Axisymmetric dynamo action is possible (with anisotropic conductivity)

“[…] things we thought impossible are possible.” (Axel B.)

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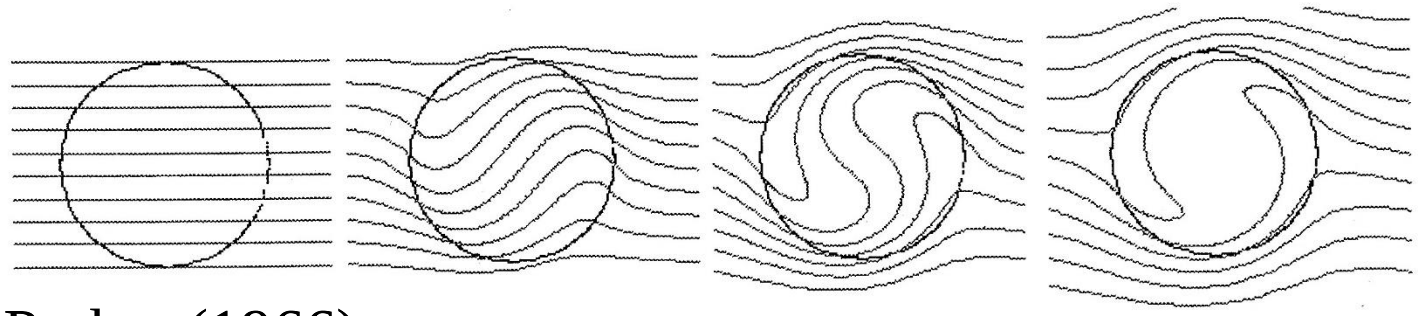
Introduction

"[…] things we thought impossible …"

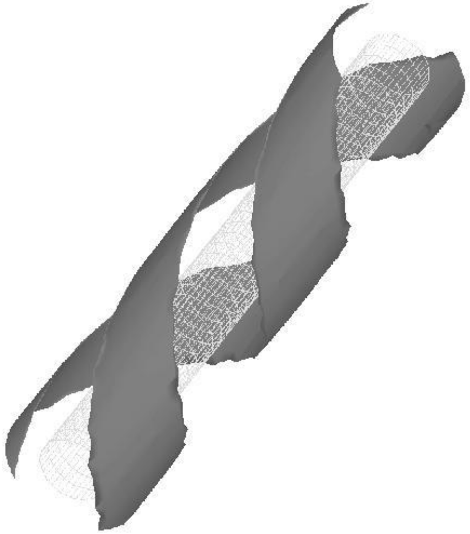
Dynamo action from **2D azimuthal shear** is impossible !



Parker (1966)

In addition, from **Cowling's theorem**, axisymmetric dynamo is impossible !

Helical shear
⇒ dynamo action

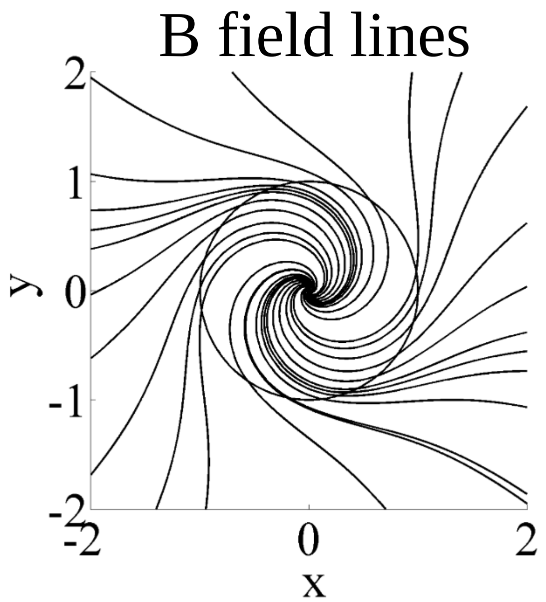


Ponomarenko (1973)

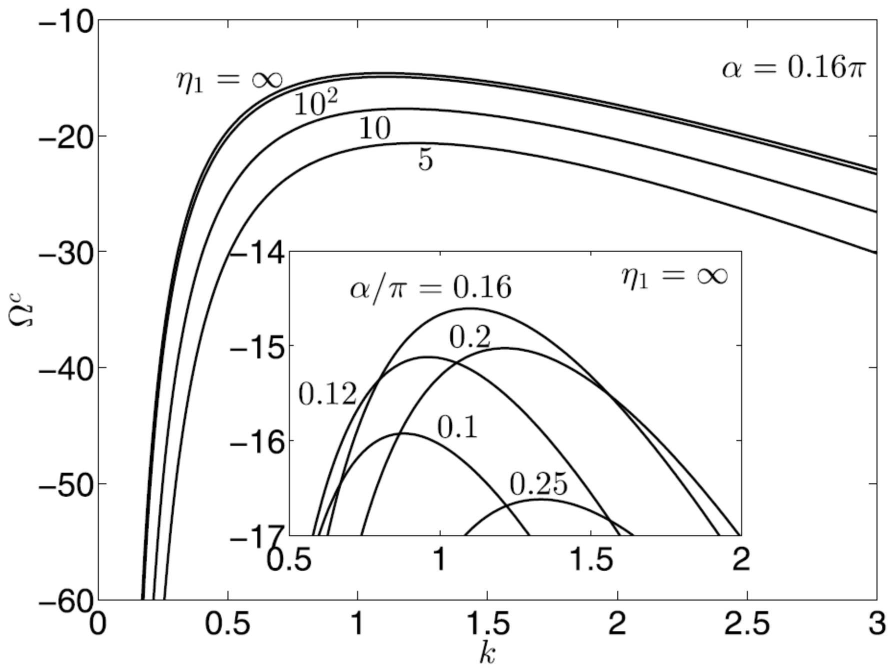
m=1

Introduction

“[…] things we thought impossible … are possible”



We will demonstrate that **2D azimuthal shear** can produce **axisymmetric dynamo** action, provided the electrical **conductivity is anisotropic**



Critical magnetic Reynolds number

$$\Omega^c = \frac{c}{s}\left(I_1(\tilde{k})K_1(\tilde{k}) - I_1(k)K_1(k)\right)^{-1}$$

1. **How can anisotropic conductivity beat Cowling’s antidynamo theorem ?**
2. Exact solution of the induction equation
3. Conclusion and outlook

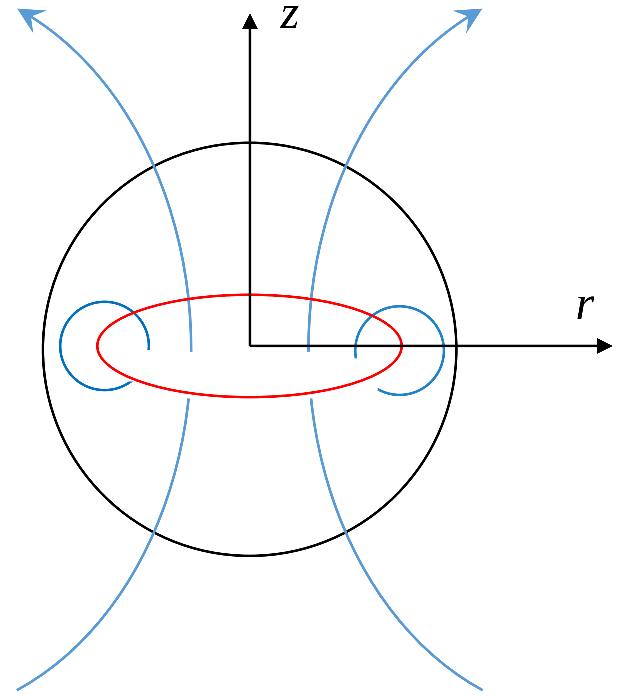
Cowling's antidynamo theorem (1934)

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \left(\frac{1}{\sigma} \nabla \times \left(\frac{\mathbf{B}}{\mu} \right) \right)$$

« An **axisymmetric magnetic field cannot** be generated by dynamo action under the assumption of **axisymmetry** of
- velocity field,
- electrical conductivity,
- magnetic permeability and
- shape of the conductor »

(Kaiser & Tilgner, 2014)

Cowling (1934)
Backus (1957)
Braginskii (1964)
Lortz (1968)
Moffatt (1978)
Ivers & James (1984)
Fearn (1988)
Proctor (2007)

