Bonuses on G_2 structures

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Based on 2101.12605 and ongoing work
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Motivation

Compactifications and moduli

- Background of form $(X \times Y, g = g_X \oplus g_Y, \dots)$
- Perturbation on internal manifold, eg. $g \to g + \delta g_Y$, carries mass in effective theory on X;
- Mass depends on details of metric kinetic operator, which is difficult;
- Supersymmetry simplifies by looking at SUSY preserving solutions.
- Effective particles $\leftrightarrow$ (formal) moduli of SUSY preserving metrics

Pictures

Background

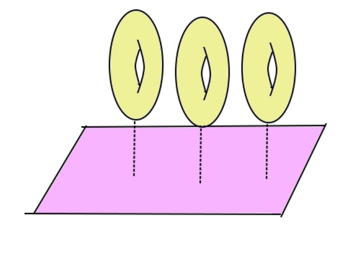

Figure: Pink background = effective spacetime, yellow torus is the internal manifold

Pictures

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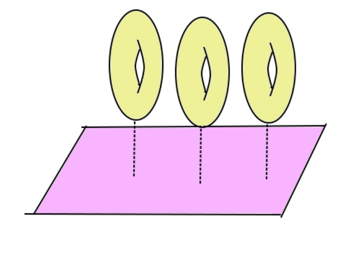

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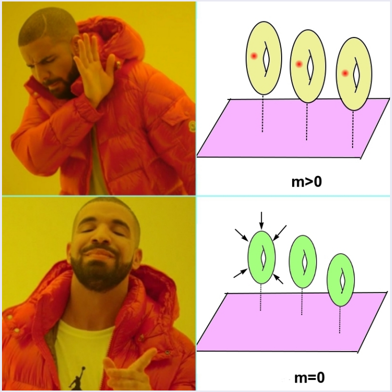
Perturbations



G_2 structure preserves SUSY

Definition

A G_2 structure manifold is a seven dimensional, Riemannian manifold equipped with a reduction of its tangent structure group to G_2 group.

More useful definitions

1. Y equipped with 3-form φ, such that locally:

$$\varphi = (e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6) \wedge e^7 + \\ + e^1 \wedge (e^3 \wedge e^5 - e^4 \wedge e^6) - e^2 \wedge (e^3 \wedge e^6 - e^4 \wedge e^5)$$

   Defines metric g_φ and vector product $\times_\varphi$, and Hodge dual four-form $\psi = *_\varphi \varphi$.

2. Y equipped with choice of spin structure + nowhere vanishing spinor.

G_2 structure preserves SUSY

Covariant derivative

From second definition: Y equipped with choice of spin structure + nowhere vanishing spinor, Ψ.

- Exists covariant derivative, ∇, killing Ψ, i.e.

$$\nabla\Psi = 0$$

- Preserves SUSY in compactification (if torsion is accounted for).

An extra spinor

From topology

- Odd dimension manifolds possess non-vanishing vector field, since Euler number vanishes;
- Clifford multiplication with the distinguished spinor $\rightsquigarrow$ new spinor:

$$\Psi' = v \cdot \Psi \,.$$

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But not extra SUSY

Extra SUSY requires the new spinor is constant with respect to same covariant derivative

$$\nabla(v \cdot \Psi) = \frac{1}{|v|^2}((\nabla v^\#) \wedge v^\#) \cdot (v \cdot \Psi) \,,$$

so, equivalent to $\nabla v \wedge v = 0$.

(Sometimes you can find such a v, see our paper for an example)

Definition of ACS

Definition

For general odd-dimensional, Riemannian manifold, M, an *almost contact structure* (ACS) is a triple $(R, J, \sigma) \in \left(\Gamma_M(TM), \Gamma_M(\mathrm{End(TM)}), \Omega^1(M)\right)$ such that:

- R is unit length;
- $J^2 = -\mathbf{1} + R \otimes \sigma$;
- $\sigma(R) = 1$.

ACMS

An ACS is said to metric if, in addition,

$$g(Ju, Jv) = g(u, v) - \sigma(u)\sigma(v)$$

$G_2 \to ACS$

Proposition (M. F. Arikan, H. Cho, S. Salur, 2011)

Let v arbitrary non-zero vector field on Y, G_2 structure manifold. $(v, v \times_\varphi -, v^\#)$ defines an ACMS on Y.

