Scattering amplitudes and algebraic geometry — Elliptics and beyond

Nils Matthes

University of Copenhagen, DFG

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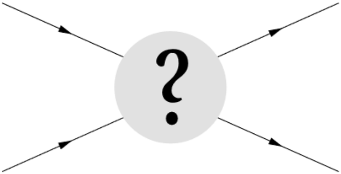
Overview

Scattering amplitudes in QFT

Goal: Understand interactions of elementary particles



Obstacle: Only initial and final state are measurable

Basic paradigm of perturbative QFT

Associate Feynman diagrams/integrals to all possible particle exchanges

Periods, I

Meta-theorem cf. Bogner–Weinzierl (2007)

Scattering amplitudes are periods in the sense of algebraic geometry.

A period is a complex number whose real and imaginary part can be written

$$\int_D P(x_1,\ldots,x_n)dx_1\ldots dx_n,$$

where $P(x_1,\ldots,x_n)$ is a rational function, and $D \subset \mathbb{R}^n$ is a subset defined by polynomial inequalities (with $\mathbb{Q}$-coefficients). Kontsevich–Zagier (1999)

Examples

(i) $\pi = \int_{x^2+y^2\leq 1} dxdy$

(ii) $\log(n) = \int_{1\leq x\leq n} \frac{dx}{x}$, for $n = 1,2,3,\ldots$

(iii) $\zeta(n) = \int_{0\leq x_1\leq\ldots\ldots\leq x_n\leq 1} \frac{dx_1}{1-x_1}\frac{dx_2}{x_2}\ldots\frac{dx_n}{x_n}$, for $n = 2,3,4,\ldots$

Recent occurrences of periods in amplitude computations

QCD

- Higgs cross section, N^3LO Anastasiou–Duhr–Dulat–Herzog–Mistlberger (2015)
- double-real gluon emission, N^3LO Melnikov–Rietkerk–Tancredi–Wever (2018)
- double-soft limits, massless partons Caola–Delto–Frellesvig–Melnikov (2018)

$\mathcal{N} = 4$ SYM

- two-loop hexagon amplitude Del Duca–Duhr–Smirnov (2008), Goncharov–Spradlin–Vergu–Volovich (2010)

Superstring theory

- tree-level: Schlotterer–Stieberger (2012), Broedel–Schlotterer–Stieberger (2013)
- one-loop, closed: Green–Russo–Vanhove (2008), D'Hoker–Green–Vanhove (2015)
- one-loop, open: Broedel–Mafra–M.–Schlotterer (2014), Broedel–M.–Richter–Schlotterer (2017)

Periods, II

Holy grail Grothendieck, André,. . .

A systematic and algorithmic way to decompose every period into irreducible constituents.

The only game in town is the motivic coaction. Goncharov, Brown

Caveats

- Construction requires deep results from arithmetic algebraic geometry
- Even then, explicitly only known in very special cases.

Why would physicists care?

1. For practitioners: The motivic coaction drastically simplifies amplitude computations; compresses many dozens of pages to just a few lines.
   $\rightsquigarrow$ higher efficiency & faster computations
2. For theorists: Periods form an essential part of the mathematical structure of QFT.
   $\rightsquigarrow$ If one wants to describe amplitudes conceptually, one needs to understand the intervening periods.

Why would mathematicians care?

Conversely, amplitudes often inspire new mathematics.

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(i) Double copy: 'gravity=gauge x gauge' Bern–Carrasco–Johansson (2010)

$\rightsquigarrow$ Double copy for periods Brown–Dupont (2018), e.g.

$$\underbrace{\log |z|^2}_{single-valued} = \log z + \log \bar{z}$$

(ii) 4-loop $g-2$ in QED, Bessel moments Laporta (2017)

$\rightsquigarrow$ Broadhurst–Roberts conjecture (2017), proved using new methods in algebraic geometry Fresán–Sabbah–Yu (2020)

(iii) One-loop (and beyond) closed string theory, modular graph functions Green–Russo–Vanhove (2008), D'Hoker–Green–Pioline (2017)

$\rightsquigarrow$ New class of modular forms Brown (2017); higher genus?

Multiple polylogarithms

Arguably the best studied class of periods are (special values of)

Multiple polylogarithms Leibniz (1696)

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{k_1}\ldots n_r^{k_r}}, \qquad k_1,\ldots,k_r \geq 1.$$

Their motivic coaction is well understood. Goncharov (1996), Brown (2011)

Slogan

Amplitudes involving only MPLs are completely understood.
Symbol/motivic coaction techniques Goncharov–Spradlin–Vergu–Volovich (2010), Duhr (2012),...

