Towards $T\bar{T}$ Deformations of Warped CFTs

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Motivation

Warped CFT

- WCFTs are simple examples of non-relativistic QFT with lots of symmetry.
- Their holographic duals are Warped AdS_3 spaces. $WAdS_3$ black holes are a prototype for non-AdS holography.

$T\overline{T}$ deformations

- Example of an irrelevant operator which is well defined up to arbitrary scales.
- Many quantities can be exactly computed in the deformed theory
- The construction can be generalized to other deformations such as $J\overline{T}$.

Factorization of the $T\overline{T}$ operator

- The $T\overline{T}$ operator is defined as the determinant of the stress tensor in 2 dimensions:

$$\mathcal{O}_{T\overline{T}} = \det T = \epsilon^{\mu\rho}\epsilon^{\nu\sigma} T_{\mu\nu} T_{\rho\sigma}$$

where the product of operators is carefully handled by point splitting.

- Using translational invariance and conservation of the stress tensor, we can show that the expectation value of the determinant of the stress tensor in a 2d QFT is a constant independent of position:

$$\epsilon^{\mu\rho}\epsilon^{\nu\sigma} \langle T_{\mu\nu}(z) T_{\rho\sigma}(w)\rangle \equiv \langle \mathcal{O}_{T\overline{T}}\rangle = \text{constant}$$

- Using cluster decomposition one can factorize the matrix elements of the above vacuum expectation value:

$$\langle n|\mathcal{O}_{T\overline{T}}|n\rangle = \epsilon^{\mu\rho}\epsilon^{\nu\sigma} \langle n|T_{\mu\nu}(z)\,|n\rangle\langle n|\,T_{\rho\sigma}(w)|n\rangle = \text{constant}_n$$

[Zamalodchikov: hep-th/0401146]

Deforming a QFT

- One can deform a theory with this irrelevant operator:

$$\frac{\partial S_\lambda}{\partial \lambda} = \int d^2 z\, \mathcal{O}^{(\lambda)}_{T\overline{T}} \implies \partial_\lambda E_n = \langle n | \mathcal{O}^{(\lambda)}_{T\overline{T}} | n \rangle$$

- On a cylinder of circumference R, this relation takes the form of a differential equation for the deformed energy spectrum:

$$\frac{\partial E_n}{\partial \lambda} = E_n \frac{\partial E_n}{\partial R} + \frac{P_n^2}{R}$$

- This is a universal relation which holds true in any $T\overline{T}$ deformed QFT.
- Other quantities which can be calculated in the deformed theory are the Lagrangian, S-matrix, torus partition function, etc.

[Zamalodchikov: hep-th/0401146]

Holographic interpretations of $T\overline{T}$ deformations

There are a few interpretations of how a gravitational dual theory deforms when the boundary theory undergoes a $T\overline{T}$ deformation.

- **Random boundary geometry:**
  The dual geometry is an ensemble of AdS_3 spaces or BTZ black holes without a finite cutoff, but instead with randomly fluctuating boundary diffeomorphisms through a Hubbard-Stratanovich transformation. [Cardy: 1801.06895][Hirano, Shigemori: 2003.06300]
- **Finite radial cutoff with Dirichlet boundary conditions:**
  The deformation parameter and the finite radial cutoff of a BTZ black hole with Dirichlet boundary conditions obey $\lambda \sim r_c^{-2}$.
  [McGough, Mezei and Verlinde: 1611.03470]
- **Mixed boundary conditions through a variational principle:**
  This approach derives "mixed" boundary conditions without introducing a finite radial cutoff. If you exclude matter fields, the mixed boundary conditions can be interpreted as the finite radial cutoff result. [Guica, Monten: 1906.1151]

Holographic $T\overline{T}$ via the Variational Principle

- In the large N limit:

$$\partial_\lambda \delta S = \delta \partial_\lambda S \implies \partial_\lambda \frac{1}{2} \int_{\partial \mathcal{M}} d^2x \sqrt{\gamma}\, T^{(\lambda)}_{\mu\nu} \delta \gamma^{(\lambda)\mu\nu} = \delta \int_{\partial \mathcal{M}} d^2x \sqrt{\gamma}\, \mathcal{O}^{(\lambda)}_{T\overline{T}}$$

