int_\Sigma Q_\pm$

Kerr: $$\alpha_\pm = \frac{\omega - \Omega_\pm m}{2\kappa_\pm}$$

$$\zeta^\pm \Phi = 2\alpha_\pm \Phi$$

Potential thermodynamic interpretation of Killing tower objects!

Future Directions

- **Globally defined symmetry generators.** In 2103.01234 (Charalambous, Dubovsky, Ivanov) present globally defined symmetry generators of the near-region Klein-Gordon equation on a Kerr background. They possess a smooth Schwarzschild limit $a \to 0$ and reproduce the hidden symmetry generators for Schwarzschild found in 1106.0999 (Bertini, Cacciatori, Klemm). These were used to study tidal Love numbers.

- **Thermodynamic interpretation of Killing objects.** That is, can we use Killing tensors to build up the Wald Noether charges in a way that gives us meaningful Physical insight?

- **Log CFTs.** How is hidden symmetry manifest when we change the dynamics? For example, study higher derivative interactions, which are known to be instrumental in holographic duals of log CFTs (1605.03959: Hogervorst, Paulos and Vichi).

- **Possible classical double-copy story for Kerr-NUT-(A)dS black holes.**