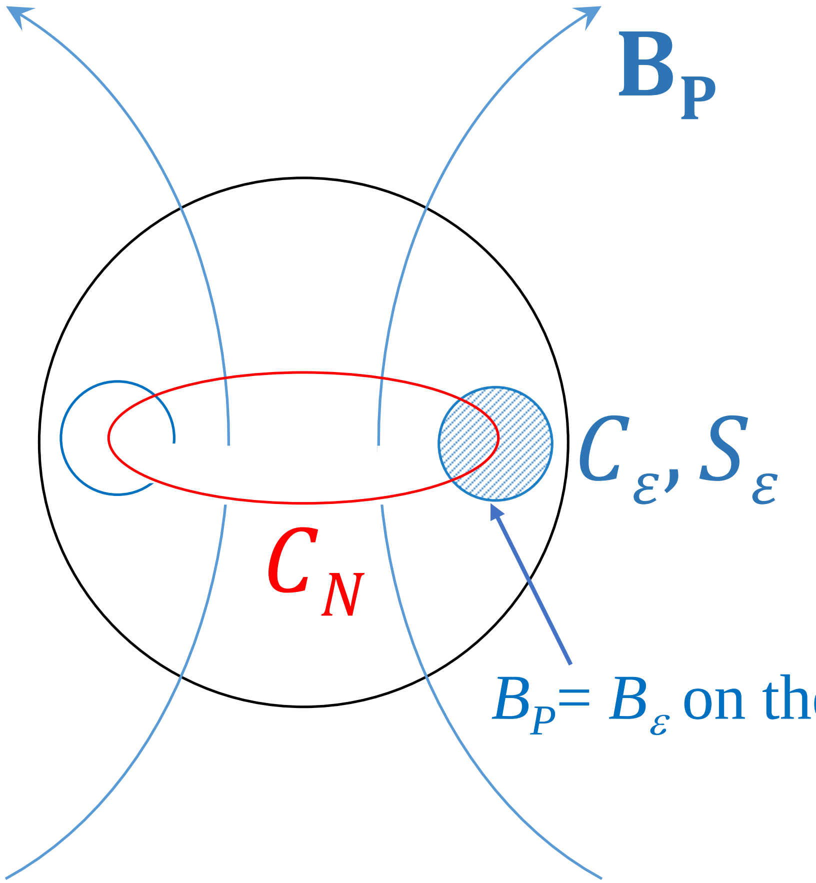


An axisymmetric world

Generalized to time-dependent / nonsolenoidal velocity, variable conductivity, …

Cowling's neutral point argument

Poloidal / Toroidal decomposition



Assume $\mathbf{B_P}$

$$\mathbf{j_T} = \nabla \times \frac{\mathbf{B_P}}{\mu} = \sigma(\mathbf{E_T} + \mathbf{u_P} \times \mathbf{B_P})$$

$$\oint_{C_\varepsilon} \frac{\mathbf{B_P}}{\mu} dl = \int_{S_\varepsilon} \sigma(\mathbf{E_T} + \underbrace{\mathbf{u_P} \times \mathbf{B_P}}_{\leq U_P B_\varepsilon}).\mathbf{dS}$$

Axisymmetry $\Rightarrow \mathbf{E_T} = 0$

$$\Rightarrow 2\pi\varepsilon \frac{B_\varepsilon}{\mu} \leq \sigma U_P B_\varepsilon \pi \varepsilon^2$$

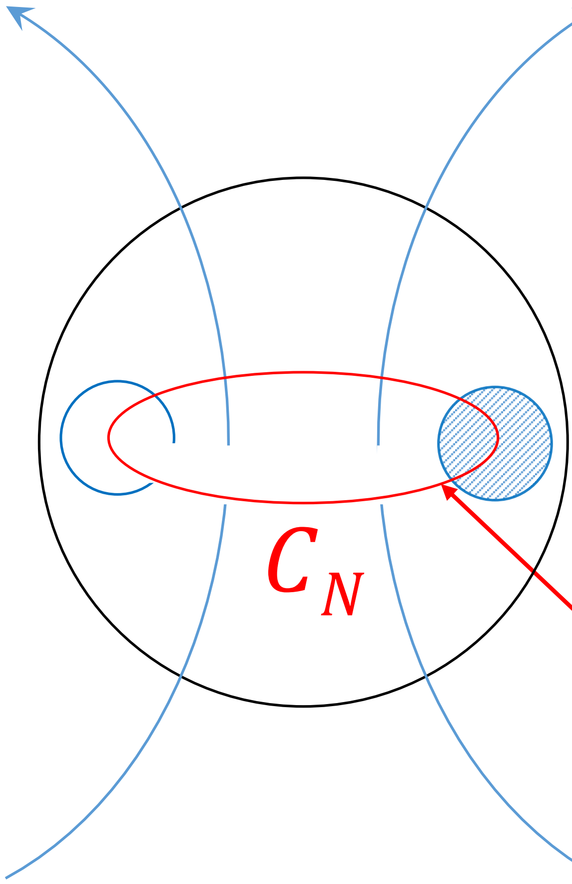
$\lim \varepsilon = 0 \quad \Rightarrow B_\varepsilon = 0$

$\Rightarrow \mathbf{j_T} = \mathbf{0}$ on C_N

$\Rightarrow \mathbf{B_P} = \mathbf{0}$

Cowling's neutral point argument

Poloidal / Toroidal decomposition



$$\mathbf{j_T} = \nabla \times \frac{\mathbf{B_P}}{\mu} = \sigma(\mathbf{E_T} + \mathbf{u_P} \times \mathbf{B_P})$$

Suppose σ is a tensor (anisotropic conductivity)

$$\mathbf{j_T} = [\sigma](\mathbf{E_T} + \mathbf{u_P} \times \mathbf{B_P} + \boldsymbol{E_P} + \mathbf{u_T} \times \mathbf{B_P} + \mathbf{u_P} \times \mathbf{B_T})$$

$\Rightarrow$ Cowling's neutral point argument fails

Anisotropic conductivity

Consider $\mathbf{q}$ such that

$$\sigma = \begin{cases} \sigma^{\parallel} \text{ in the direction of } \mathbf{q} \\ \sigma^{\perp} \text{ in the directions } \mathbf{q}_1 \text{ and } \mathbf{q}_2, \text{ perpendicular to } \mathbf{q} \end{cases}$$

For example, in plasmas:

- $\sigma^{\parallel} = 1.96\, \sigma^{\perp}$, with $\mathbf{q}$ in the direction of the magnetic field

S. I. Braginskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovich, Vol. 1 (Consultants Bureau, New York, 1965), pp.205–311.

Anisotropic conductivity

Consider a unit vector **q** such that

$$\sigma = \begin{cases} \sigma^{\parallel} \text{ in the direction of } \mathbf{q} \\ \sigma^{\perp} \text{ in the directions } \mathbf{q}_1 \text{ and } \mathbf{q}_2\text{, perpendicular to } \mathbf{q} \end{cases}$$

Look for $[\sigma_{ij}]$ satisfying $\mathbf{j} = [\sigma_{ij}]\,\mathbf{E}$, with

- If $\mathbf{E} = \mathbf{q}$ then $\mathbf{j} = \sigma^{\parallel}\,\mathbf{E}$ ⇨ $[\sigma_{ij}]\mathbf{q} = \sigma^{\parallel}\,\mathbf{q}$
- If $\mathbf{E} \,.\, \mathbf{q} = 0$ then $\mathbf{j} = \sigma^{\perp}\,\mathbf{E}$ ⇨ $[\sigma_{ij}]\mathbf{q}_1 = \sigma^{\perp}\,\mathbf{q}_1$
  $[\sigma_{ij}]\mathbf{q}_2 = \sigma^{\perp}\,\mathbf{q}_2$

⇨ $$\boxed{\sigma_{ij} = \sigma^{\perp}\delta_{ij} + (\sigma^{\parallel} - \sigma^{\perp})q_i q_j}$$