- Solving this, the boundary metric γ is quadratic in the deformation parameter λ:

$$\gamma^{(\lambda)}_{\mu\nu} = \gamma^{(0)}_{\mu\nu} - 2\lambda \hat{T}^{(0)}_{\mu\nu} + \lambda^2 \hat{T}^{(0)}_{\mu\rho} \hat{T}^{(0)}_{\nu\sigma} \gamma^{(0)\rho\sigma}$$
$$\hat{T}^{(\lambda)}_{\mu\nu} = \hat{T}^{(0)}_{\mu\nu} - \lambda \hat{T}^{(0)}_{\mu\rho} \hat{T}^{(0)}_{\nu\sigma} \gamma^{(0)\rho\sigma}$$
$$(\hat{T}_{\mu\nu} \equiv T_{\mu\nu} - \gamma_{\mu\nu} T^{\rho}{}_{\rho})$$

- If there are no matter fields, the solution of the boundary metric can be written as a Fefferman-Graham expansion with

$$\rho_c = -\frac{\lambda}{4\pi G l}$$

which agrees with the finite cutoff result.

[Guica, Monten: 1906.11251]

AdS$_3$ gravity in the Chern-Simons formalism

- The Einstein Hilbert action with AdS$_3$ boundary conditions can be written as an SL$_2(\mathbb{R})\times$ SL$_2(\mathbb{R})$ Chern-Simons theory: [Witten '88]

$$S_{\mathsf{EH}} = I_{\mathsf{CS}}(A) - I_{\mathsf{CS}}(\bar{A}); \quad I_{\mathsf{CS}} = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right)$$

where the level $k = \frac{l}{4G}$, and the gauge fields are $A = \omega + e$ and $\bar{A} = \omega - e$ in terms of the spin connection and vielbein.

- Solutions for the gauge fields are of the following form:

$$A = b(\mathrm{d}+a)b^{-1}, \quad \bar{A} = b(\mathrm{d}+\bar{a})b^{-1}$$
$$a = (L_{-1} + \mathfrak{L}(z)L_1)dz, \quad \bar{a} = (L_1 + \bar{\mathfrak{L}}(\bar{z})L_{-1})d\bar{z}, \quad b = e^{\rho L_0}$$

which reproduces a Bañados metric, i.e. all asymptotically AdS solutions with a flat boundary parametrized by $\mathfrak{L}$ and $\bar{\mathfrak{L}}$ (L_i are SL$_2(\mathbb{R})$ generators). [Bañados: hep-th/9901148]

Holographic $T\overline{T}$ deformation in Chern-Simons formulation

- Approach 1: use a "state dependent" transformation of the boundary coordinates in $A, \bar{A}$ [He, He, Gao: 2109.12885]

$$dz \to \frac{dz - \lambda\bar{\mathfrak{L}}\, d\bar{z}}{1-\lambda^2\mathfrak{L}\bar{\mathfrak{L}}}, \quad d\bar{z} \to \frac{d\bar{z} - \lambda\mathfrak{L}\, dz}{1-\lambda^2\mathfrak{L}\bar{\mathfrak{L}}}$$

- Approach 2: generalize the solution of the gauge fields

$$a_\mu = 2e^1_\mu L_1 - f^{-1}_\mu L_{-1} + \omega_\mu L_0, \quad \bar{a}_\mu = f^1_\mu L_1 - 2e^{-1}_\mu L_{-1} + \omega_\mu L_0$$

and include the appropriate boundary term for the variation of the action to vanish on the boundary. This allows you to write the boundary stress tensor and $\mathcal{O}_{T\overline{T}}$ as :

$$T^\mu_a = \frac{k}{\pi}\epsilon_{ab}\epsilon^{\mu\nu} f^b_\nu; \quad \partial_\lambda S = \frac{1}{2}\left(\frac{k}{\pi}\right)^2 \int_{\partial\mathcal{M}} f^{-1}\wedge f^{+1}$$

Using $\partial_\lambda \delta S = \delta\partial_\lambda S$ one can derive the appropriate flow equations in the Chern-Simons formalism. [Llabrés: 1912.13330]

Symmetries of Warped CFTs

Warped CFTs are 2 dimensional non-Lorentz invariant QFTs with the following properties:

- Translation invariance in both coordinates x, t
- Scaling invariance in one coordinate, x
- Non-Relativistic Boost invariance
- Not Lorentz Invariant $\implies$ Do not couple to (Pseudo)-Riemannian metrics.
- Conserved charges of the translation currents form a U(1) Virasoro-Kac Moody algebras.

