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Color-Kinematics duality for Chern-Simons theory

Maor Ben-Shahar

Nordic Network Meeting, November 2021

Based on:

[M.B.S, M. Guillen, 2108.11708]

[M.B.S, H. Johansson, 21XX.XXXXX]

Outline

- Motivation
- Review of BCJ relations and double copy
- $\mathcal{N} = 4$ Chern-Simons matter and $\mathcal{N} = 8$ DBI
- Off-shell color-kinematics duality
- Kinematic algebra in pure CS theory

Motivation and main results

- Ongoing search for gauge theories obeying the color-kinematics duality [recent review: 1909.01358]
- Off-shell color-kinematics is still an open problem.
- Observation: typically conformal theories [C. Cheung, J. Mangan,C-H. Shen]
- Suggests 3d CS theory + matter should be re-examined
- 3-Lie algebra in $\mathcal{N} = 6$ ABJM and $\mathcal{N} = 8$ BLG studied before [T. Bargheer, S. He, T. McLoughlin; H. Johansson, Y-T. Huang].

We find:

- $\mathcal{N} = 4$ CS-matter theory obeys the standard CK duality
- $\mathcal{N} = 8$ Dirac-Born-Infeld (DBI) obtained as double copy
- Pure-CS theory has off-shell CK duality, off-shell double copy

Review: Bern-Carassco-Johansson (BCJ) relations

Amplitude decomposition for adjoint particles:

$$\mathcal{A}_n = \sum_\sigma A(1, \sigma_2, \ldots, \sigma_n) \mathrm{Tr}(T^{a_1} T^{a_{\sigma_2}} \ldots T^{a_{\sigma_n}})$$

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BCJ relations: $(n-3)!$ basis of partial amplitudes for many gauge theories; (S)YM, YM+matter etc.

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Example ($s_{ij} \equiv (p_i + p_j)^2$):

$$s_{23} A(1234) = s_{13} A(1243) \; .$$

Relations imply underlying color-kinematics duality/double copy.

Double Copy

KLT double copy [Kawai-Lewellen-Tye]

$$\mathcal{M}_4 = s_{12} A(1234) A(1243)$$

$$\vdots$$

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Double copy is a bi-linear operation: "$\mathrm{GT_L} \otimes \mathrm{GT_R}$".

Applies for Einstein gravity+matter, supergravity.

Chern-Simons matter

Want to find CS matter theory obeying BCJ amplitude relations

$$\mathcal{L} = \frac{\epsilon_{\mu\nu\rho}}{2}\left(A^{a\mu}\partial^{\nu}A^{a\rho} - \frac{gi}{3}f^{abc}A^{a\mu}A^{b\nu}A^{c\rho}\right) + (D_\mu\bar{\phi})^a(D^\mu\phi)^a \ .$$

Coupling: $g = \sqrt{\frac{4\pi}{k}}$

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$$\begin{aligned} A_4(\phi_1\bar{\phi}_2\phi_3\bar{\phi}_4) &= 2\epsilon^{p_1p_2p_3}\left(\frac{(-1)^{|\phi|}}{s_{23}} - \frac{1}{s_{12}}\right) \;, \\ A_4(\phi_1\bar{\phi}_2\bar{\phi}_4\phi_3) &= \frac{2\epsilon^{p_1p_2p_3}}{s_{12}} \;. \end{aligned}$$

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BCJ relations: $s_{23}A(1234) = s_{13}A(1243)$ imply $|\phi| = 1$.

Chern-Simons matter

Now consider adding fermions:

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$$+ \alpha^{(1)}g^4\phi^a\bar{\phi}^b\bar{\phi}^d\phi^c\phi^e\bar{\phi}^h f^{abx}f^{xdy}f^{ycz}f^{zeh}$$

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BCJ relations imply:

- $\alpha^{(i)} = 1$
- Opposite statistics (!)
- $\mathcal{N} = 4$ SUSY, Q^{α}_{A}

$$A(\Psi_1\Psi_2\Psi_3\Psi_4) = \delta^4(Q)\frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

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Same partial amplitudes as $\mathcal{N} = 4$ CS with bi-fundamental matter [D. Gaiotto,E. Witten], and $\mathcal{N} = 4$ truncations of ABJM

Double Copy

What is the double copy of this $\mathcal{N}=4$ CS theory?

$$\mathcal{M}_4 \quad = \quad s_{12}A(1234)A(1243) = \delta^8(Q)$$

Match DBI with Maximal SUSY in 3D.

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Match DBI with Maximal SUSY in 3D.