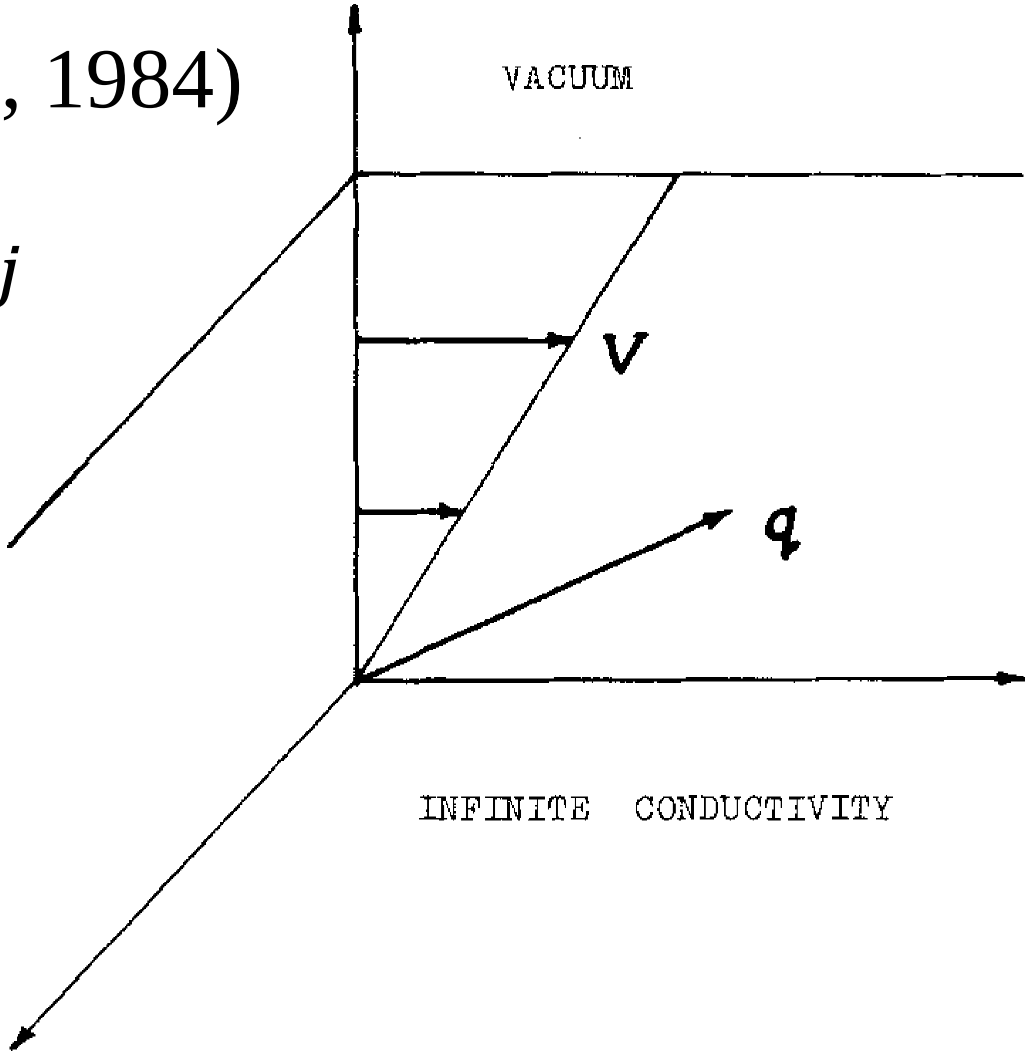
Other dynamo studies with anisotropic conductivity

Ruderman & Ruzmaikin (GAFD, 1984)

$$\sigma_{ij} = \sigma^{\perp}\delta_{ij} + (\sigma^{\parallel} - \sigma^{\perp})q_i q_j$$

- Cartesian geometry
- Linear shear
- Asymptotic study

$$\sigma^{\perp} \gg \sigma^{\parallel}$$

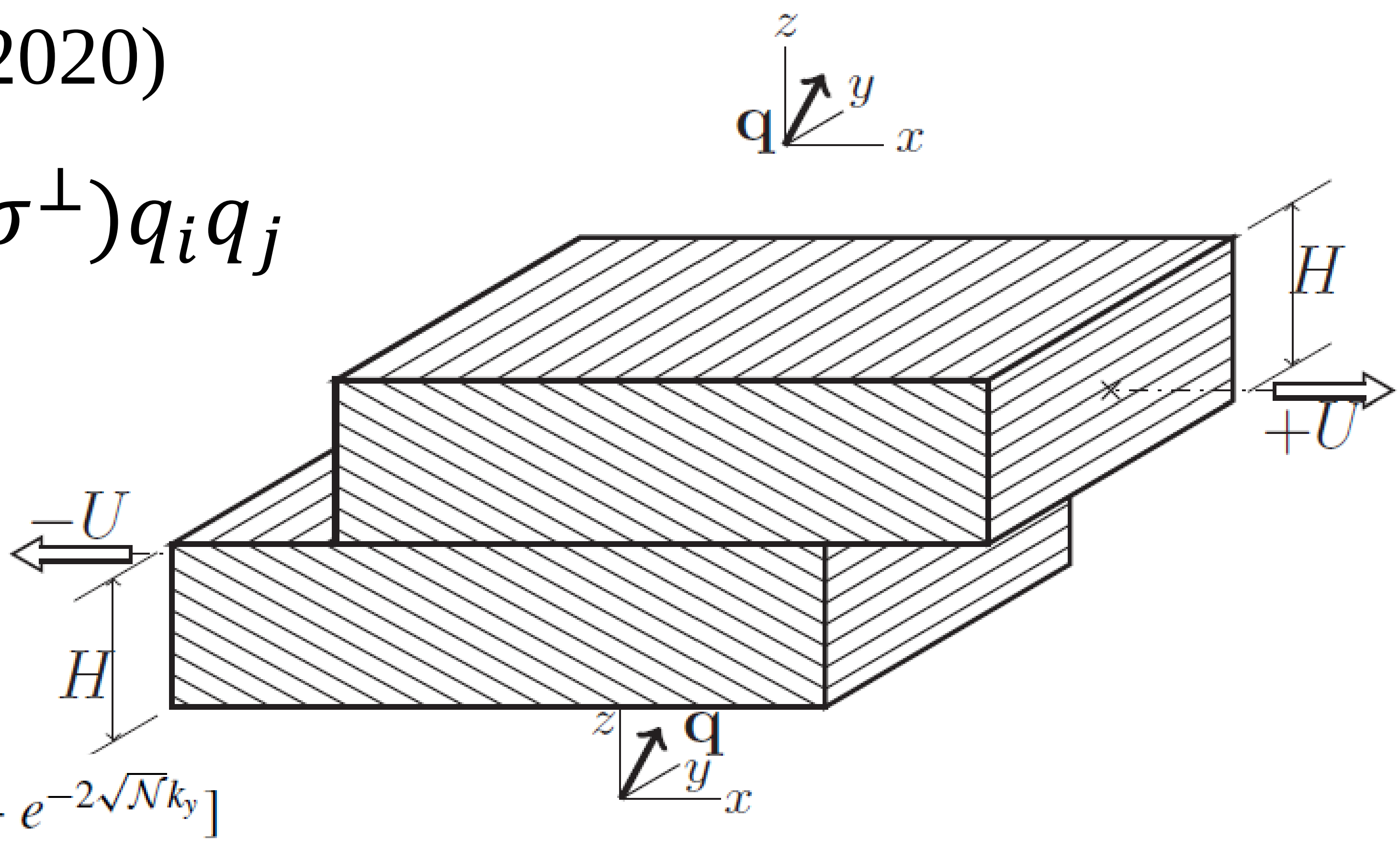


Other dynamo studies with anisotropic conductivity

Alboussière et al (PRE, 2020)

$$\sigma_{ij} = \sigma^{\perp}\delta_{ij} + (\sigma^{\parallel} - \sigma^{\perp})q_i q_j$$

- Cartesian geometry
- Sliding plates
- Exact solutions



$$R_{mc} = \frac{k_y \mathcal{F}(\mathcal{N}-1)[1+e^{-2\sqrt{\mathcal{N}}k_y}]}{1+e^{-2\sqrt{\mathcal{N}}k_y}\left[1+\frac{1}{\sqrt{\mathcal{N}}}\right]-\frac{1}{\sqrt{\mathcal{N}}}-2e^{-(1+\sqrt{\mathcal{N}})k_y}}.$$

Other dynamo studies with anisotropic conductivity

Lortz (Z. Naturforsch, 1989)

$$[\sigma] = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & e \\ 0 & e & 1 \end{pmatrix}_{(\mathbf{e_r}, \mathbf{e}_{\boldsymbol{\theta}}, \mathbf{e}_{\boldsymbol{z}})}$$

- Torus geometry
- Asymptotic study in the limit of zero torus curvature
- Formal proof, without specifying any velocity field.