[Hofman, Rollier: 1411.0672]

Warped Geometry

- We can define a boost invariant vector and 1 form : $\bar{q}^a, q_a; q_a\bar{q}^a = 0$
- Using these one can define a boost invariant 2-form which acts as a metric: $g_{ab} = q_a q_b$, however this object is degenerate. One can define non-degenerate anti-symmetric tensors: h_{ab}, h^{ab} to raise and lower indices.
- τ_μ^a are warped vielbeins mapping spacetime vectors to warped geometry variables.
- The warped volume form is defined as $H = \epsilon^{\mu\nu}\tau_\mu^a\tau_\nu^b h_{ab}$. Notice that we can not use $\det g$ since it is 0.
- Coupling a Warped CFT to a warped background, we get:

$$\delta S = \int d^2x H(J_a^\mu \delta\tau_\mu^a) \stackrel{*}{=} \int dx\, dt (T(x)\delta q_t + P(x)\delta\bar{q}_t)$$

- The non-zero components of the translation currents T and P generate the Virasoro-Kac Moody algebras respectively.

Holographic construction of a WCFT

- The bulk dual of a WCFT is a Warped AdS$_3$. The (lower spin) gravity theory can be written as a SL$_2(\mathbb{R})\times$ U(1) Chern-Simons theory. [Hofman, Rollier: 1411.0672]

$$S_W = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{tr}\left(\mathcal{A}\wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) + \frac{\kappa}{8\pi}\int_{\mathcal{M}} \mathcal{C}\wedge \mathrm{d}\mathcal{C}$$

- The appropriate solutions of the gauge fields which allow for black hole solutions in the bulk are

$$\mathcal{A} = b^{-1}(\mathrm{d}+a)b, \quad \mathcal{C} = \frac{4\pi}{\kappa}\mathcal{K}d\phi + \left(\nu + \frac{4\pi}{\kappa}\mathcal{K}\mu\right)dt$$

$$a = (L_1 + \mathfrak{L}L_{-1})d\phi + (\mu L_1 + \omega_1 L_0 + \omega_2 L_{-1})dt, \quad b = e^{\rho L_0}$$

where $\mathfrak{L} = \frac{2\pi}{k}(\mathcal{L} - \frac{2\pi}{\kappa}\mathcal{K}^2)$.

- This corresponds to the "vacuum" WAdS$_3$ solution if $\mu = \omega_i = 0$ and $\nu = 1$. [Azeyanagi, Detournay, Riegler: 1801.07263]

Variational Principle and asymptotic symmetries

- Computing the boundary charges in terms of the functions $\mathcal{L}$ and $\mathcal{K}$, one obtains the U(1) Virasoro-Kac Moody algebra once again, with charges of $\mathcal{L}$ generating the Virasoro algebra and charges of $\mathcal{K}$ generating the U(1) Kac-Moody algebra.
- Adding an appropriate boundary term to ensure the total variation of the action is 0 when chemical potentials μ and ν are fixed on the boundary, one can show that the on-shell variation of the action when the chemical potentials are allowed to vary is

$$\delta S_W = \int_{\partial\mathcal{M}} (\mathcal{L}\delta\mu + \mathcal{K}\delta\nu)$$

- Comparing with the variation of a WCFT action, one can identify $\mathcal{L}, \mathcal{K}, \mu, \nu$ with $T, P, q_t\bar{q}_t$ respectively.

Towards $T\overline{T}$ deformations of WCFT

- There are a few obstacles in understanding how to perform a $T\overline{T}$ deformation of a WCFT. For example, due to the lack of a non-singular background metric, it is not possible to employ the holographic variational methods by Guica and Monten to compute the flow equations of the Warped geometry variables and other WCFT quantities.
- However, the Chern-Simons formulation of WCFT and $T\overline{T}$ give us a window into a possible way of overcoming this obstacle.
- A promising line of approach is to use the generalization of the solutions of $\mathrm{SL}_2(\mathbb{R})\times$ $\mathrm{SL}_2(\mathbb{R})$ gauge fields given by Llabrès to construct a $T\overline{T}$ operator in the Chern-Simons formalism and adapt it to the the $\mathrm{SL}_2(\mathbb{R})\times$ U(1) case of WAdS_3.
- The mass of WAdS_3 black holes dual to $T\overline{T}$ deformed WCFT should solve the Burgers' flow equations, which should arise from the flow equations derived holographically.

Thank you!