Constructing the 6 Loop 4 Point $\mathcal{N} = 4$ sYM Integrand

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with

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Roadmap

Motivations

Goal of program: UV behavior of 7 loop $\mathcal{N} = 8$ SUGRA
Why?

- SUSY arguments predict $L = 7$ counterterm in $D_c = 4$ (Bossard, Howe, Stelle; Green, Russo, Vanhove; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger; many more)
- Similar counterterms proven absent for $\mathcal{N} = 4, 5$ at $L = \mathcal{N} - 1$ (Bossard, Howe, Stelle, Vanhove; Bern, Davies, Dennen; Bern, Davies, Dennen, Huang)
- Improved behavior observed in $D = 4$ kinematics (AE, Hermann, Parra-Martinez, Trnka)

History of direct calculations:

- 1&2 loops '80 -'90s (Green, Schwarz, Brink; Bern, Dixon, Dunbar, Perelstein, Rozowsky)
- 3 loops '07-'10 (Bern, Carrasco, Dixon, Johansson, Kosower, Roiban)
- 4 loops '09-'12 (Bern, Carrasco, Dixon, Johansson, Roiban)
- 5 loops 2018 (Bern, Carrasco, Chen, AE, Johansson, Parra-Martinez, Roiban, Zeng)

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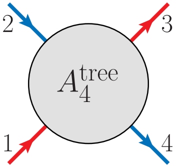
Computational Challenges

Obstacle	Solution
Many Feynman diagrams, cancellations between diagrams	Cuts contain minimal needed data
Cuts from state sums: 256 states per cut propagator	Double copy: $\text{GR} = \text{YM}^2$
sYM state sums still hard, rapidly exploding γ traces	Color-kinematics + tricks for special cuts
Problems with CK at 5L – still need to solve previous problems	New recursive tools for cuts & integrands

X Identity

Four-point ordered YM tree amplitudes only have s and t channel poles.

What if we try to "sit on the u pole" anyway, via $p_3 \to p_1, p_4 \to p_2$?



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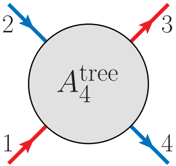

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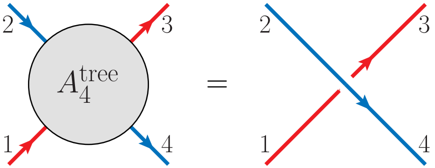
Dimensionless, must respect all symmetries: Can only get identity insertions! The diagram disconnects!



e.g.: $$A_4^{\text{tree, YM}} = \frac{t_8 F^4}{s\ t} \xrightarrow{p_3=p_1} (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4)$$

Same for all other supersymmetric states

H Identity

Can we find a similar identity that maintains planarity?

- Consider $s_{13}A(1,3,2,4)|_{p_3=p_1}$: cut with zero momentum exchange
- Apply $s_{13}A(1,3,2,4) = -s_{12}A(1,2,3,4)$
- Use X ID on RHS: $s_{12}A(1,2,3,4) \to -s_{14}A(1,3)A(2,4)$

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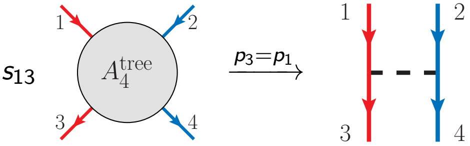

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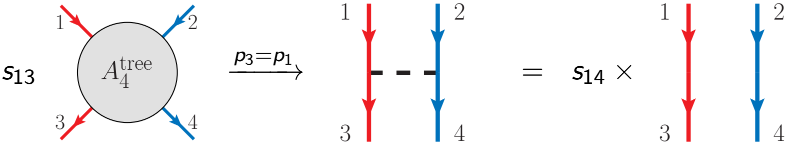

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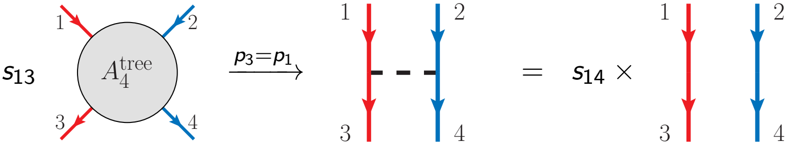

e.g.: $s_{13}A(1,3,2,4) = \dfrac{t_8 F^4}{s_{23}} \xrightarrow{p_3=p_1} s_{14}(\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4)$

Same for all other supersymmetric states. Extends to (super)gravity.
N.B.: Physical identity (soft factorization), not heuristic rule.