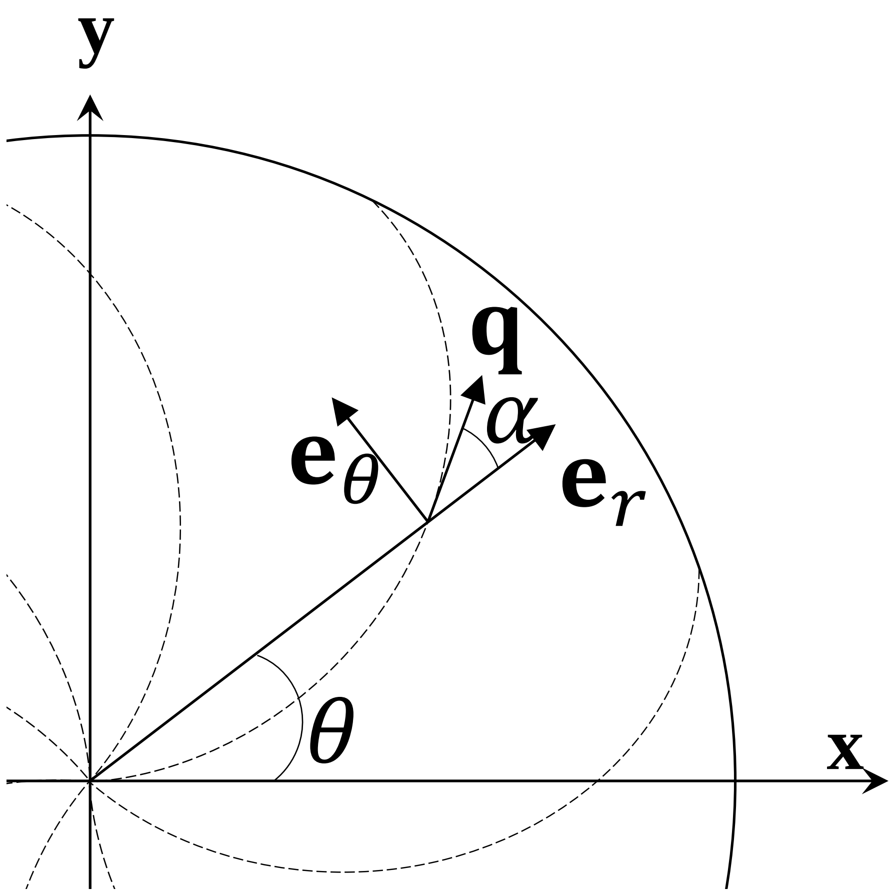
Logarithmic spiral anisotropy

We choose a logarithmic spiral anisotropy

$$\mathbf{q} = \cos\alpha\,\mathbf{e_r} + \sin\alpha\,\mathbf{e}_{\boldsymbol{\theta}}$$

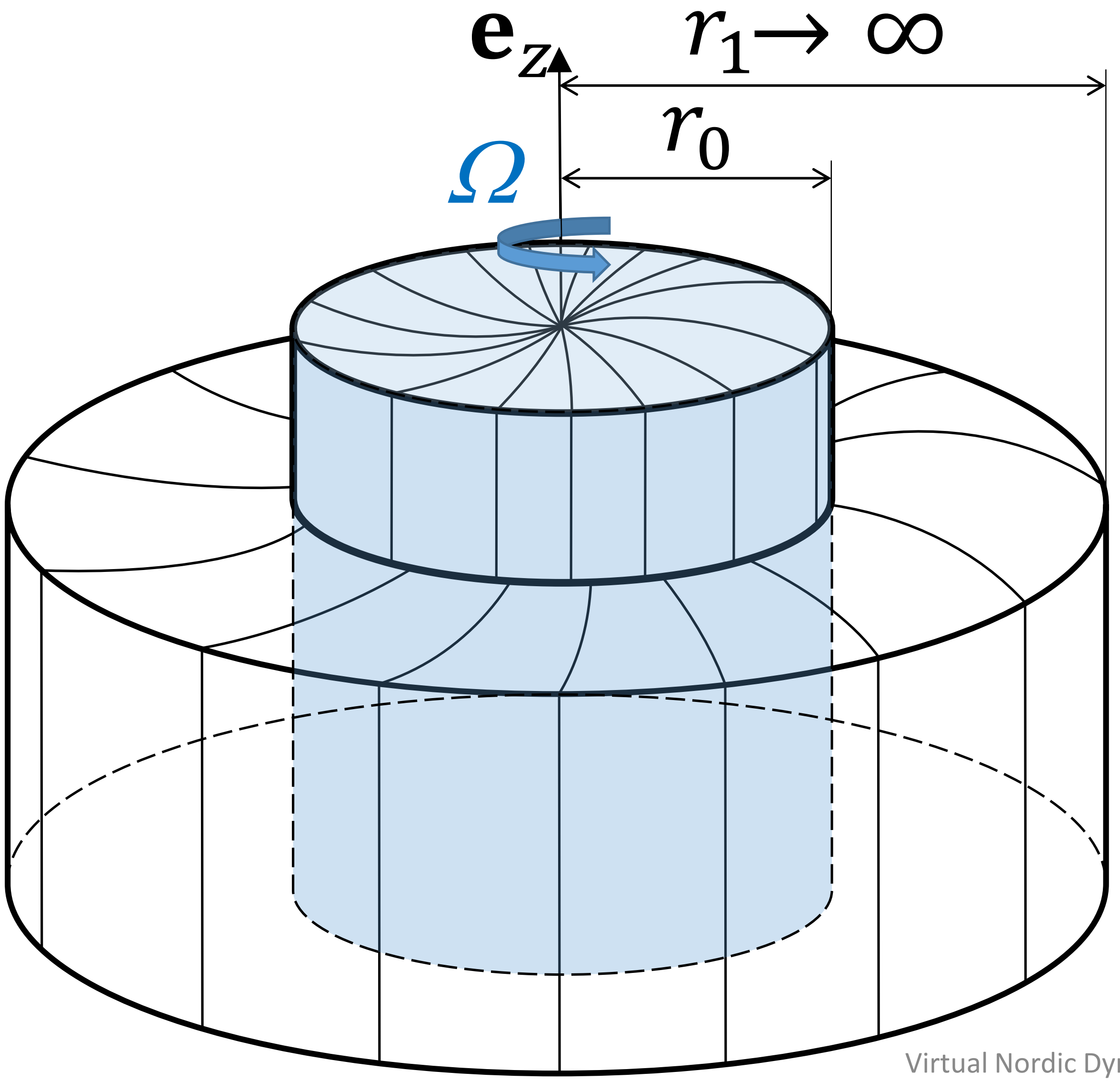
$$[\sigma] = \begin{pmatrix} \sigma^{\parallel}c^2 + \sigma^{\perp}s^2 & (\sigma^{\parallel} - \sigma^{\perp})cs & 0 \\ (\sigma^{\parallel} - \sigma^{\perp})cs & \sigma^{\parallel}s^2 + \sigma^{\perp}c^2 & 0 \\ 0 & 0 & \sigma^{\perp} \end{pmatrix}_{(\mathbf{e_r}, \mathbf{e}_{\boldsymbol{\theta}}, \mathbf{e}_{\boldsymbol{z}})}$$

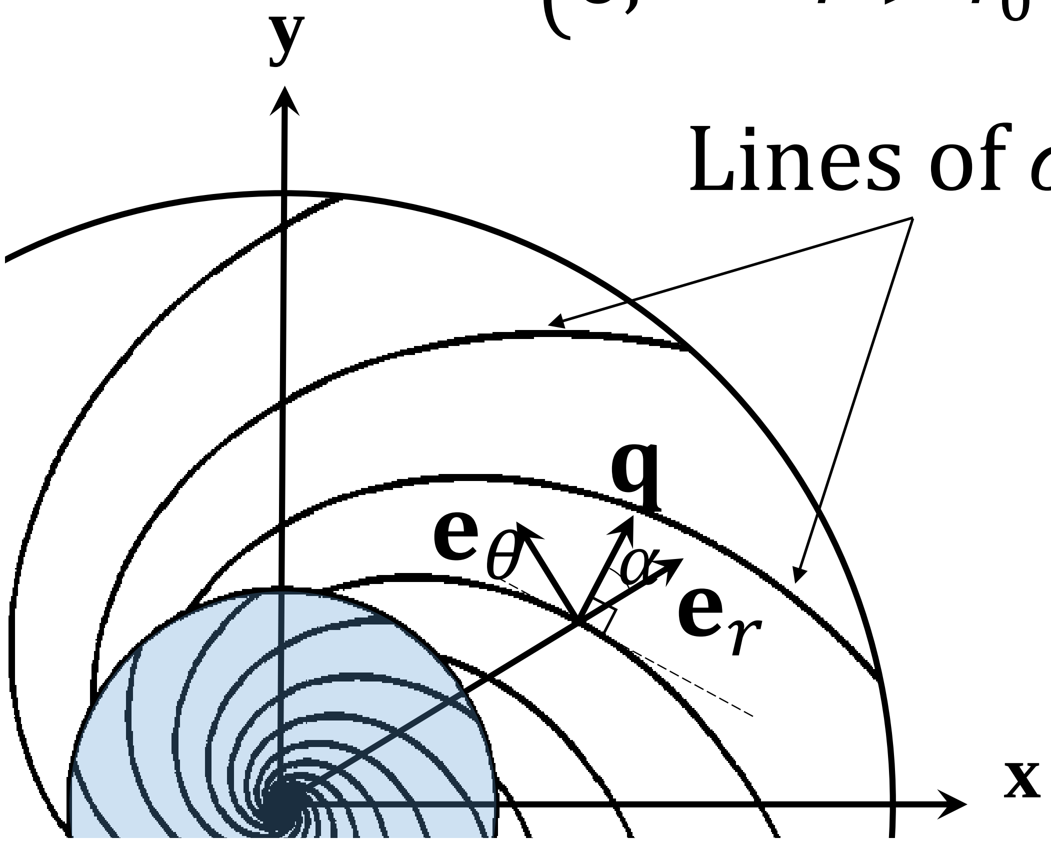
$$(c, s) = (\cos\alpha, \sin\alpha)$$



Inner cylinder in solid body rotation

$$\mathbf{u} = \begin{cases} r\Omega\mathbf{e}_\theta, & r \le r_0 \\ \mathbf{0}, & r > r_0 \end{cases}$$