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Fact

But MPLs are (vastly!) insufficient to describe all scattering amplitudes in QFT.

Beyond MPLs

An example from QED Sabry (1962)

There is a two-loop diagram for the one-electron propagator that involves elliptic integrals:

$$\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Recent abundance of (generalized) elliptic integrals in QFT, e.g.

- two-loop amplitudes for $t\bar{t}$ v. Manteuffel–Tancredi (2017)
- 4-loop contribution to electron $g-2$ in QED Laporta (2017)
- Higgs boson via gluon fusion at $\mathrm{N}^3\mathrm{LO}$ Mistlberger (2018)

Upshot

To understand these Feynman integrals analytically, need appropriate mathematical language: multiple elliptic polylogarithms Brown–Levin (2011)

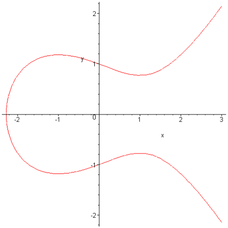
Elliptic curves

Elliptic curves are parametrized by $\tau \in \mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

Algebraic curve

Weierstrass equation:

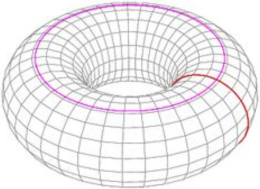
$$E : y^2 = 4x^3 + g_2(\tau)x + g_3(\tau)$$



Riemann surface

Torus uniformization:

$$\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \cong \mathbb{C}^\times / e^{2\pi i \tau \mathbb{Z}}$$



Multiple elliptic polylogarithms/zeta values

Iterated integrals of the real-analytic Kronecker function Kronecker (1891), Zagier (1990), Brown–Levin (2011)

$$\Omega_\tau(\xi,\alpha) = \exp\left(2\pi i \frac{\mathrm{Im}(\xi)}{\mathrm{Im}(\tau)}\alpha\right) \frac{\theta'_\tau(0)\theta_\tau(\xi+\alpha)}{\theta_\tau(\xi)\theta_\tau(\alpha)}.$$

Expansion coefficients give integration kernels for MEPLs

$$\Omega_\tau(\xi,\alpha) = \sum_{n\geq 0} f_\tau^{(n)}(\xi)\alpha^{n-1} = \frac{1}{\alpha} + \frac{\theta'_\tau(\xi)}{\theta_\tau(\xi)} + 2\pi i \frac{\mathrm{Im}(\xi)}{\mathrm{Im}(\tau)} + O(\alpha).$$

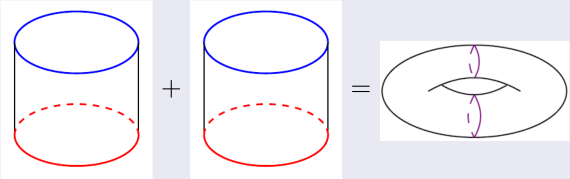
For superstring amplitudes, need:

Elliptic multiple zeta values Enriquez (2013)

$$I(n_1,\ldots,n_r;\tau) = \int_0^1 d\xi_1 f_\tau^{(n_1)}(\xi_1) \int_0^{\xi_1} d\xi_2 f_\tau^{(n_2)}(\xi_2) \ldots \int_0^{\xi_{r-1}} d\xi_r f_\tau^{(n_r)}(\xi_r).$$

One-loop string amplitudes

Worldsheets for open/closed superstring amplitudes at one-loop.



KLT relations Kawai–Lewellen–Tye (1986)

'Double copy': closed string = (open string) × (open string)

- Caveat: Presently only known at tree-level, but expected to hold more generally Casali–Mizera–Tourkine (2019)
- Expectation: Enough to study open string amplitudes

One-loop open superstring amplitudes

In the simplest case, one considers the four point function

Brink–Green–Schwarz (1982)

$$\mathcal{I}_{4\mathrm{pt}}(1,2,3,4) \equiv \int_0^1 d\xi_4 \int_0^{\xi_4} d\xi_3 \int_0^{\xi_3} d\xi_2 \int_0^{\xi_2} d\xi_1 \delta(\xi_1) \prod_{j<k}^{4} \exp\left[s_{jk} P_{jk}\right],$$

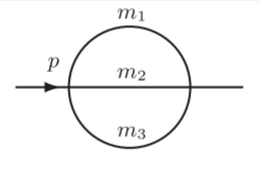
where $s_{jk} \equiv \alpha'(k_i + k_j)^2$ are dimensionless Mandelstam variables, and $P_{jk} = P(\xi_j - \xi_k)$ is the genus one Green's function.