Method of Maximal Cuts (MMC)

(Bern et al)

Systematic integrand construction from cuts

1. Enumerate diagram basis, striate by *cut depth* (k)
   1. Build all cubic vacuums
   2. Attach four external legs
   3. Collapse internal legs
   4. Cubic = max; one quartic = next-to-max; ...
2. Each diagram γ corresponds to both a cut and a numerator
3. Proceed by cut level, matching cuts by inheriting poles from lower-k and constructing new numerators

$$\mathcal{P}^{(k)}_{\gamma,\text{ans}} = \mathcal{C}^{(k)}_{\gamma} - \mathcal{R}^{(k)}_{\gamma,\text{MMC}}$$

Local polynomial — Cut – truth — Rational

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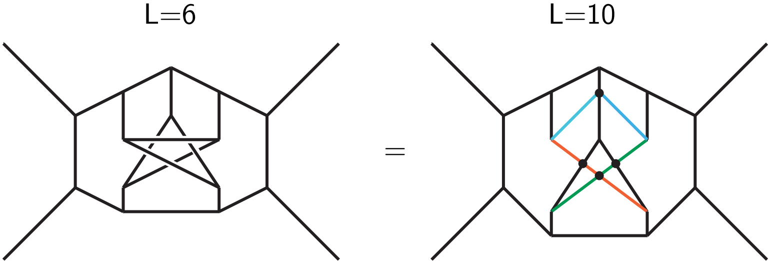
Trying to determine — Need to eval — Known higher cuts: $\sum \frac{n}{p^2}$

Applications of X ID in MMC

X Identity: Evaluate $\mathcal{C}^{(k)}$ *directly* from limit of higher-loop cut



Challenges:

1. Edge crossing is NP-Hard[1]
2. Quickly outpace known planar cuts (11+ loops)
3. Only works for color-ordered cuts

[1] ex: There're better crossing schemes than in the diagram. Can you find one?

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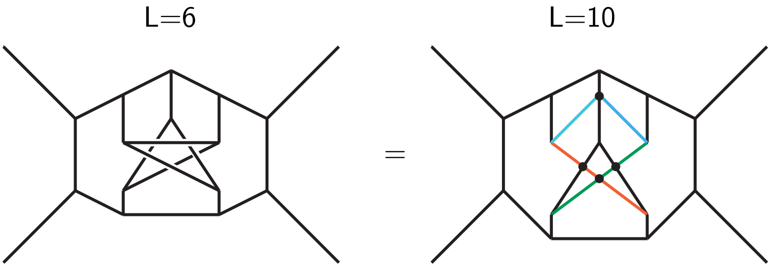

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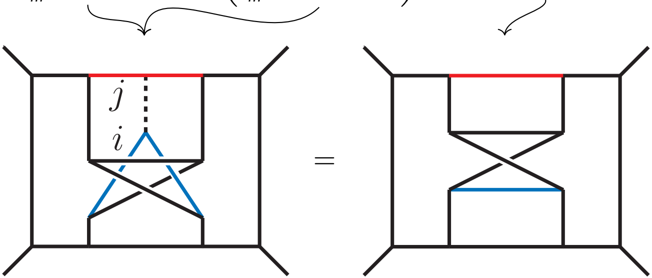

Challenges:

1. Need to merge conditions
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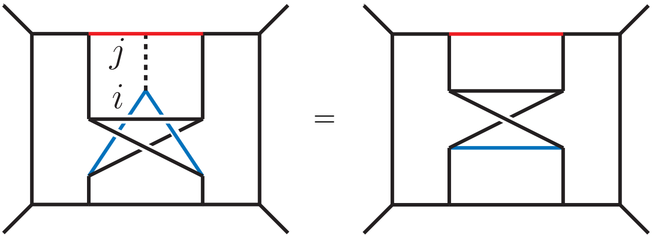

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Resolving the conditions

Fix a basis of momentum invariants

$\lim_{\ell_m \to 0}$ induces linear relations between invariants: π_{ℓ_m}

- Brute force: use an ansatz
  - $\mathcal{P}_{ans}$ as literal polynomial ansatz in basis
  - H ID gives linear equations between ansatz parameters
- More clever: intersection of polynomial ideals

$$\pi_{\ell_m}\mathcal{P}^{(k)}_{\gamma,ans} = h_{\gamma,\ell_m}\mathcal{C}^{(k)}_{\gamma\backslash\ell_m} - \pi_{\ell_m}\mathcal{R}^{(k)}_{\gamma,MMC}$$

$$\Rightarrow \mathcal{P}^{(k)}_{\gamma,ans} = \pi_{\ell_m}\mathcal{P}^{(k)}_{\gamma,ans} + \ker\pi_{\ell_m} \subset \left\langle \pi_{\ell_m}\mathcal{P}^{(k)}_{\gamma,ans}\,,\pi_{\ell_m}\right\rangle$$

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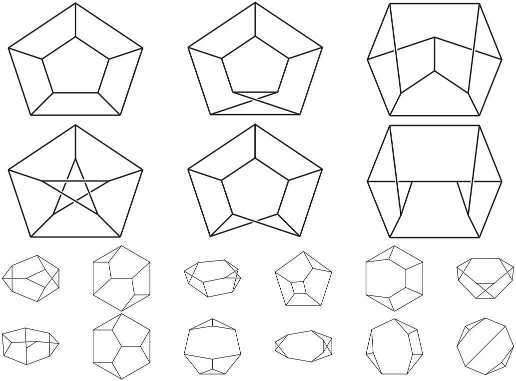