1. How can anisotropic conductivity beat Cowling’s antidynamo theorem ?
2. **Exact solution of the induction equation**
3. Conclusion and outlook

Dimensionless equations

Induction equation $\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times ([\eta] \nabla \times \mathbf{B})$

Anisotropic diffusivity $\eta_{ij} = \delta_{ij} + \eta_1 q_i q_j, \ \ \eta_1 = \frac{\sigma^{\perp}}{\sigma^{\parallel}} - 1$

Solid body rotation $\mathbf{u} = \begin{cases} r\Omega \mathbf{e}_\theta, r \le 1 \\ \mathbf{0}, \quad\quad r > 1 \end{cases}$

Axisymmetry $\mathbf{B}(r, z, t) = \tilde{B}\mathbf{e}_\theta + \nabla \times \left(\tilde{A}\mathbf{e}_\theta\right)$

$\Rightarrow \nabla \times (\mathbf{u} \times \mathbf{B}) = 0$

Stationary and z-independent flow conjugated to linearity of Ind. Eq. implies

$$\mathbf{B} = \left(-\mathrm{i}kA, B, \frac{1}{r}\partial_r(rA)\right) \exp(\gamma t + \mathrm{i}kz)$$

$\Rightarrow$ 1D problem

Resolution

$$\mathbf{B} = \left(-\mathrm{i}kA, B, \frac{1}{r}\partial_r(rA) \right) \exp(\gamma t + \mathrm{i}kz)$$

$$\gamma A + D_k(A) = i\eta_1 cskB - \eta_1 s^2 D_k(A),$$

$$\gamma B + D_k(B) = -i\eta_1 cskD_k(A) - \eta_1 c^2 k^2 B,$$

$$\eta_1 = \frac{\sigma^{\perp}}{\sigma^{\parallel}} - 1$$

where $D_\nu(X) = \nu^2 X - \partial_r[\frac{1}{r}\partial_r(rX)]$, $c = \cos\alpha$ and $s = \sin\alpha$.

Looking for the dynamo threshold ($\gamma = 0$) leads to

$$D_{\tilde{k}}(B) = D_k\left(B - i\frac{ck}{s}A\right) = 0, \qquad \tilde{k} = k\left(\frac{1+\eta_1}{1+\eta_1 s^2}\right)^{1/2}$$

Solution at threshold

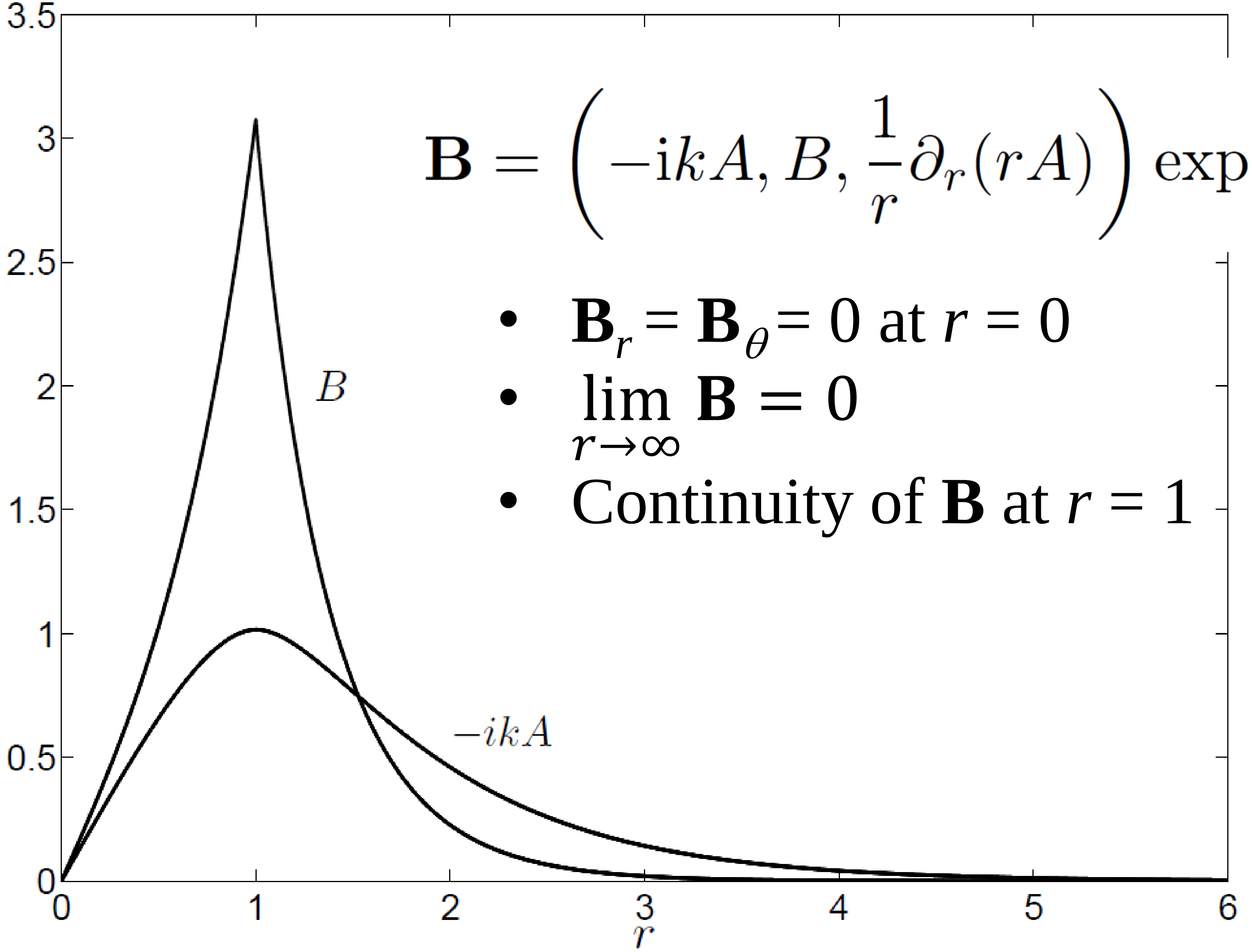
The solutions of $D_\nu(X) = 0$ being a linear combination of $I_1(\nu r)$ and $K_1(\nu r)$, we find

$$r < 1, \quad \begin{cases} A = \frac{s}{ick}\left(\lambda \frac{I_1(\tilde{k}r)}{I_1(\tilde{k})} + \mu \frac{I_1(kr)}{I_1(k)}\right) \\ B = \lambda \frac{I_1(\tilde{k}r)}{I_1(\tilde{k})} \end{cases} \tag{13}$$

$$r > 1, \quad \begin{cases} A = \frac{s}{ick}\left(\lambda \frac{K_1(\tilde{k}r)}{K_1(\tilde{k})} + \mu \frac{K_1(kr)}{K_1(k)}\right) \\ B = \lambda \frac{K_1(\tilde{k}r)}{K_1(\tilde{k})}, \end{cases} \tag{14}$$

with $\lambda\Gamma(\tilde{k}) + \mu\Gamma(k) = 0$

where I_1 and K_1 are modified Bessel functions of first and second kind. In (13) and (14) the following boundary conditions have been applied to A and B: finite values at $r = 0$, continuity at $r = 1$, and $\lim_{r\to\infty} A, B = 0$.



$$\mathbf{B} = \left(-\mathrm{i}kA, B, \frac{1}{r}\partial_r(rA)\right)\exp(\gamma t + \mathrm{i}kz)$$

- $\mathbf{B}_r = \mathbf{B}_\theta = 0$ at $r = 0$
- $\lim_{r\to\infty} \mathbf{B} = 0$
- Continuity of **B** at $r = 1$

A crucial boundary condition

The continuity of $\mathbf{E}_z$ implies the following identity

$$(\partial_r B - \mathrm{i}k\Omega A)(r = 1^-) = \partial_r B(r = 1^+).$$

Here is the velocity!

