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SYM Theories on $\mathbb{CP}^2$

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Motivation

Compute $d = 4$ $\mathcal{N} = 2$ SYM partition functions on compact manifolds with a T^2-action and a Killing vector with isolated fixed points

Two conjectures:

- Fluxes enter the partition functions as a shift of the Coulomb branch parameter [Nekrasov (2005)]
- Any distribution of instanton or anti-instantons at each fixed point gives rise to a supersymmetric theory [Festuccia, Qiu, Winding, Zabzine (2018)]

  $\Rightarrow$ Donaldson-Witten theory on S^4 corresponds to instantons at both poles. Flipping one of them to an anti-instanton gives rise to Pestun theory on S^4

Introduction: $5d$ $\mathcal{N} = 1$ SYM

Five dimensional $\mathcal{N} = 1$ SYM on simply connected Sasaki-Einstein manifolds with T^3-action and a Killing vector v with isolated fixed fibers

$\Rightarrow$ Only configuration invariant under the T^3-action contribute, and localize around the fixed orbits of the Killing vector [Qiu, Zabzine (2016)]

$$F_H^+ = 0, \quad \iota_v F = 0 \Rightarrow \text{Contact instanton:} \star F = -\kappa \wedge F$$
$$D_\mu \sigma = 0$$

Around each fixed fiber the manifold looks like twisted $\mathbb{C}^2 \times S^1$

$$Z = \int_{\mathfrak{h}} d\sigma_0 e^{-\frac{8\pi^3 r}{g_{YM}^2} \varrho \mathrm{Tr}[\sigma_0^2]} \cdot \det{}'_{adj} S_3^C(i\sigma_0|v) \prod_{i=1}^{r} Z^{\mathrm{Nekrasov}}_{\mathbb{C}^2 \times S^1}(\beta_i, \epsilon_i, \epsilon_i')$$

Introduction: S^5

Perturbative partition function on squashed $S^1 \hookrightarrow S^5 \to \mathbb{CP}^2$

$$Z_{S^5}^{pert} = \prod_{\alpha \in roots} \prod_{n_1,n_2,n_3 \geq 0} (\boldsymbol{n} \cdot \boldsymbol{\omega} + i\alpha(\sigma_0)) \prod_{n_1,n_2,n_3 \geq 1} (\boldsymbol{n} \cdot \boldsymbol{\omega} - i\alpha(\sigma_0))$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$ are eigenvalues under T^3-action and squashing parameters $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$

To localize one needs to pick a pair of supercharges $\mathcal{Q} + \overline{\mathcal{Q}}$, $\mathcal{Q} - \overline{\mathcal{Q}}$ squaring to bosonic transformations which include a $U(1) \subset T^3$

$\Rightarrow$ The Killing vector field generating this transformation

$$v = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$$

where (e_1, e_2, e_3) is the vector field of T^3-action on $\mathbb{C}^3$

Dimensional Reduction 1

Two choices of fiber, with respect to v, for dimensional reduction

- $x^{top} = +e_1 + e_2 + e_3 \sim v$
- $x^{ex} = -e_1 + e_2 + e_3$ orthogonal to v

Two different quantum numbers for rotation along the fiber

$$t^{top,ex} = \pm n_1 + n_2 + n_3$$

On the (squashed) $\mathbb{CP}^2$ base manifold we find two theories

- (Equivariant) topologically twisted Donaldson-Witten theory, $v_{4d} = 0$ in the unsquashed limit
- Exotic theory, closely related to Pestun's theory on S^4, $v_{4d} \neq 0$ in the unsquashed limit

Perturbative Partition Function on S^5

We substitute t^{top}, t^{ex} in the perturbative partition function

$$Z^{top}_{S^5} = \prod_{t^{top}\geq n_2+n_3} \prod_{n_2,n_3\geq 0} (\omega_1 t^{top} + \epsilon_1^{top} n_2 + \epsilon_2^{top} n_3 + i\alpha(\sigma_0)) \times (\dots)$$

$$Z^{ex}_{S^5} = \prod_{t^{ex}\leq n_2+n_3} \prod_{n_2,n_3\geq 0} (-\omega_1 t^{ex} + \epsilon_1^{ex} n_2 + \epsilon_2^{ex} n_3 + i\alpha(\sigma_0)) \times (\dots)$$

$$\epsilon_1^{top} = \omega_2 - \omega_1, \quad \epsilon_2^{top} = \omega_3 - \omega_1$$

$$\epsilon_1^{ex} = \omega_2 + \omega_1, \quad \epsilon_2^{ex} = \omega_3 + \omega_1$$

Around each of the three fixed fibers the manifold looks like $\mathbb{C}^2 \times S^1$

$\Rightarrow$ Factorized result: $\quad Z^{pert}_{S^5} = \prod_{\alpha\in roots} \prod_{i=1}^{3} Z^{pert}_{\mathbb{C}^2\times S^1}$

Lens Space

To perform dimensional reduction we introduce a $\mathbb{Z}_p$ quotient acting freely on the fiber $L^5(p, \pm 1) \equiv S^5/\mathbb{Z}_p$

$\Rightarrow \pi_1(L^5(p, \pm 1)) \cong \mathbb{Z}_p$, p inequivalent complex line bundles

$$A = \mathrm{diag}(A_p^{m_1}, \dots, A_p^{m_k}), \quad 0 \leq m_i < p \text{ and } i = 1, \dots, k$$