Can use this to conclude:

- Reduced structure group $G_2 \to SU(3) \times \mathbf{1}$; and
- Decompose G_2 reps to $SU(3) \times \mathbf{1}$ reps;
- Example: $\varphi \to \sigma \wedge \omega + \Omega_+$ (familiar invariants from complex geometry).

Perturbative moduli

ACS and local coordinates

- A *choice* of ACS can be used as a tool to construct local coordinates on moduli space:
  - Split $T(\mathfrak{G}_2)$ into variations *along* the vector "preserving the ACS", and orthogonal "deformation of ACS".
  - Both are easier to control: preserving ACS $\implies$ $SU(3)$ deformations; orthogonal directions are like δv.
  - Expect better control of $SU(3)$ deformations $\sim \mathcal{N} = 2$.

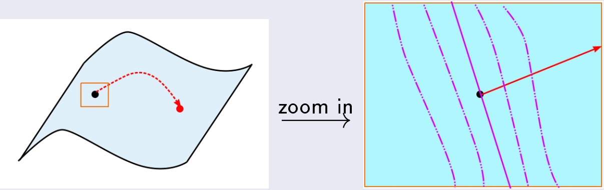

Figure: The blue background represents G_2 moduli space, the red arrow a path of G_2 structures, connecting Ψ with $v \cdot \Psi$

Several extra spinors!

- Earlier: a basic topological invariant (Euler number) + geometry (G_2 structure) lead to ACMS;
- More sophisticated topological obstructions ([E. Thomas, 1969]) + G_2 leads to more sophisticated structure, *almost contact 3-structure* ([Kuo, 1970]) .

Sketch definition

An almost contact three-structure is a triple of unit vectors (v_1, v_2, v_3) satisfying $v_1 \times_\varphi v_2 = v_3$.

- That this actually defines AC3S is due to [Todd, 2015]
- The upshot is that we have *four* nowhere vanishing spinors, and reduced structure group to $SU(2)$.

What is $\mathbf{3}\mathcal{AC}$?

Fibrewise

For each $y \in Y$, we have that an AC3S is given by a triple of ON vectors, satisfying an algebraic condition:

$$v_1 \times v_2 = v_3 \,.$$

So, uniquely fixed by *two* ON vectors.
This space is well understood, a "Stiefel manifold":

$$V_2 := G_2 / SU(2)$$

Harvey,Lawson,1981

Globally, fibre bundle $\mathcal{V}_2$, associated to TY with typical fibre V_2.
An AC3S is a section of this bundle:

$$\mathbf{3}\mathcal{AC} = \Gamma_Y(\mathcal{V}_2) \,.$$

Relation to branes

Integrability

- An AC3S defines two subbundles of tangent bundle:
  1. $\mathcal{T} := \mathrm{span}(v_1, v_2, v_3) \subset TY$, trivial rank 3 bundle;
  2. $\mathcal{T}^{\perp} \subset TY$, a rank four bundle, trivial if and only if it has a nowhere zero section (that's pretty weird for a vector bundle).
- Natural to ask about integrability, i.e. are they actually tangent to a submanifold at any given point? Globally?
- Answer is given by Frobenius theorem.
- Our interest is the following statement: let $X \subset Y$ a three-cycle such that $TX \subset TY$ is equal to $\mathcal{T}|_X$. Then
  1. $[v_i, v_j] = f_{ij}^k v_k$ (definition of integrability);
  2. X is an associative cycle (follows from definition of AC3S).

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  1. $[v_i, v_j] = f_{ij}^k v_k$ (definition of integrability);
  2. X is an associative cycle (follows from definition of AC3S).
- Similarly $Z^4 \subset Y$ is such that $TZ = \mathcal{T}^\perp|_Z$ only if Z is coassociative.

Conclusion and Outlook

Conclusions

- G_2 structure manifolds host a range of bonus structures;
- We can use these bonuses to probe topology of moduli space, locally and globally;
- The geometry of these bonuses can be related to extremely interesting geometry of G_2 structure

Outlook

- Better understanding of the comparison between G_2 structure and these spaces;
- Explore these geometric aspects;