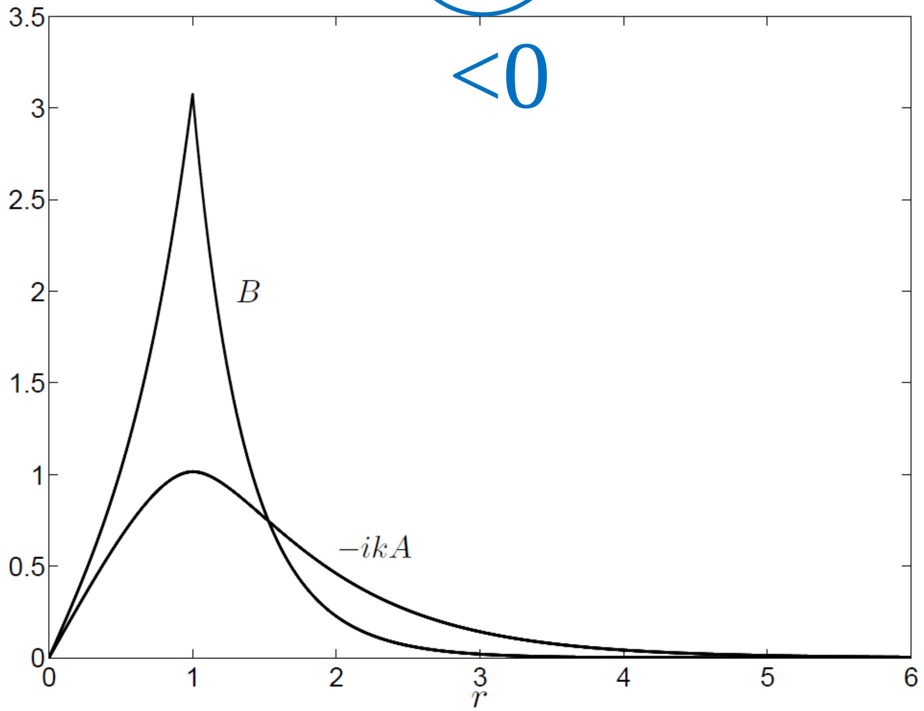
It corresponds to the differential rotation between the inner and outer cylinders

A crucial boundary condition

The continuity of $\mathbf{E}_z$ implies the following identity

$$(\partial_r B - ik\Omega A)(r = 1^-) = \partial_r B(r = 1^+).$$

$>0 \quad >0 \quad\quad <0 \quad \Rightarrow \Omega < 0$



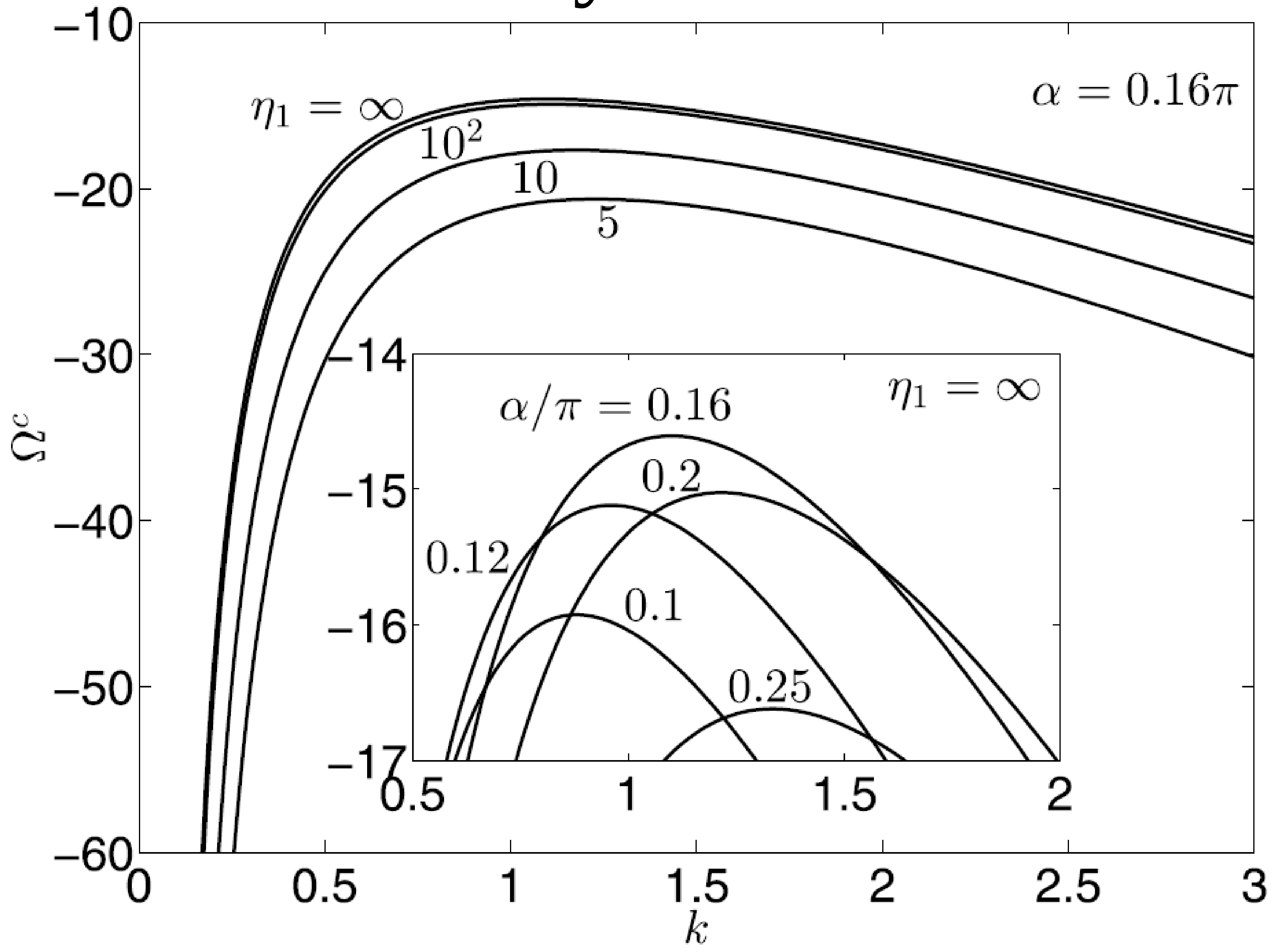
Dynamo threshold

$$\Omega^c = \frac{c}{s}\left(I_1(\tilde{k})K_1(\tilde{k}) - I_1(k)K_1(k)\right)^{-1}$$

Explicit 
expression !

with $\tilde{k} = k\left(\frac{1+\eta_1}{1+\eta_1 s^2}\right)^{1/2}$ $\eta_1 = \frac{\sigma^\perp}{\sigma^\parallel} - 1$

Dynamo threshold



$\sigma^{\perp} > \sigma^{\parallel}$

$$\eta_1 = \frac{\sigma^{\perp}}{\sigma^{\parallel}} - 1 > 0$$

Dynamo threshold

The minimum value of $|\Omega^c|$ is obtained for $\eta_1 \to \infty$, $k^* = 1.1$ and $\alpha^* = 0.16\pi$,

$$\Omega^* = \min_{\eta,k,\alpha} |\Omega^c| = 14.6$$

results. Considering an inner cylinder of radius $r_0 = 0.05$ m, taking the conductivity of copper $\mu_0\sigma_0 \approx 72.9$ s m^{-2} leads to a dynamo threshold $f^* = \Omega^*(2\pi\mu_0\sigma_0 r_0^2)^{-1} \approx 12.8$ Hz. Provided the cylinder height and outer radius r_1 are sufficiently large, this is experimentally achievable. Such an anisotropic

⇨ experimentally feasible

Understanding

Crucial term

$$\gamma \mathbf{B}_r = \eta_1 c s k^2 \mathbf{B}_\theta - (1 + \eta_1 s^2) D_k(\mathbf{B}_r),$$

$$\gamma \mathbf{B}_\theta = \eta_1 c s D_k(\mathbf{B}_r) - (D_k + \eta_1 c^2 k^2) \mathbf{B}_\theta. + \frac{s}{c}(\partial_r \Omega)\mathbf{B}_r$$