Quotient introduces a projection condition on the modes

$$t^{top,ex} = \pm n_1 + n_2 + n_3 = \alpha(\mathfrak{m}) \bmod p, \quad \mathfrak{m} = \mathrm{diag}\,(m_1, \dots, m_k)$$

and a sum over inequivalent flat connections in the partition function

$$Z_{L^5(p,\pm 1)} = \sum_{[\mathfrak{m}]} \int d\sigma_0 e^{-S_{cl}} Z^{pert}_{L^5(p,\pm 1)}(\sigma_0, \mathfrak{m}) Z^{non-pert}_{L^5(p,\pm 1)}(\sigma_0, \mathfrak{m})$$

Dimensional Reduction 2

At large p the quotient shrinks the fiber to a point and the manifold effectively resembles $\mathbb{CP}^2$

- The projection conditions set $t^{top} = \alpha(\mathfrak{m})$ and $t^{ex} = \alpha(\mathfrak{m})$
- The sum over flat connections becomes a sum over flux sectors labeled by their magnetic charge $\mathfrak{m}$

$$Z_{\mathbb{CP}^2} = \sum_{[\mathfrak{m}]} \int d\sigma_0 e^{-S_{cl}} Z^{pert}_{\mathbb{CP}^2}(\sigma_0, \mathfrak{m}) Z^{non-pert}_{\mathbb{CP}^2}(\sigma_0, \mathfrak{m})$$

- Reducing along $v \sim x^{top}$ contact instantons give anti-self-dual instantons at each of the three fixed points
- Reducing along x^{ex} a contact instanton at one fixed fiber of S^5 is flipped into a self-dual anti-instanton at one fixed point of $\mathbb{CP}^2$

Factorized Expressions

Reducing along $v \sim x^{top}$ we find an equivariant version of topologically twisted Donaldson-Witten theory

$$Z^{pert,top}_{\mathbb{CP}^2} = \prod_{\alpha(\mathfrak{m}) \geq n_2+n_3} \prod_{n_2,n_3 \geq 0} \left(\epsilon_1 n_2 + \epsilon_2 n_3 + i\alpha(\sigma_0) + \left(1 - \frac{\epsilon_1+\epsilon_2}{3}\right)\alpha(\mathfrak{m}) \right)(\dots)$$

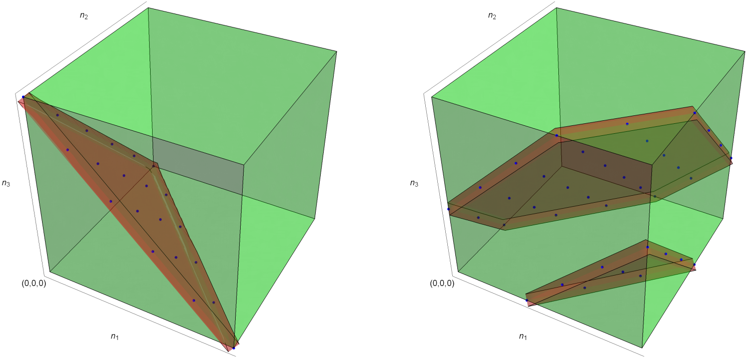
Reducing along x^{ex} we find exotic theories, closely related to Pestun's theory on S^4

$$Z^{pert,ex}_{\mathbb{CP}^2} = \prod_{\alpha(\mathfrak{m}) \leq n_2+n_3} \prod_{n_2,n_3 \geq 0} \left(\epsilon_1 n_2 + \epsilon_2 n_3 + i\alpha(\sigma_0) + \left(\frac{1}{3} - \frac{\epsilon_1+\epsilon_2}{3}\right)\alpha(\mathfrak{m}) \right)(\dots)$$

Difference due to fermionic generators $\mathcal{Q}$, $\overline{\mathcal{Q}}$ squaring to different isometries on the base manifold

More on Unfactorized Expressions

At values of $t^{top,ex} = \alpha(\mathfrak{m})$ we plot the slices of (n_2, n_3) contributing to $Z^{pert}_{\mathbb{CP}^2}$



Left plot is a finite slice for $t^{top} = 5$ and right plot are infinite slices for $t^{ex} = \pm 2$.

Factorized Expression

Similarly we can reduce the factorized expression $\Rightarrow$ at each fixed point of $\mathbb{CP}^2$ the manifold looks like $\mathbb{C}^2$

$\Rightarrow$ Fluxes only enter as a shift of the Coulomb branch parameter σ_0

$$l = 1 \quad \text{top: } i\alpha(\sigma_0) + \alpha(\mathfrak{m})\left(1 - \frac{\epsilon_1 + \epsilon_2}{3}\right)$$
$$\text{ex: } i\alpha(\sigma_0) + \alpha(\mathfrak{m})\left(\frac{1}{3} - \frac{\epsilon_1 + \epsilon_2}{3}\right),$$

Similarly for $l = 2, 3$ fixed points.

Factorized expressions need to be regularized. We use regularization coming from S^5

$\Rightarrow$ Ongoing work with R. Mauch [2112.xxxxx] to derive a purely $d = 4$ approach to regularization

Conclusions

We have found results for the perturbative partition function, at all flux sector, on $\mathbb{CP}^2$, both for topologically twisted and exotic theories:

- We find shifts due to fluxes consistent with Nekrasov's conjecture
- We show how both DW and Pestun-like theories arise from dimensional reduction of same $5d$ $\mathcal{N} = 1$ SYM theory on S^5
- We find sum over a single integer labeling different flux sectors

The procedure is generalizable into two directions:

- Extend reduction to generic $d = 5$ Sasaki-Einstein manifolds
- Consider quaternion projective space $\mathbb{HP}^1 = S^4$

  $\Rightarrow$ Hopf fibration $S^3 \hookrightarrow S^7 \rightarrow S^4$