Correct statistics after double copy!

$$(\phi + \psi + c.c.) \otimes (\phi + \psi + c.c.) \to \phi_i + \psi_i$$

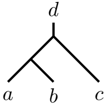
R-symmetry: $SO(4) \times SO(4) \to SO(8)$

Off-shell color-Kinematics duality

Relationship between scattering amplitude building blocks:

[Bern-Carassco-Johansson]

$$\mathcal{A}_n = \sum_{i \in \Gamma_n} \frac{c_i n_i}{D_i}$$



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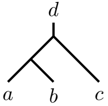

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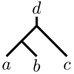
$$f^{abx} f^{xcd} + f^{cax} f^{xbd} + f^{bcx} f^{xad} = 0$$

$c\,\Big($


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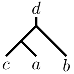

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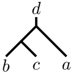

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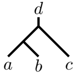
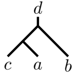
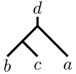
$$c_i + c_j + c_k = 0 \quad \Leftrightarrow \quad n_i + n_j + n_k = 0 \ ,$$

For the previous example we need:

$n\Bigg($  $\Bigg) + n\Bigg($  $\Bigg) + n\Bigg($  $\Bigg) = 0$

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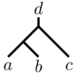
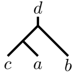
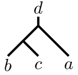
$n\Bigg($ 

$\Bigg) + n\Bigg($ 

$\Bigg) + n\Bigg($ 

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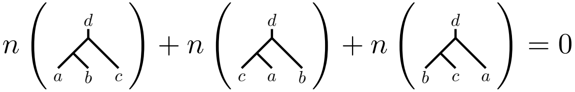

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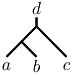
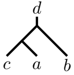
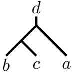
$n\Big($ 

$\Big) + n\Big($ 

$\Big) + n\Big($ 

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Q: is there a kinematic algebra responsible for the kinematic identities?

Kinematic algebra in Chern-Simons theory

Consider Axelrod-Singer formulation:

$$S = \frac{k}{2\pi} \int d^3x d^3\theta \, \mathrm{Tr} \left(\frac{1}{2} \Psi Q \Psi + \frac{i}{3} \Psi \Psi \Psi \right) \ ,$$

Superfield:

$$\Psi = c + \theta_\mu A^\mu + \theta_\mu \theta_\nu \epsilon^{\mu\nu\rho} \partial_\rho \bar{c} \ ,$$

(World line) BRST operator

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To define the propagator:

- $b = \frac{\partial}{\partial \theta^\mu}\partial^\mu$:
  - $b^2 = 0$
  - $bQ + Qb = \partial_\mu \partial^\mu = \Box$

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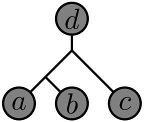

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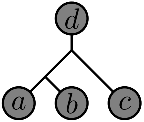

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$$b(b(\Psi_a \Psi_b)\Psi_c) + \text{cyclic}(a, b, c) = 0$$

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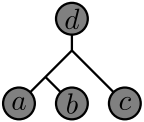

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Using $b^2 = 0$ which implies $b(\Psi_i) = 0$. Jacobi identity of

$$L_\psi(f) \equiv b(\psi f)\ , \qquad [L_\psi, L_\phi] = L_{b(\psi\phi)}$$

$$f(x,\theta) \to f(x + \partial^{(\theta)}\phi, \theta + \partial\phi) = f + b(\phi f) + b(\phi b(\phi f)) + ...$$

diffeos in (x,θ) space that preserve the gauge $bf = 0$,

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Linearized cotton tensor appears in expansion of LHS

$$C^{\mu\nu} = \epsilon^{\mu\alpha\beta} D_\alpha R^\nu_\beta + (\mu \leftrightarrow \nu) \ ,$$

$C^{\mu\nu} = 0$ follows from the action

$$S_{\mathrm{CSgrav}} = \frac{1}{2\pi} \int d^3 \epsilon^{\mu\nu\rho} \left(\Gamma^\alpha_{\mu\beta} \partial_\nu \Gamma^\beta_{\rho\alpha} + \frac{2}{3} \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\gamma} \Gamma^\gamma_{\rho\alpha} \right) \ .$$

Conclusion

- Obtained CS-matter theories satisfying CK duality, with SUSY
- Double copy to DBI, up to maximal $\mathcal{N} = 8$ SUSY
- Off-shell CK duality follows from Feynman rules of

$$S = \langle \Psi Q \Psi + \Psi^3 \rangle$$

- Works in SYM with Berkovits formulation [M. Guillen, MBS. 2108.11708]
- Similar actions for other theories exist
- Open problems, e.g. Wilson loops, double copy actions
- Off-shell double is possible, and kinematic algebra exists!