Diffusive terms

Differential rotation

$$\eta = \begin{pmatrix} 1 + \eta_1 c^2 & \eta_1 c s & 0 \\ \eta_1 c s & 1 + \eta_1 s^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

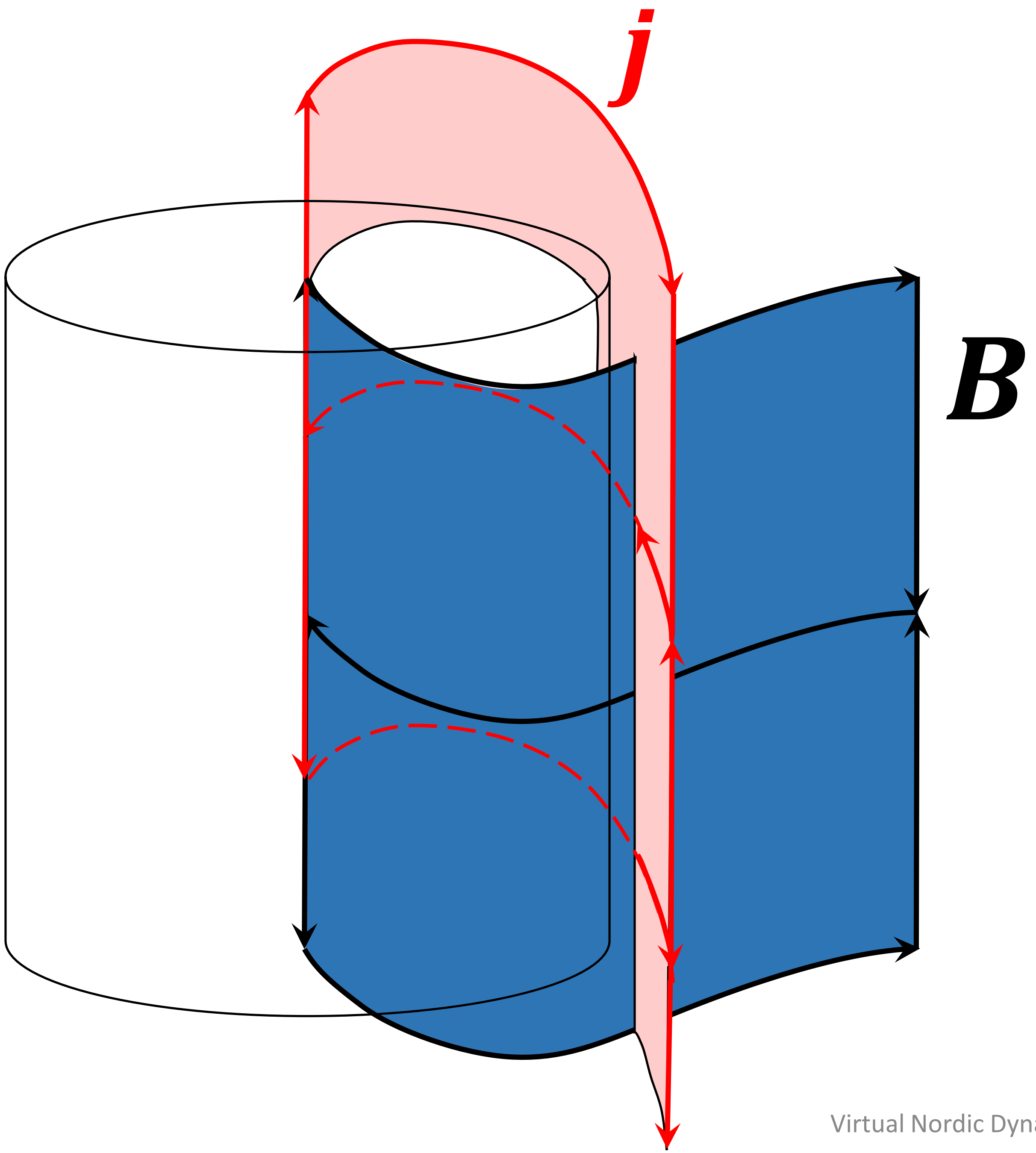
Spiraling currents

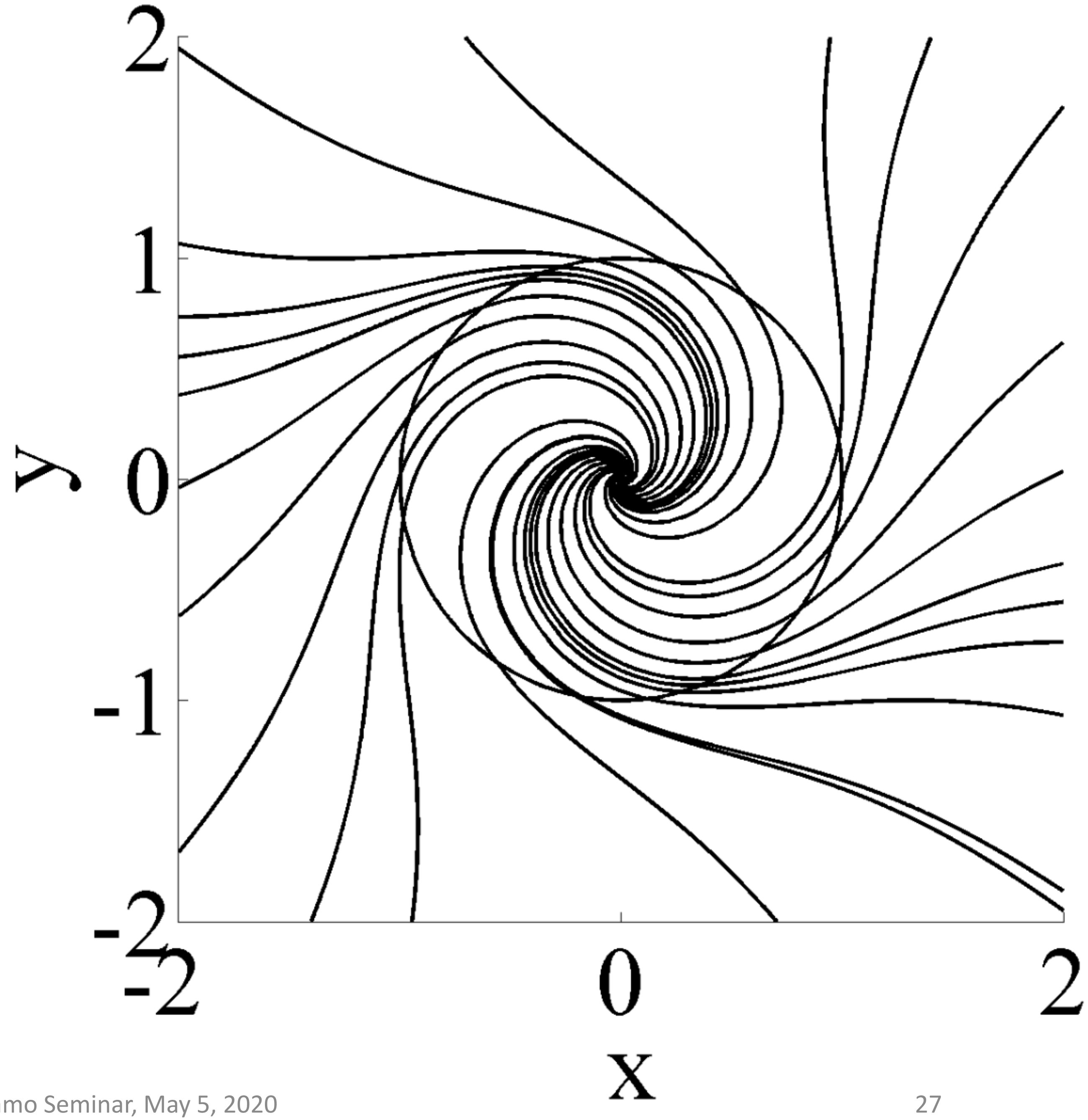
$$\mathbf{B} = \left(-ikA,\, B,\, \frac{1}{r}\partial_r(rA)\right)\exp(\gamma t + ikz)$$

$$\mathbf{j} = \left(-ikB,\, D_k(A),\, \frac{1}{r}\partial_r(rB)\right)\exp(\gamma t + ikz)$$

At threshold: $\frac{j_\theta}{j_r} = -\frac{\eta_1 cs}{1+\eta_1 s^2}$ $\Rightarrow$ spiraling currents

$$\eta_1 \to \infty \Rightarrow \mathbf{j}.\,\mathbf{q} = 0$$


j
B


2
1
0
-1
-2
y
-2
0
2
x

1. How can anisotropic conductivity beat Cowling's antidynamo theorem ?
2. Exact solution of the induction equation
3. **Conclusion and outlook**