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# cuts	5548	41649	156853	363963	576582	706,281

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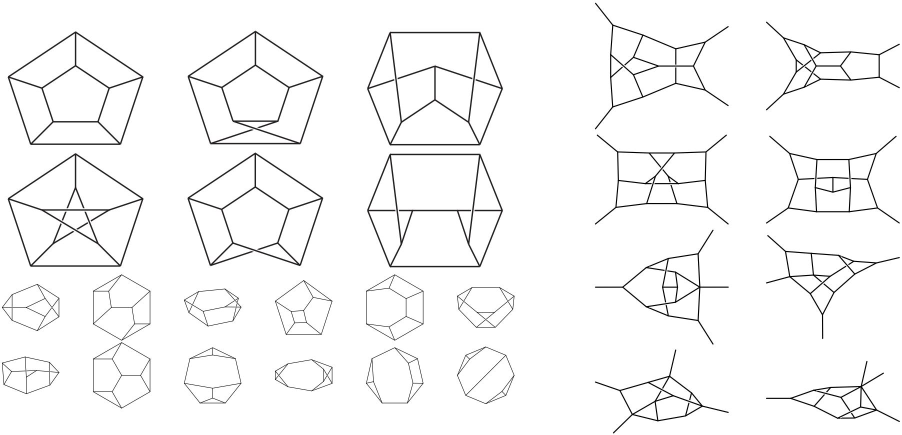

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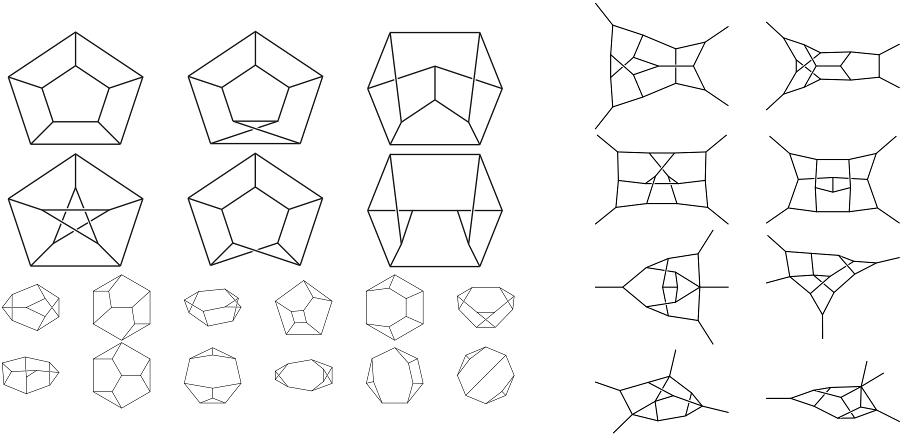

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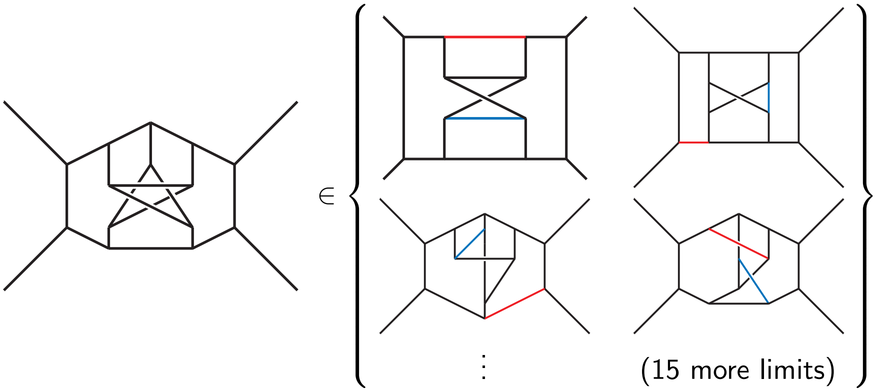
6L Numerator Construction



The 6L Integrand

N^kM	0	1	2	3	4	$\sum$
cuts	5548	41649	156853	363963	576582	1.14×10^6
non-zero contacts	4420	16776	38000	54000	33000	1.46×10^5

Fits on a CD!

Contact terms carry half-ladder color factors

- Longest numerator: 62,511 terms
- Shortest numerator: 1 term (ladder diagrams)
- Average numerator: 90 terms

Looking Forward

- UV Integration
- Prepping tools for SUGRA: KLT, IBPs
- Improve efficiency for 7 loops
  - Intersection of ideals is senstive to many superficial choices
  - Minimize number of limits to evaluate
  - Ansatz requires efficient inversion/row reduction
- Application to other theories: QCD, open string eff.
- Cubic representation: generalized double-copy, color-kinematics duality?

Thanks!

Questions?

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