Result Broedel–Mafra–M.–Schlotterer (2014)

The Taylor coefficients of $\mathcal{I}_{4\mathrm{pt}}(1,2,3,4)$ can be expressed in terms of elliptic multiple zeta values.

Applications to two-loop amplitudes in QFT

Sunrise diagram



- Arises naturally in, e.g., higher order corrections in electroweak precision calculations Bauberger–Böhn–Weiglein–Berends–Buza (1994).
- Can be expressed in terms of elliptic dilogarithms Bloch–Vanhove (2013).

Result for equal masses Adams–Bogner–Weinzierl (2014)

$$3\frac{\psi_1(q)}{\pi i}\left\{\frac{1}{2}\left[\mathrm{Li}_2(\zeta_3)-\mathrm{Li}_2(\zeta_3^{-1})\right]+\sum_{k=1}^{\infty}(-1)^k\left[\mathrm{Li}_2\left(\zeta_3(-q)^k\right)-\mathrm{Li}_2\left(\zeta_3^{-1}(-q)^k\right)\right]\right\},$$

where $\psi_1(q)$ is a period, and $\zeta_3=\exp(2\pi i/3)$

Towards a motivic elliptic coaction, I

- Higher loop computations in QFT are notoriously difficult, and the end result often very complicated.
- Given the success of the motivic coaction for MPLs:

Want

A motivic coaction for elliptic Feynman integrals.

- First step: elliptic symbol (=rough approximation to motivic coaction) Broedel–Duhr–Dulat–Penante–Tancredi (2017).

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Problem

Integrands for MEPLs are not algebraic (not even meromorphic!) functions on the elliptic curve, due to the presence of $\mathrm{Im}(\xi)$ terms.

$\Rightarrow$ not clear they are periods; motivic coaction?

Towards a motivic elliptic coaction, II

Let $\pi : E^\natural \to E$ be the universal vector extension of E. It is the moduli space of line bundles equipped with an integrable connection on E.

Key insight Fonseca–M. (2020)

The integrands for MEPLs are algebraic on $E^\natural$.

Application: new construction of elliptic KZB equation from conformal field theory Knizhnik–Zamolodchikov (1984), Bernard (1988).

Theorem Fonseca–M. (2021)

The universal elliptic KZB equation agrees with the nonabelian Gauss–Manin connection on $E^\natural \setminus \pi^{-1}(0)$.

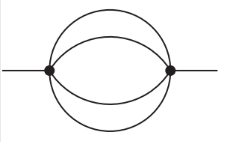
Work in progress: Use this to define a motivic coaction for MEPLs.

Dream goal

An explicit coaction principle for elliptic (Feynman) integrals cf. Abreu–Britto–Duhr–Gardi–Matthew (2021)

Beyond elliptics, I

Three-loop banana diagram, all masses equal



To compute the corresponding Feynman integrals analytically, need

Definition

A meromorphic modular form of weight k for $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a meromorphic[a] function f on $\mathfrak{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

[a] Including at cusps

Beyond elliptics, II

Theorem Broedel–Duhr–M. (2021)

The three-loop equal-mass banana integral can be expressed analytically using iterated period integrals of meromorphic modular forms for the group

$$\Gamma_1(6) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod 6 \right\}.$$

This result uses that the three-loop banana integral satisfies a Picard–Fuchs equation which is a symmetric square of a rank two equation.

Remark

In the three-loop unequal mass case or beyond three loops, one encounters Calabi–Yau periods Bloch–Kerr–Vanhove (2014), Bourjaily–He–MacLeod–von Hippel–Wilhelm, Bönisch–Fischbach–Klemm–Nega–Safari, Bönisch–Duhr–Fischbach–Klemm–Nega (2021).

Summary

- Amplitudes are central objects in QFT, and other areas (superstring theory, etc.).
- Their precise and efficient understanding requires sophisticated mathematical tools.
- If only MPLs are involved, the problem is basically solved; uses motivic coaction.
- Next frontier: MEPLs, need a motivic coaction for elliptic Feynman integrals/MEPLs
  $\rightsquigarrow$ work in progress
- At higher loop-order, one must go beyond elliptic curves (meromorphic modular forms, Calabi–Yau periods); their mathematical structure awaits to be fully understood

Key message of this talk

The understanding of amplitudes is a joint enterprise between physicists and mathematicians!