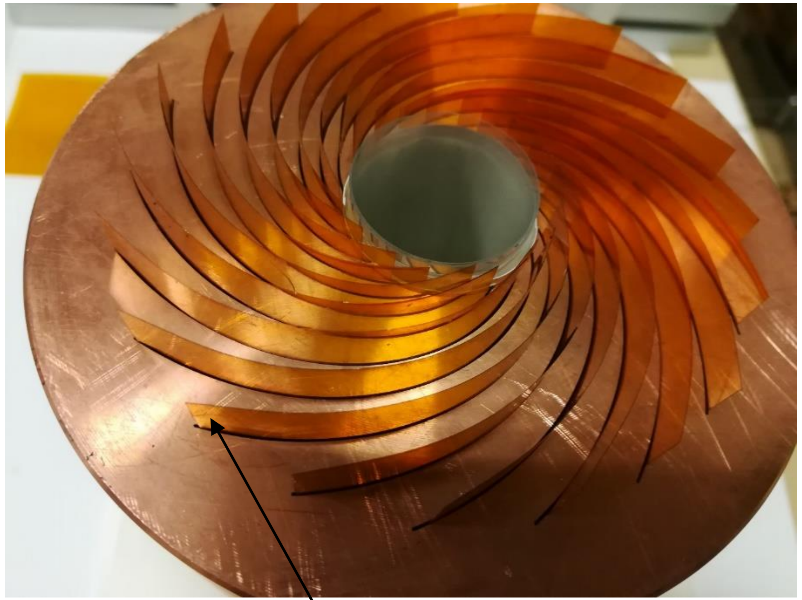
Conclusion and outlook

- **Axismmetric dynamo** action is possible with anisotropic conductivity
- **Exact solution** for logarithmic spiral anisotropy and simple azimuthal shear
- Threshold sufficiently low to be **experimentally achievable**
- Application to plasmas ?

An experimental demonstration

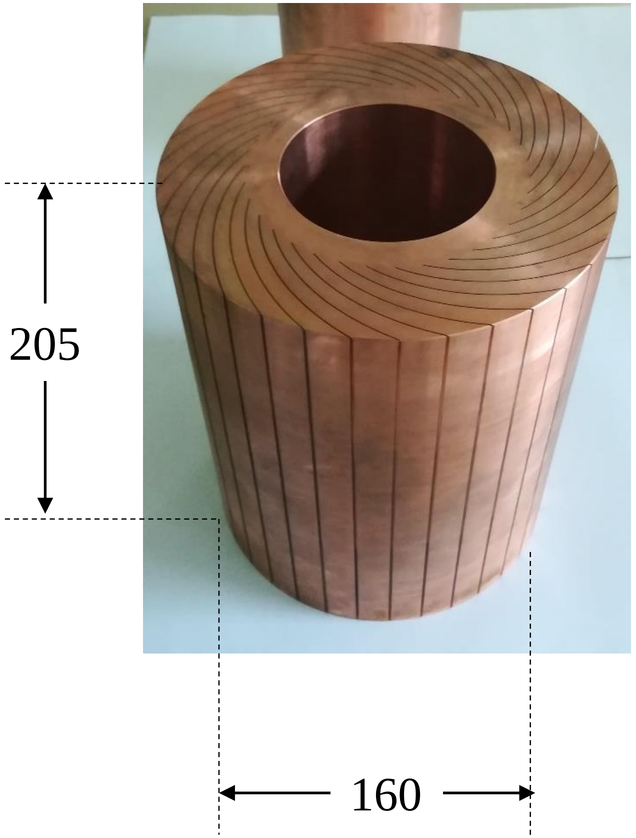
$$\sigma^{\perp} > \sigma^{\parallel}$$

Inner cylinder



Spiraling insulator
(kapton + epoxy)

Outer cylinder



Both cylinders before final electroerosion

Alboussière 2020

Application to plasmas ?

where the electrical conductivities are

$$\sigma_\perp = \frac{e^2 n_e \tau_e}{m_e} = \sigma_1 T_e^{3/2},$$

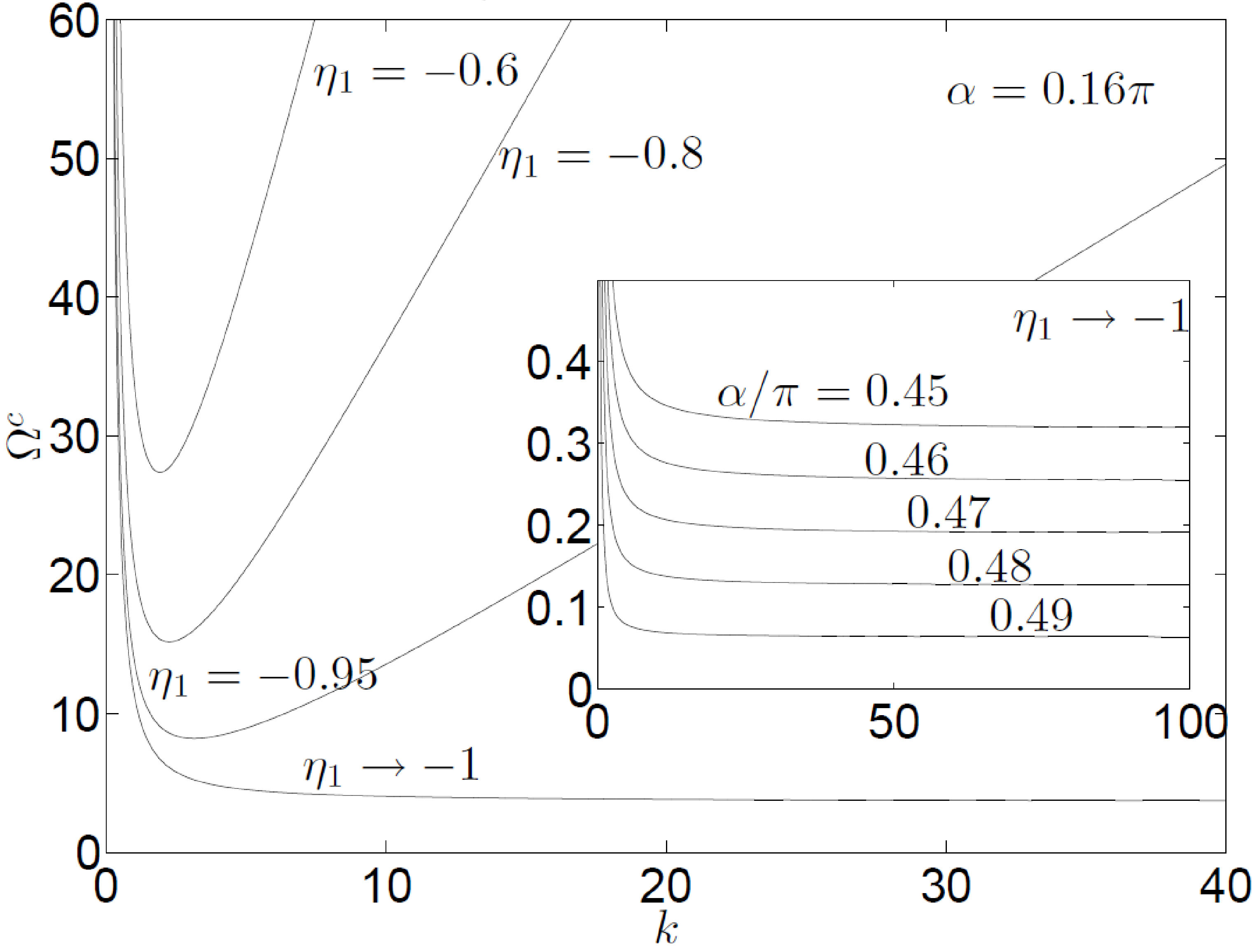
$$\sigma_\| = 1.96\sigma_\perp = 1.96\sigma_1 T_e^{3/2},$$

$\eta_1 \sim -0,5$

- S. I. Braginskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovich, Vol. 1 (Consultants Bureau, New York, 1965), pp.205–311.

$$\sigma^\perp < \sigma^\| \Rightarrow \eta_1 = \frac{\sigma^\perp}{\sigma^\|} - 1 < 0$$

Dynamo threshold



$\sigma^{\perp} < \sigma^{\parallel}$

$$\eta_1 = \frac{\sigma^{\perp}}{\sigma^{\parallel}} - 1 < 0$$